MASSACHUSETTS INSTITUTE OF TECHNOLOGY

6.436J/15.085J Fall 2018
Problem Set 3

Readings:
Notes from Lecture 4 and 5.

Supplementary readings:
[GS], Sections 3.1-3.7.
[C], Section 2.1
[BT], for background on counting (which we don't cover) go through the last section of Ch. 1 of [BT], available at:
http://athenasc.com/Prob-2nd-Ch1.pdf.

Exercise 1. Suppose that $X_1, X_2, \ldots, X_n, \ldots$ are random variables defined on the same probability space. Show that $\max\{X_1, X_2\}$, $\sup_n X_n$, and $\limsup_{n\to\infty} X_n$ are random variables, using only Definition 1 in the notes for Lecture 4, and first principles, without quoting any other known facts about measurability.

Solution: For any $c \in \mathbb{R}$, we have

$$\{\omega : \max\{X_1(\omega), X_2(\omega)\} \leq c\} = \{\omega : X_1(\omega) \leq c\} \bigcap \{\omega : X_2(\omega) \leq c\}\,.$$

Since X_1 and X_2 are random variables, we know that the set on the left-hand side is measurable, i.e., $\max\{X_1, X_2\}$ is a random variable. We also have

$$\left\{\omega : \sup_n X_n(\omega) \leq c\right\} = \bigcap_n \{\omega : X_n(\omega) \leq c\}\,,$$

and thus $\sup_n X_n$ is a random variable.

To show $\limsup_n X_n$ is a random variable, define $g_k(\omega) = \sup_{n\geq k} X_n(\omega)$. We have shown that these functions are measurable for all k. We now argue that

$$\left\{\omega : \limsup_n X_n(\omega) \geq c\right\} = \bigcap_k \{\omega : g_k(\omega) \geq c\},$$

which will immediately imply that $\limsup_n X_n$ is measurable. Indeed, suppose that ω belongs to the set on the right-hand side, i.e., $g_k(\omega) \geq c$, for all k. Since $\limsup X_n(\omega) = \lim_k g_k(\omega)$, and using the definition of $\limsup$, it follows that $\limsup X_n(\omega) \geq c$, and ω belongs to the set on the left-hand side.

Conversely, suppose that ω belongs to the set on the left-hand side. Then, $\lim_k g_k(\omega) =$
$\limsup_n X_n(\omega) \geq c$. However, $g_k(\omega)$ is a nonincreasing sequence of numbers, because the supremum is being progressively taken over smaller sets. It follows that $g_k(\omega) \geq c$ for all k, and ω belongs to set on the right-hand side.

Exercise 2. Given a distribution function F_X show that $F_X(x)$ is continuous at x_0 if and only if $\mathbb{P}(X = x_0) = 0$.

Solution: Since a distribution function is always right continuous, F_X is continuous at x_0 if and only if it is left-continuous at x_0, that is $\lim_{n\to\infty} F_X(x_n) = F_X(x_0)$ for any sequence $x_n \uparrow x_0$, or equivalently

$$\lim_{n\to\infty} \mathbb{P}(X^{-1}(-\infty, x_n]) = \mathbb{P}(X^{-1}(-\infty, x_0]).$$

The sequence of sets $\{X^{-1}(-\infty, x_n]\}$ is increasing, and by continuity of the probability measures $\mathbb{P}$ and action of the inverse image under unions

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(X^{-1}(-\infty, x_n]) &= \mathbb{P}(X^{-1}\left(\bigcup_{n=1}^{\infty}(-\infty, x_n]\right)\\ &= \mathbb{P}(X^{-1}(-\infty, x_0))\\ &= \mathbb{P}(X^{-1}(-\infty, x_0]) - \mathbb{P}(X = x_0).\end{aligned}$$

Hence the desired equality is satisfied if and only if $\mathbb{P}(X = x_0) = 0$.

Exercise 3. The probabilistic method. A party of $n = 20$ people is gathered. A host selects some of the $n(n-1)/2$ pairs of people and introduces them to each other. Show that the host can do the introduction in such a way that for every group of 7 people there are at least two who are introduced to each other and there are at least two who are not.

Hint: Consider introducing people randomly and independently.

Note: The probabilistic method is a general method for proving existence: if you can prove that a randomly selected structure has certain desired properties with some positive probability (no matter how small), then a structure with these properties is guaranteed to exist.

Solution: Assume that all possible pairs are introduced independently with probability $1/2$. Let $\mathcal{S}$ be the set of all subsets of size 7 out of the 20 people at the party. For each $S \in \mathcal{S}$, let A_S be the event that there is a pair in S that was introduced and a pair in S that was not introduced. The probability of the complementary event is

$$\mathbb{P}(A_S^c) = 2\left(\frac{1}{2}\right)^{\binom{7}{2}} = \frac{1}{2^{20}},$$

as A_S^c occurs if either all of the $\binom{7}{2}$ pairs get introduced or non of the $\binom{7}{2}$ pairs get introduced.

By the probabilistic method it suffices to show that $A = \bigcap_{S\in\mathcal{S}} A_S$ has positive probability, as this event would have zero probability if no such configuration existed, or equivalently, the complementary event does not have unit probability. Applying a union bound

$$\mathbb{P}(A^c) = \mathbb{P}\left(\bigcup_{S\in\mathcal{S}} A_S^c\right) \leq \sum_{S\in\mathcal{S}} \mathbb{P}(A_S^c) = \frac{|\mathcal{S}|}{2^{20}} = \frac{\binom{20}{7}}{2^{20}} < 0.08 < 1,$$

as desired.

Exercise 4. Let X be a nonnegative integer random variable. Show that

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

Be careful in citing whatever results from the lecture notes are needed to justify the steps in your derivation.

Solution: By definition,

$$\begin{aligned} E[X] &= \sum_{a=1}^{\infty} a\mathbb{P}(X = a) \\ &= \sum_{a=1}^{\infty}\sum_{n=1}^{a} \mathbb{P}(X = a) \\ &= \sum_{n=1}^{\infty}\sum_{a=n}^{\infty} \mathbb{P}(X = a) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(X \geq n), \end{aligned}$$

where the third inequality follows from Eq. (1) in Lecture 5, which allows us to interchange the order of summation in double sums of nonnegative numbers.

Exercise 5. Let F_1 and F_2 be two CDFs, and suppose that $F_1(t) < F_2(t)$, for all t. Assume that F_1 and F_2 are continuous and strictly increasing. Show that there exist random variables X_1 and X_2, with CDFs F_1 and F_2, respectively, defined on the same probability space such that $X_1 > X_2$. *Hint:* Think of simulating X_1 and X_2 using a common "random number generator".

Solution: Let $((0,1), \mathcal{B}, \lambda)$ be the Lebesgue probability space on $(0,1)$, where $\mathcal{B}$ is the Borel σ-algebra on $(0,1)$. By assumption F_i is strictly increasing and continuous on $\mathbb{R}$. Therefore, $F_i^{\;\;1}$ exists and is strictly increasing on $(0,1)$. Let $X_i : ((0,1), \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$, $X_i = F_i^{\;\;1}$. Then the X_i are random variables, as the F_i^{-1} are continuous which implies measurable, with CDFs

$$F_{X_i}(t) = \lambda(X_i^{-1}(-\infty, t]) = \lambda(F_i(-\infty, t]) = \lambda((0, F_i(t)]) = F_i(t).$$

Moreover, for $t \in (0,1)$

$$\begin{aligned} & t = F_1(F_1^{-1}(t)) < F_2(F_1^{-1}(t)) \\ \implies \quad & F_2^{-1}(t) < F_2^{-1}(F_2(F_1^{-1}(t))) = F_1^{-1}(t). \end{aligned}$$

Hence $X_1(\omega) = F_1^{-1}(\omega) > F_2^{-1}(\omega) = X_2(\omega)$ for all $\omega \in (0,1)$ and thusly, $X_1 > X_2$.

Exercise 6. Let $\{X_n\}$ be a sequence of independent non-negative random variables. Show that sequence X_n is almost surely bounded if and only if $\sum_{n=1}^{\infty} \mathbb{P}(X_n > c) < \infty$ for some c. (Hint: X_n a.s. bounded simply means $\mathbb{P}(\sup_n X_n = \infty) = 0$.)

Solution: Suppose that there exists such a c. Then, with probability 1, the set $S = \{n \mid X_n > c\}$ is finite. Then, $\sup_n X_n \le \max\{c, \max_{n \in S} X_n\} < \infty$, a.s.

Conversely, if if no such c exists, then $X_n > c$, i.o. By letting k range over the integers, we see that except for a countable union of zero measure sets, then for all k, there exists n_k such that $X_{n_k} > k$, so that $\sup_n X_n = \infty$, a.s.

(Alternate Solution). The event $\{\sup_n X_n = \infty\}$ can be expressed as

$$\{\sup_n X_n = \infty\} = \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{X_k > m\}.$$

Suppose $\sup_n X_n(\omega) = \infty$. Suppose there exists natural numbers m_0 and n_0 so that $\sup_{k \ge n_0} \le m_0$, then

$$\begin{aligned} \sup_n X_n(\omega) &= \max\left\{ \max_{j=1,\ldots,n_0-1} X_j(\omega), \sup_{k \ge n_0} X_k(\omega) \right\} \\ &\le \max\left\{ \max_{j=1,\ldots,n_0-1} X_j(\omega), m_0 \right\} < \infty, \end{aligned}$$

a contradiction. Conversely suppose for all $M \in \mathbb{N}$ and for all $N \in \mathbb{N}$ there exists $k_N \geq N$ such that $X_{k_N}(\omega) > M$, then

$$\sup_n X_n(\omega) \geq X_{k_N} > M.$$

Therefore the $\sup$ is larger than any natural number and must be infinite. Hence the desired relation holds.

Rewriting the above expression

$$\left\{\sup_n X_n = \infty\right\} = \bigcap_{m=1}^{\infty} \{X_n > m \text{ i.o. }\}.$$

Hence, for all $m \in \mathbb{N}$

$$\mathbb{P}\left(\sup_n X_n = \infty\right) \leq \mathbb{P}\left(X_n > m \text{ i.o. }\right\}. \tag{1}$$

Moreover, as the events $\{X_n > m\}$ are nested, $\{X_n > m+1\} \subset \{X_n > m\}$, by continuity of probability

$$\mathbb{P}\left(\sup_n X_n = \infty\right) = \lim_{m\to\infty} \mathbb{P}\left(X_n > m \text{ i.o }\right). \tag{2}$$

Suppose X_n is almost surely bounded. Suppose for all $c \in \mathbb{R}$

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n > c) = \infty.$$

then as the $\{X_n\}$ are independent by the Borel Cantelli Lemma, for all c,

$$\mathbb{P}(X_n > c \text{ i.o }) = 1.$$

In particular this holds for all $m \in \mathbb{N}$. Applying (2)

$$\mathbb{P}\left(\sup_n X_n = \infty\right) = \lim_{m\to\infty} \mathbb{P}\left(X_n > m \text{ i.o }\right) = \lim_{m\to\infty} 1 = 1,$$

a contradiction.

Conversely suppose there exists $c \in \mathbb{R}$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n > c) < \infty \quad \Longrightarrow \quad \mathbb{P}\left(X_n > c \text{ i.o}\right) = 0,$$

by Borel Cantelli. Therefore, there exists an $m_0 \in \mathbb{N}$, $m_0 \geq c$ and by monotonicity of $\mathbb{P}$ and (1)

$$\mathbb{P}(\sup_n X_n = \infty) \leq \mathbb{P}(X_n \geq m_0) \leq \mathbb{P}(X_n \geq c) = 0.$$

Hence X_n is almost surely bounded.

Exercise 7. (Another application of the probabilistic method.) Let G be an undirected graph with neither loops nor multiple edges, and write d_v for the degree of vertex v (i.e., the number of edges incident on v). An independent set is a set of vertices no pair of which is joined by an edge. Let $\alpha(G)$ be the size of the largest independent set of G. Use the probabilistic method to show that $\alpha(G) \geq \sum_v 1/(1+d_v)$. *Hint:* Order the nodes at random, and examine the nodes one at a time, putting them in the independent set as long as there are no conflicts with previously examined nodes. Find the expected value of the resulting set.

Solution: The result in this problem is known as Turan's theorem. Let V be the set of vertices of G and note $|V| = n$. Consider ordering the n nodes of a graph according to a random permutation. Nodes in this ordering have neighbors that come earlier, or later in the ordering.

Let I be the set of nodes whose neighbors all come after them in the randomly selected ordering. Note that I must be an independent set. We can write the random variable $|I|$ as the sum of indicator variables X_v, where X_v is 1 if and only if node v is in I, and zero otherwise.

Under a random ordering, the probability that a node v is in I, is at least the probability that a node is the first among it and all its d_v neighbors. This is $1/(d_v + 1)$. Therefore,

$$E(|I|) \geq \sum_{v \in V} \frac{1}{d_v + 1},$$

and therefore there must exist some ordering for which the size of I is at least as big as this. This gives a lower bound on $\alpha(G)$, and the result follows.

In addition, you need to be sure that you can solve elementary problems. As a check, make sure you are able to solve the next problem (not to be handed in).

Drill problem: At his workplace, the first thing Oscar does every morning is to go to the supply room and pick up one, two, or three pens with equal probability 1/3. If he picks up three pens, he does not return to the supply room again that day. If he picks up one or two pens, he will make one additional trip to the supply room, where he again will pick up one, two, or three pens with equal probability 1/3. (The number of pens taken in one trip will not affect the number of pens taken in any other trip.) Calculate the following:

(a) The probability that Oscar gets a total of three pens on any particular day.

(b) The conditional probability that he visited the supply room twice on a given day, given that it is a day in which he got a total of three pens.

(c) $\mathbb{E}[N]$ and $\mathbb{E}[N \mid C]$, where $\mathbb{E}[N]$ is the unconditional expectation of N, the total number of pens Oscar gets on any given day, and $\mathbb{E}[N \mid C]$ is the conditional expectation of N given the event $C = \{N > 3\}$.

(d) $\sigma_{N|C}$, the conditional standard deviation of the total number of pens Oscar gets on a particular day, where N and C are as in part (c).

(e) The probability that he gets more than three pens on each of the next 16 days.

(f) The conditional standard deviation of the total number of pens he gets in the next 16 days given that he gets more than three pens on each of those days.