System Identification

6.435

SET 10

– Instrumental Variable Methods
– Identification in Closed Loop
– Asymptotic Results

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Instrumental Variable Method

- Connects parametric methods and correlation methods.
- Least squares revisited:

$$y(t) = \Phi^T(t)\theta_o + v(t) \qquad v : WN$$

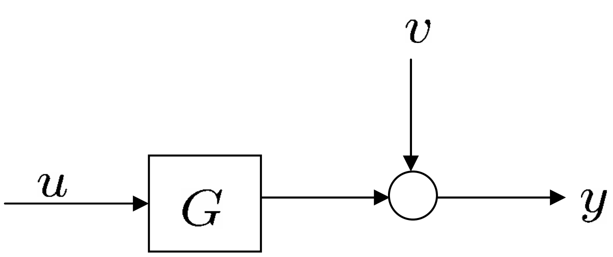
$\widehat{\theta}_N$ satisfies

$$\frac{1}{N}\sum_{t=1}^{N}\Phi(t)y(t) = \left[\frac{1}{N}\sum_{t=1}^{N}\Phi(t)\Phi^T(t)\right]\widehat{\theta}_N$$

- We can arrive to this by correlating both sides with

$$\Phi(t);\ \frac{1}{N}\sum\Phi(t)v(t) \simeq 0$$

- If $v(t)$ is not white, the above method will yield a biased estimate.
- Main Idea: Correlate with a vector $\xi(t)$ which is uncorrelated from $v(t)$.
- $\xi(t) \triangleq$ Instrument
- If experiment is open loop:



$\xi(t)$ is constructed from u.

- In closed loop, $\xi(t)$ is constructed from reference signals.

- <u>Detail</u>: $\xi(t) : (n_\xi \times 1) - \text{vector.}$

$$\frac{1}{N}\sum_{t=1}^{N}\xi(t)y(t) = \frac{1}{N}\sum_{t=1}^{N}\xi(t)\Phi^T(t)\theta + \underbrace{\frac{1}{N}\sum_{t=1}^{N}\xi(t)v(t)}_{\simeq 0}$$

- Define:

$$\theta_{IV}^N = \text{sol}\left[\frac{1}{N}\sum_{t=1}^{N}\xi(t)\Phi^T(t)\theta - \frac{1}{N}\sum_{t=1}^{N}\xi(t)y(t) \equiv 0\right]$$

$$= \text{sol}\left[f_N\left(\xi, \theta, Z^N\right) = 0\right]$$

- Extended IV

$$\varepsilon_F(t,\theta) = L\varepsilon(t,\theta) = L\left(y - \Phi^T\theta\right) : \text{ Filtered error}$$

$$f_N\left(\theta, Z^N, \xi\right) = \frac{1}{N}\sum_{t=1}^{N} \xi(t)\alpha(\varepsilon(t,\theta))$$

some function

$$\theta_{IV}^N = \text{sol}\left[f_N \equiv 0\right]$$

or $\theta_{IV}^N = \min_\theta \; ||f_N|| = \min_\theta \; f_N^T Q f_N$

- In LS $\xi(t) = \Phi(t)$.
- If $\sum_{t=1}^{N} \xi(t)\Phi^T(t)$ has an inverse, then

$$\theta_{IV}^N = \left[\frac{1}{N}\sum_{t=1}^{N} \xi(t)\Phi^T(t)\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N} \xi(t)y(t)\right]$$

Choice of Instrument

System:

$Ay = Bu + v$ “$\boldsymbol{v}$ may not be white”

<u>Open Loop Operation</u> $\boldsymbol{u}$ & $\boldsymbol{v}$ are independent

Define the instrument as a function of $\boldsymbol{u}$

$$\xi\left(t, u^{t-1}\right) = K(q)\left[-x(t-1), \ldots, -x(t-n_a), u(t-1), \ldots, u(t-n_b)\right]$$

$$N(q)x = M(q)u$$

$\text{dim} = n_n$ $\text{dim } n_m$

To solve for θ_{IV}^{N}, we need the inverse of

$$\left[\frac{1}{N}\sum_{t=1}^{N}\xi(t)\Phi^T(t)\right]$$

<u>Recall</u>: If $\xi(t) = \Phi(t)$. The inverse exists if u is p.e. of order n_b (or order $n_a + n_b$ if $v = 0$)

Convergence Results

- Assume Data generated in closed loop as before. Define

$$\varepsilon_F(t,\theta) = L\varepsilon(t,\theta)$$

$$f_N(t,\theta) = \frac{1}{N}\sum_{t=1}^{N} \xi(t,\theta)\varepsilon(t,\theta)$$

$$\xi(t,\theta) = K_u(q,\theta)u + K_y(q,\theta)y \quad \text{(Past Data)}$$

$$\theta_{IV}^N = \underset{\theta \in D_m}{\text{sol}} \left[f_N\left(\xi,\theta,Z^N\right)\right] = 0$$

- Results: $f_N\left(\theta, Z^N\right) \xrightarrow[\text{uniformly in } \theta]{\text{w.p.1}} \bar{E}\xi(t,\theta)\varepsilon_F(t,\theta) = \bar{f}(\theta)$

$$\theta_{IV}^N \xrightarrow[\text{w.p.1}]{} D_c = \{\theta \in D_m | \bar{f}(\theta) = 0\}$$

Consistency

δ : $\quad Ay = Bu + v \qquad v = H_o e$

Ass. $\exists \quad \theta_o \in D_m \quad \ni \quad m(\theta_o) = G_o(q) = \dfrac{B_o(q)}{A_o(q)}$

Let $\xi(t) = K_u(q,\theta)u + K_y(q,\theta)y$

$$\bar{f}(\theta) = \bar{E}\left(\xi(t)\left[L\left(\Phi^T(\theta_o - \theta) + v\right)\right]\right)$$

$$= \left(\bar{E}\xi(t)\Phi_F^T(t)\right)(\theta_o - \theta).$$

$$\Rightarrow \theta_o \in D_c. \quad \Rightarrow \quad D_T \subseteq D_c$$

<u>Question</u>: Is $D_c \subseteq D_T$?

True if $\bar{E}\xi(t)\Phi_F^T(t)$ is non singular

Conditions under which $\bar{E}\xi(t)\Phi_F^T(t)$ is non singular are hard to find.

Theorem: Suppose the true system $\frac{B_o}{A_o}$ has degrees n_b^o, n_a^o. Suppose the model $\frac{B}{A}$, with degrees n_b, n_a. Also, suppose the instrument is given by

$$\xi(t) = L\left[x(t-1), \ldots, x(t-n_a), u(t-1), \ldots, u(t-n_b)\right]$$

with $M_x = N_u$

Then:

1) If $\min\left(n_a - n_a^o, n_b - n_b^o\right) > 0 \Rightarrow R$ is singular

2) If $\min\left(n_a - n_n, n_b - n_m\right) > 0 \Rightarrow R$ is singular.

with $R = \bar{E}\xi(t)\Phi^T(t)$

<u>Proof</u>: Notice that $\bar{E}\xi(t)\Phi^T(t) = \bar{E}\xi(t)\tilde{\Phi}^T(t)$ where $\tilde{\Phi}(t)$ is

$$\tilde{\Phi}^T(t) = (-z_o(t-1), \ldots, -z_o(t-n_a), u(t-1), \ldots, u(t-n_b))$$

& $\quad z_o(t) = \dfrac{B_o(q)}{A_o(q)}u \quad$ (noiseless)

Since $n_a > n_a^o, \quad n_b > n_b^o, \quad \exists \quad S \quad \ni \quad \forall \quad t \quad (S \neq 0)$

$\tilde{\Phi}^T(t)S = 0 \qquad$ (pole/zero cancellation at θ_o)

$\Rightarrow \quad \left(\bar{E}\xi(t)\tilde{\Phi}^T\right)S = 0 \quad \Rightarrow \quad \bar{E}\xi(t)\Phi^T(t)S = 0$

$\Rightarrow \quad \bar{E}\xi(t)\Phi^T(t)$ is singular $\qquad$ [similarly for ξ]

Result:

If $\min\left(n_a - n_a^o, n_b - n_b^o\right) = 0$

$\min\left(n_a - n_n, n_b - n_m\right) = 0$

$-\quad \Phi_u(\omega) > 0$ (or $\boldsymbol{u}$ is p.e.)

Then $\bar{E}\xi(t)\Phi^T(t)$ is generically non singular.

Proof: The set of parameters that satisfy singularity have measure zero.

Asymptotic Distribution

Suppose $\bar{E}\xi(t)\Phi_F^T(t) = R$ is non singular

$$\theta_{IV}^N = \left(\frac{1}{N}\sum_{t=1}^{N}\xi(t)\Phi_F(t,\theta)\right)^{-1}\left(\frac{1}{N}\sum_{t=1}^{N}\xi(t)y(t)\right)$$

$$\downarrow \qquad\qquad \downarrow$$

$$R^{-1} \qquad\qquad \frac{1}{N}\sum_{t=1}^{N}\xi(t)\left(\Phi_F^T\theta_o + v_o\right)$$

$$\simeq \quad \theta_o \quad + \quad R^{-1}\frac{1}{\sqrt{N}}\left(\frac{1}{\sqrt{N}}\sum_{t=1}^{N}\xi(t)v_o(t)\right)$$

$$\searrow$$

$$N(0,P)$$

$$P = \lim_{N\to\infty} \frac{1}{N} \bar{E} \sum_{t,s=1}^{N} \xi(t) v_o(t) \xi^T(s) v_o(s)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{t,s=1}^{N} \bar{E}\xi(t)\xi^T(s) E v_o(t) v_o(s)$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{t,s=1}^{N} \bar{E}\xi(t)\xi^T(s) R_v(t-s)$$

$$\sqrt{N}\left(\theta_{IV}^N - \theta_o\right) \sim \text{Asym } N(0, P_\theta)$$

$$P_\theta = R^{-1} P R^{-1}$$

Frequency Domain Characterization

$$\varepsilon(t,\theta) = y(t) - \Phi^T(t)\theta$$
$$= y(t) - (1-A)y - Bu = Ay - Bu$$
$$= A\left[G_o u + H_o e - \frac{B}{A}u\right]$$

$$\bar{f}(\theta) = \bar{E}\xi(t,\theta)\varepsilon_F(t,\theta) \qquad \xi(t,\theta) = K_u\left(e^{iw},\theta\right)u$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[G_o\left(e^{iw}\right) - G\left(e^{iw},\theta\right)\right]\Phi_u(\omega)A\left(e^{iw}\right)L\left(e^{iw}\right)K_u\left(e^{-iw},\theta\right)$$

$$G = \frac{B}{A}$$

$\Rightarrow$ a different kind of fit!