Readings:
Notes from Lecture 2 and 3.

Supplementary readings:
[GS], Sections 1.4-1.7.
[C], Chapter 1.3
[W], Chapter 1.

Exercise 1. Consider a probabilistic experiment involving infinitely many coin tosses, and let $\Omega = \{0,1\}^\infty$ (think of 0 and 1 corresponding to heads and tails, respectively). A typical element $\omega \in \Omega$ is of the form $\omega = (\omega_1, \omega_2, \ldots)$, with $\omega_i \in \{0,1\}$.

As in the notes for Lecture 2, we define $\mathcal{F}_n$ as the σ-field consisting of all sets whose occurrence or nonoccurrence can be determined by looking at the result of the first n coin flips. The σ-field $\mathcal{F}$ for this model is defined as the smallest σ-field that contains all of the $\mathcal{F}_n$.

(a) Consider the event H consisting of all ω with the following property. There exists some time t at which the number of ones so far is greater than or equal to the number of zeros so far. Show that $H \in \mathcal{F}$.

(b) (Harder) Consider the set A of all ω for which the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \omega_i$$

exists. Show that $A \in \mathcal{F}$.
Note: This is important because, once we have also chosen a probability measure, it allows us to make statements about the probability that this limit (the long-term fraction of heads) exists.
Hint: The event A_x "the limit defined above exists and is equal to x" belongs to $\mathcal{F}$. However, this does not imply that $\bigcup_x A_x \in \mathcal{F}$ (why?). You need to find some other way of describing the event A in terms of unions, complements, etc., of events in the $\mathcal{F}_n$. For example, use the fact that a sequence converges if and only if it is a "Cauchy sequence."

Exercise 2. Suppose that the events A_n satisfy $\mathbb{P}(A_n) \to 0$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n^c \cap A_{n+1}) < \infty$. Show that $\mathbb{P}(A_n \text{ i.o.}) = 0$. *Note:* A_n i.o., stands for "A_n occurs infinitely often", or "infinitely many of the A_n occur", or just $\limsup_n A_n$. ***Hint:*** Borel-Cantelli.

Exercise 3. Consider one of our standard probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$, with $\Omega = (0, 1]$, $\mathcal{F}$ – Borel and $\mathbb{P}$ – the Lebesgue measure. To every element $\omega \in \Omega$ we assign its infinite decimal representation. We disallow decimal representations that end with an infinite string of nines. Under this condition, every number has a unique decimal representation.

(a) Let A be the set of points in $(0, 1]$ whose decimal representation contains at least one digit equal to 9. Find $\mathbb{P}[A]$.

(b) Let B be the set of points that have infinitely many 9's in the decimal representation. Find $\mathbb{P}[B]$. (Hint: Borel-Cantelli).

Exercise 4. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let A be an event (element of $\mathcal{F}$). Let $\mathcal{G}$ be collection of all events that are independent from A. Show that $\mathcal{G}$ need not be a σ-algebra.

Exercise 5. Let $A_1, A_2, \ldots$ and B be events.

(a) Suppose that $A_k \searrow A$, i.e. $A_k \supset A_{k+1}$ and $A = \cap_{k=1}^{\infty} A_k$. Assume B is independent of A_k. Show that B is independent of A.

(b) Suppose that A_1 is independent of B and also that A_2 is independent of B. Is it true that $A_1 \cap A_2$ is independent of B? Prove or give a counterexample.

Exercise 6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Show that function

$$d(A, B) \triangleq \mathbb{P}[A \triangle B]$$

satisfies the triangle inequality (i.e. $d(A, B) \leq d(A, C) + d(C, B)$ for any A, B, C).

Fun fact: Under this pseudo-metric any algebra is dense in the σ-algebra it generates. Thus, any event in a complicated σ-algebra (such as Borel) can be approximated arbitrarily well by events in a simple algebra (like finite unions of $[a, b)$).

Exercise 7. ***[Optional, not to be graded]*** Let $\Omega_1 \subset \Omega$ and let $\mathcal{C}$ be some collection of subsets of Ω. Let

$$\mathcal{C}_1 = \mathcal{C} \cap \Omega_1 \triangleq \{A \cap \Omega_1 : A \in \mathcal{C}\}$$

and denote by $\mathcal{F}_1$ ($\mathcal{F}$) the minimal σ-algebra on Ω_1 (Ω) generated by $\mathcal{C}_1$ ($\mathcal{C}$). Also define

$$\mathcal{F}_2 = \mathcal{F} \cap \Omega_1 \triangleq \{A \cap \Omega_1 : A \in \mathcal{F}\}.$$

$\mathcal{F}_2$ is called *a trace* of $\mathcal{F}$ on Ω_1. Show $\mathcal{F}_1 = \mathcal{F}_2$. (*Hint:* show that collection $\mathcal{G} = \{E \in \mathcal{F} : E \cap \Omega_1 \in \mathcal{F}_1\}$ is a monotone class.)

Exercise 8. ***[Optional, not to be graded]*** Let $\Omega = [0, 1)$ and let $\mathcal{F}_0$ be the collection of finite unions $\cup_{i=1}^{N}[a_i, b_i)$ for $a_i, b_i \in [0, 1]$. For any $A \in \mathcal{F}_0$, let $\mathbb{P}[A] = 1$ if one of the $b_i = 1$, and $\mathbb{P}[A] = 0$ otherwise. In Lectures we showed that $\mathcal{F}_0$ is an algebra but not a σ-algebra.

(a) Show that $\mathbb{P}$ is a non-negative (finitely) additive set-function on $\mathcal{F}_0$.

(b) Show that $\mathbb{P}$ is not countably additive on $\mathcal{F}_0$.