Readings:

(a) Notes from Lecture 10 and 11.

(b) [Grimmett-Stirzaker]: Section 4.1-4.10. Optionally, Section 4.11.

Exercise 1. The probabilistic method. Twelve per cent of the circumference of a circle is colored blue, the rest is red. Show that, irrespective of the manner in which the colors are distributed, it is possible to inscribe a regular octagon in the circle with all its vertices red.

Hint: The probabilistic method is a general method for proving existence: if you can prove that a randomly selected structure has certain desired properties with some positive probability (no matter how small), then a structure with these properties is guaranteed to exist.

Solution: We pick an inscribed regular octagon at random by choosing uniformly over the circle the position of a vertex. Let event V_i be the event that vertex i is red, $i = 1, \ldots, 8$. We are interested in showing that $\mathbb{P}(V_1 \cap \cdots \cap V_8) > 0$. Note that for any i, $\mathbb{P}(V_i) = 0.88$. Unfortunately, our events are not independent, so we cannot multiply probabilities. Instead, we have:

$$\begin{aligned}
\mathbb{P}(V_1 \cap \cdots \cap V_8) &= 1 - \mathbb{P}((V_1 \cap \cdots \cap V_8)^c) \\
&= 1 - \mathbb{P}(V_1^c \cup \cdots \cup V_8^c) \\
&\geq 1 - \sum_{i=1}^{8} \mathbb{P}(V_i^c) \\
&= 1 - 8 \times 0.12 = 0.04 > 0.
\end{aligned}$$

Exercise 2. Suppose X is a continuous random variable with a power law distribution. Namely there exists $c > 0$ and $\alpha > 0$ such that $\mathbb{P}(X > x) = \frac{c}{x}$, for every $x \geq c$. Consider the r-th moment of X, namely $\mathbb{E}[X^r]$, where $r > 0$ is any real value. Find necessary and sufficient conditions for r in terms of c and α for the r-th moment to be finite.

Solution: The pdf for X is

$$f_X(x) = \frac{d}{dx} F_X(x) = \alpha\, c^{\alpha} x^{-(1+\alpha)}.$$

Thus, we have

$$\mathbb{E}[X^r] = \int_c^{\infty} x^r \alpha c^{\alpha} x^{-(1+\alpha)} dx,$$

which is finite iff $r < \alpha$.

Exercise 3. We have a stick of unit length $[0, 1]$, and break it at X, where X is uniformly distributed on $[0, 1]$. Given the value x of X, we let Y be uniformly distributed on $[0, x]$, and let Z be uniformly distributed on $[0, 1-x]$. We assume that conditioned on $X = x$, the random variables Y and Z are independent. Find the joint PDF of Y and Z. Find $\mathbb{E}[X|Y]$, $\mathbb{E}[X|Z]$, and $\rho(Y, Z)$.

Solution: The conditional PDFs for $X|Y$ and $X|Z$ are

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{(1/x) 1_{\{y \leq x\}} 1_{x \in [0,1]}}{\log 1/y},$$

and

$$f_{X|Z}(x|z) = \frac{f_{Z|X}(z|x) f_X(x)}{f_Z(z)} = \frac{(1/(1-x)) 1_{z \leq 1-x} 1_{x \in [0,1]}}{\log 1/z}.$$

Therefore, the joint PDF of X, Y, Z is

$$\begin{aligned} f_{Y,Z,X}(y, z, x) &= f_{Y,Z|X}(y, z|x) f_X(x) = f_{Y|X}(y|x) f_{Z|X}(z|x) f_X(x) \\ &= \frac{1}{x} \frac{1}{1-x} 1_{\{y \leq x\}} 1_{\{z \leq 1-x\}}, \end{aligned}$$

and integrating the joint pdf of Y and Z is

$$\begin{aligned} f_{Y,Z}(y, z) &= \int_0^1 f_{Y,Z,X}(y, z, x) dx \\ &= \int_y^{1-z} \frac{1}{x(1-x)} dx \\ &= \log(1-z) - \log y + \log(1-y) - \log z \end{aligned}$$

when $y \leq 1 - z$, and 0 otherwise.

Moreover, the conditional expectations for $X|Y$ and $X|Z$ are

$$E[X|Y] = \int_0^1 x f_{X|Y}(x|y)dx = \int_0^1 \frac{1_{\{y \leq x\}}}{\log 1/y} dx = \frac{1-y}{\log 1/y} = \frac{y-1}{\log y},$$

and

$$\begin{aligned} E[X|Z] &= \int_0^1 x f_{X|Z}(x|z)dx \\ &= \int_0^1 \frac{x}{1-x} \frac{1_{z \leq 1-x}}{\log 1/z} dx \\ &= \frac{1}{\log 1/z} \int_0^{1-z} \frac{x}{1-x} dx \\ &= \frac{-1 + z - \log z}{\log 1/z} \\ &= \frac{1 - z + \log z}{\log z}. \end{aligned}$$

Finally, observe that X and $1 - X$ are identically distributed, and therefore Y and Z are identically distributed. See lecture notes for Lecture 9 for the computation $E[Y] = 1/4$, $E[YZ] = 1/24$. It follows that $E[Z] = 1/4$, and thus,

$$\operatorname{cov}(Y, Z) = \frac{1}{24} - \frac{1}{4}\frac{1}{4} = -\frac{1}{48}.$$

Now we compute the variances. We have that

$$E[Y^2] = E[E[Y^2|X]] = E[(1/3)X^2] = \frac{1}{9},$$

where we used the fact that the uniform random variable on $[0, x]$ has square-expectation of $x^2/3$. Next,

$$\operatorname{var}(Y) = E[Y^2] - E[Y]^2 = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

Thus the correlation coefficient is

$$\rho(Y, Z) = \frac{\operatorname{cov}(Y, Z)}{\sqrt{\operatorname{var}(Y)\operatorname{var}(\mathrm{Z})}} = \frac{-1/48}{\sqrt{(7/144)(7/144)}} = -\frac{3}{7}.$$

Exercise 4. Assume that $X_1, \dots, X_n$ are independent continuous random variables with common density function function f. Let $X^{(1)}, \dots, X^{(n)}$ be the ordered statistics of $X_1, \dots, X_n$. Namely, $X^{(1)}$ is the smallest of $X_1, \dots, X_n$, $X^{(2)}$ is the second smallest, etc., and $X^{(n)}$ is the largest of them all. Establish that the joint distribution of $X^{(1)}, \dots, X^{(n)}$ is given by the joint density

$$f_{X^{(1)},\dots,X^{(n)}}(x_1, \dots, x_n) = n! f(x_1) \cdots f(x_n), \qquad x_1 < x_2 < \cdots < x_n,$$

and $f_{X^{(1)},\dots,X^{(n)}}(x_1, \dots, x_n) = 0$, otherwise. Use this to derive the densities for $\max_j X_j$ and $\min_j X_j$.

Solution: Define

$$g(x_1, \dots, x_n) := n! f(x_1) f(x_2) \cdots f(x_n) 1_{x_1 < x_2 < \cdots < x_n}.$$

For $x_1 < x_2 < \cdots < x_n$

$$\begin{aligned}
&\mathbb{P}(X^{(1)} \le x_1, X^{(2)} \le x_2, \dots, X^{(n)} \le x_n) \\
&\quad = n! P(X_1 \le x_1, X_1 < X_2 \le x_2, \dots, X_{n-1} < X_n \le x_n), \\
&\quad = \int_{z_1 \le x_1, z_1 < z_2 \le x_2, \cdots, z_{n-1} < z_n \le x_n} n! f(z_1) f(z_2) \cdots f(z_n) \\
&\quad = \int_{z_1 \le x_1, z_2 \le x_2, \dots, z_n \le x_n} g(z_1, \dots, z_n),
\end{aligned}$$

where the first line follows by explicitly enumerating the $n!$ ways in which the event $\{X^{(1)} \le x_1, X^{(2)} \le x_2, \dots, X^{(n)} \le x_n\}$ could occur; the second line follows since f is a density for each X_i, and the X_i are independent; and the third line follows by definition of g. This last equality implies that g is the joint density of the random variables $X^{(1)}, X^{(2)}, \dots, X^{(n)}$.

By definition, $\max_j X_j = X^{(n)}$ and the distribution for $X^{(n)}$ is obtained by integrating out the other variables. More specifically, the joint density of $X^{(2)}, \dots, X^{(n)}$ is

$$\begin{aligned}
g(y_2, \dots, y_n) &= \int_{-\infty}^{+\infty} g(y_1, y_2, \dots, y_n) dy_1 \\
&= \int_{-\infty}^{y_2} n! f(y_1) f(y_2) \cdots f(y_n) 1_{y_1 < y_2 < \cdots y_n} dy_1 \\
&= n! F(y_2) f(y_2) \cdots f(y_n) 1_{y_2 < y_3 < \cdots < y_n}.
\end{aligned}$$

Similarly, the joint density of $X^{(3)}, \dots, X^{(n)}$ is

$$\begin{aligned}
g(y_3, \dots, y_n) &= \int_{-\infty}^{+\infty} n! F(y_2) f(y_2) \cdots f(y_n) 1_{y_2 < y_3 < \cdots y_n} dy_2 \\
&= n! \frac{1}{2} F(y_3)^2 f(y_3) f(y_4) \cdots f(y_n) 1_{y_3 < y_4 < \cdots y_n}.
\end{aligned}$$

After, $n-1$ such integrations

$$g(y_n) = \frac{n!}{(n-1)!} F(y_n)^n f(y_n) = nF(y_n)^{n-1} f(y_n).$$

Again, by definition $\min_j X_j = X^{(1)}$ and we proceed in a similar manor. The joint density of $X^{(1)}, \ldots, X^{(n-1)}$ is

$$\begin{aligned} g(y_1, \ldots, y_{n-1}) &= \int_{y_{n-1}}^{+\infty} n! f(y_1) f(y_2) \cdots f(y_n) 1_{y_1 < y_2 < \cdots y_n} dy_n \\ &= n! f(y_1) f(y_2) \cdots f(y_{n-1}) [1 - F(y_{n-1})] 1_{y_1 < y_2 < \cdots y_{n-1}}, \end{aligned}$$

and of for $X^{(1)}, \ldots, X^{(n-2)}$

$$\begin{aligned} g(y_1, \ldots, y_{n-2}) &= \int_{y_{n-3}}^{+\infty} n! f(y_1) f(y_2) \cdots f(y_{n-1}) [1 - F(y_{n-1})] 1_{y_1 < y_2 < \cdots y_{n-1}} dy_{n-1} \\ &= n! f(y_1) f(y_2) \cdots f(y_{n-2}) [1 - F(y_{n-2})]^2 \frac{1}{2} 1_{y_1 < y_2 < \cdots y_{n-2}}. \end{aligned}$$

After $n-1$ such integrations, the resulting density is

$$g(y_1) = n[1 - F(y_1)]^{n-1} f(y_1).$$

Exercise 5. Let $X_1, \ldots, X_n$ be independent r.v. with $\mathrm{Exp}(\lambda)$ distribution. Consider $S_n = \sum_{1 \le j \le n} X_j$. The distribution of S_n is sometimes called *Erlang*.

(a) Establish that the density of S_n is $f_{S_n}(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} \exp(-\lambda x)$.
(A Gamma distribution with an integer shape parameter n.)

(b) Consider the joint distribution of $S_1, S_2, \ldots, S_{n-1}$ given $S_n = x$. Establish that this joint distribution is the same as the joint distribution of $U^{(1)}, \ldots, U^{(n-1)}$, where $U^{(1)}, \ldots, U^{(n-1)}$ is the order statistics of $n-1$ independent r.v. with $U(0, x)$ distribution.

Solution:

(a) We proceed by induction. The case of $n = 1$ holds as this is just the exponential density. Assuming for n exponentials, the density of $n+1$ exponentials is given by convolution

$$\begin{aligned} f_{S_{n+1}}(z) &= \int_0^z f_{S_n}(t) f_{X_{n+1}}(z-t) dt \\ &= \int_0^z \frac{\lambda^n t^{n-1}}{(n-1)!} \lambda \exp(-\lambda(z-t)) dt \\ &= \frac{\lambda^{n+1} t^n}{n!} \end{aligned}$$

as desired.

(b) Let A be the event that $S_1 \leq S_2 \leq \cdots \leq S_n$. Then

$$\begin{aligned} f_{S_1,\ldots,S_{n-1}|S_n}(s_1,\ldots,s_{n-1}|s_n) &= \frac{f_{S_1,\ldots,S_n}(s_1,\ldots,s_n)}{f_{S_n}(s_n)} \\ &= \frac{f_{X_1,\ldots,X_n}(s_1,s_2-s_1\ldots,s_n-s_{n-1})}{f_{S_n}(s_n)} \\ &= \frac{f_{X_1}(s_1)\cdot f_{X_2}(s_2-s_1)\cdot\cdots\cdot f_{X_n}(s_n-s_{n-1})}{f_{S_n}(s_n)} \\ &= \frac{\lambda e^{-\lambda s_1}\cdot\lambda e^{-\lambda(s_2-s_1)}\cdot\cdots\cdot\lambda e^{-\lambda(s_n-s_{n-1})}}{\lambda^n s_n^{n-1}e^{-\lambda s_n}/(n-1)!}\mathbb{1}_A \\ &= \frac{\lambda^n e^{-\lambda s_n}}{\lambda^n s_n^{n-1}e^{-\lambda s_n}/(n-1)!}\mathbb{1}_A \\ &= \frac{(n-1)!}{s_n^{n-1}}\mathbb{1}_A. \end{aligned}$$

Observing that $U(0,s_n)$ have density $f_U(u) = 1/s_n \mathbb{1}_{u\in(0,s_n)}$, we compute that

$$f_{U^{(1)},\ldots,U^{(n-1)}}(u_1,\ldots,u_{n-1}) = (n-1)!f_U^{n-1}(u)\mathbb{1}_A = \frac{(n-1)!}{s_n^{n-1}}\mathbb{1}_A,$$

giving the result.

Exercise 6. A needle of length $2s < 1$ unit is randomly tossed onto a quad-ruled sheet with horizontal and vertical lines spaced at 1 unit. Assuming the position and the angle of the needle are independent and uniform, find the average number of lines the needle intersects.

Solution: Consider the unit square $[0,1]\times[0,1]$ in the x,y plane. The center of the needle will be uniformly distributed within this square. Let X and Y be uniform random variables for the x and y coordinates, respectively. Moreover, let Θ be the angle the needles makes with the x-axis, where Θ is uniformly distributed between $[0,2\pi]$. By symmetry it suffices to consider the angle uniformly distributed between $[0,\pi/2]$. Furthermore, as the $2s < 1$, the needle may not cross both horizontal and vertical lines simultaneously. Crossing one of these lines requires that the center is beyond either the horizontal or vertical midpoint. Therefore, by symmetry, it suffices to consider the needle distributed in one of the quadrants, e.g. the $[0,1/2]\times[0,1/2]$. By independence the resulting joint

PDF is

$$
\begin{aligned}
f_{X,Y,\Theta}(x,y,\theta) &= \begin{cases} 2\cdot 2\cdot \frac{2}{\pi} & 0 \le x, y \le \frac{1}{2},\ 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{else} \end{cases} \\
&= \begin{cases} \frac{8}{\pi} & 0 \le x, y \le \frac{1}{2},\ 0 \le \theta \le \frac{\pi}{2} \\ 0 & \text{else} \end{cases}.
\end{aligned}
$$

The needle will cross the x-axis if $s \sin\Theta \ge X$ and similarly the needle will cross the y-axis if $s\cos\Theta \ge Y$. Therefore, the expected number of lines crossed is

$$
\begin{aligned}
&E\left[\mathbb{1}\{X \le s\sin\Theta\} + \mathbb{1}\{Y \le s\cos\Theta\}\right] \\
&\quad = \int_0^{\frac{\pi}{2}}\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \left[\mathbb{1}\{x \le s\sin\theta\} + \mathbb{1}\{x \le s\cos\theta\}\right]\frac{8}{\pi}\,dx\,dy\,d\theta \\
&\quad = \frac{8}{\pi}\left[\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{s\sin\theta} dx\,d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{s\cos\theta} dy\,d\theta\right] \\
&\quad = \frac{4}{\pi}\left[\int_0^{\frac{\pi}{2}} s\sin\theta\,d\theta + \int_0^{\frac{\pi}{2}} s\cos\theta\,d\theta\right] \\
&\quad = \frac{8s}{\pi}.
\end{aligned}
$$

Hence the expected number of lines cross is $\frac{8s}{\pi}$.