6.436J/15.085J Fall 2008
Problem Set 8 due 11/12/2008

Readings: Notes for Lectures 14-16.
Optional Readings:
[GS] Section 4.9 (multivariate normal)
[GS] Sections 5.7-5.8 (characteristic functions)

Exercise 1. Let X be a random variable with mean, variance, and moment generating function), denoted by $\mathbb{E}[X]$, $\mathrm{var}(X)$, and $M_X(s)$, respectively. Similarly, let Y be a random variable associated with $\mathbb{E}[X]$, $\mathrm{var}(X)$, and $M_X(s)$. Each part of this problem introduces a new random variable Q, H, G, D. Determine the means and variances of the new random variables, in terms of the means, and variances of X and Y.

(a) $M_Q(s) = [M_X(s)]^5$.

(b) $M_H(s) = [M_X(s)]^3[M_Y(s)]^2$.

(c) $M_G(s) = e^{6s}M_X(s)$.

(d) $M_D(s) = M_X(6s)$.

Exercise 2. A random (nonnegative integer) number of people K, enter a restaurant with n tables. Each person is equally likely to sit on any one of the tables, independently of where the others are sitting. Give a formula, in terms of the moment generating function $M_K(\cdot)$, for the expected number of occupied tables (i.e., tables with at least one customer).

Exercise 3. Suppose that

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}(X > x)}{x} \triangleq -\nu < 0. \tag{1}$$

Establish that $M_X(s) < \infty$ for all $s \in [0, \nu)$.

Exercise 4. Suppose that $X, Z_1, \ldots, Z_n$ have a multivariate normal distribution, and X has zero mean. Furthermore, suppose that $Z_1, \ldots, Z_n$ are independent. Show that $\mathbb{E}[X \mid Z_1, \ldots, Z_n] = \sum_{i=1}^{n} \mathbb{E}[X \mid Z_i]$.

Exercise 5. Let $X, W_1, \ldots, W_n$ be independent normal random variables, with $\mathbb{E}[X] = \mu$, $\text{var}(X) = \sigma_0^2$, $\mathbb{E}[W_i] = 0$, $\text{var}(W_i) = \sigma_i^2$. We want to estimate X on the basis of noisy observations of the form $Y_i = X + W_i$. Derive the formula for $\mathbb{E}[X \mid Y_1, \ldots, Y_n]$.

Exercise 6. Suppose that the multivariate characteristic function of two random variables X and Y is factorable; that is, there exist functions ϕ_1 and ϕ_2 such that $\phi_{X,Y}(s,t) = \phi_1(t) \cdot \phi_2(s)$, for all s, t. Show that X and Y are independent. *Hint:* Recall the proof of independence for multivariate Gaussians, and argue similarly.

Exercise 7.

(a) Find $\phi_X(t)$ if X is a Bernoulli(p) random variable.

(b) Suppose that $\phi_{X_i} = \cos(t/2^i)$. What is the distribution of X_i?

(c) Let $X_1, X_2, \ldots$ be independent and let $S_n = X_1 + \cdots + X_n$. Suppose that S_n converges almost surely to some random variable S. Show that $\phi_S(t) = \prod_{i=1}^{\infty} \phi_{X_i}(t)$.

(d) Evaluate the infinite product $\prod_{i=1}^{\infty} \cos(t/2^i)$. *Hint:* Think probabilistically; the answer is a very simple expression.

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