Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.432 Stochastic Processes, Detection and Estimation

Problem Set 7

Spring 2004

Issued: Thursday, April 1, 2004 **Due:** Thursday, April 8, 2004

Reading: For this problem set: Sections 4.3-4.6, 4.A, 4.B, Chapter 5
Next: Chapter 5, Sections 6.1 and 6.3

Problem 7.1
Let $N(t)$ be a Poisson counting process on $t \geq 0$ with rate λ. Let $\{y_i\}$ be a collection of statistically independent, identically-distributed random variables with mean and variance

$$\begin{aligned} E[y_i] &= m_y \\ \operatorname{var} y_i &= \sigma_y^2, \end{aligned}$$

respectively. Assume that the $\{y_i\}$ are statistically independent of the counting process $N(t)$ and define a new random process $y(t)$ on $t \geq 0$ via

$$y(t) = \begin{cases} 0 & N(t) = 0 \\ \displaystyle\sum_{i=1}^{N(t)} y_i & N(t) > 0 \end{cases}.$$

(a) Sketch a typical sample function of $N(t)$ and the associated typical sample function of $y(t)$.

(b) Use iterated expectation (condition on $N(t)$ in the inner average) to find $E[y(t)]$ and $E[y^2(t)]$ for $t \geq 0$.

(c) Prove that $y(t)$ is an independent-increments process on $t \geq 0$ and use this fact to find the covariance function $K_{yy}(t,s)$ for $t,s \geq 0$.

Problem 7.2 (practice)

(a) Let $x(t)$ be an independent increments process on $t \geq 0$ whose covariance function is $K_{xx}(t,s)$, for $t,s \geq 0$. Show that

$$K_{xx}(t,s) = \operatorname{var}[x(\min(t,s))] \qquad \text{for } t,s \geq 0.$$

(b) Suppose $x(t)$ in part (a) has stationary increments. Show that

$$m_x(t) = at + b, \qquad \text{for } t \geq 0,$$

and

$$K_{xx}(t, s) = c\min(t, s) + d \qquad \text{for } t, x \geq 0$$

where a, b are constants and c, d are non-negative constants.

Problem 7.3

Let $x[n]$ be a real-valued, discrete-time, zero-mean wide-sense stationary random process with correlation function $R_{xx}[m]$ and spectrum $S_{xx}(z)$.

(a) Suppose $x[n]$ is the input to two real-valued linear time-invariant systems as depicted below, producing two new processes, $y_1[n]$ and $y_2[n]$. Find $K_{y_1y_2}[n]$ and $S_{y_1y_2}(z)$.

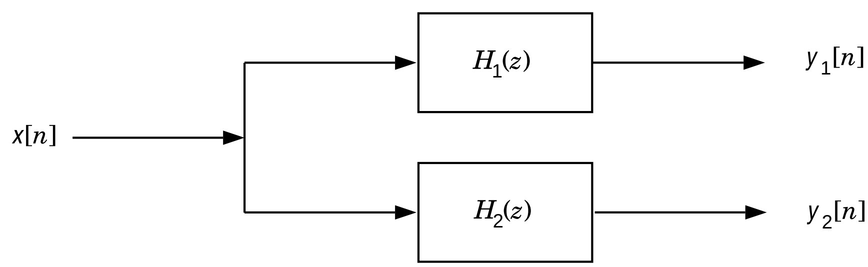

Figure 1-1

(b) Suppose next that $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ are non-overlapping frequency responses, i.e.,

$$|H_1(e^{j\omega})| \cdot |H_2(e^{j\omega})| = 0, \qquad \forall \omega.$$

Show that in this case $y_1[n]$ and $y_2[m]$ are uncorrelated for all n and m. Are $y_1[n]$ and $y_2[m]$ statistically independent (for all n and m) in the case that $x[n]$ is a Gaussian random process? Explain.

Problem 7.4

Consider a stationary process $y(t)$ satisfying the equation

$$\frac{dy(t)}{dt} + 2y(t) = u(t)$$

where $u(t)$ is a zero-mean stationary process with covariance function

$$K_{uu}(\tau) = \delta(\tau) + 4e^{-|\tau|}.$$

Also assume that the transformation from $u(t)$ to $y(t)$ is linear, time-invariant, and stable.

(a) Determine $S_{yy}(s), S_{uy}(s), K_{yy}(\tau)$, and $K_{uy}(\tau)$.

(b) Find a stable shaping filter for $y(t)$, i.e., find the system function $H(s)$ of a stable, causal system so that if the input $w(t)$ to this system is white noise with spectral height 1 ($K_{ww}(\tau) = \delta(\tau)$), then the output has power spectral density $S_{yy}(s)$ found in part (a). Is your choice of $H(s)$ unique? Why or why not?

Problem 7.5
Consider the system depicted in Fig. 3-1.

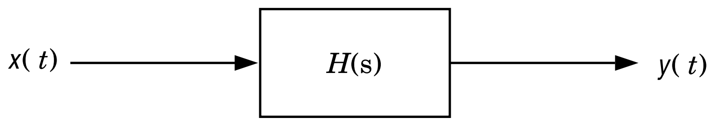

Figure 3-1

The zero-mean, wide-sense stationary stochastic process $x(t)$ has covariance function

$$R_{xx}(\tau) = e^{-|\tau|}.$$

We would like to find a linear time-invariant system with system function $H(s)$ so that the output $y(t)$ has covariance function

$$R_{yy}(\tau) = e^{-|\tau|}.$$

(a) Find a suitable $H(s)$, such that $y(t)$ *cannot* be written in the form $y(t) = x(t-\tau)$ for some τ. Is it unique? If your answer is yes, explain. If your answer is no, construct another suitable $H(s)$.

(b) Assume that we also want $R_{xy}(\tau)$ to have the form

$$R_{xy}(\tau) = Ke^{-2\tau}, \qquad \tau > 0$$

for some constant K. Does a system function $H(s)$ exist that meets the specifications? If your answer is no, explain. If your answer is yes, calculate the corresponding $R_{xy}(\tau)$ for all τ, and indicate whether it is uniquely specified.

Problem 7.6
Suppose $x[n]$ is a zero-mean, wide-sense stationary random process with

$$S_{xx}(e^{j\omega}) = 1.$$

We observe $y[n] = h[n] * x[n]$ where $h[n] = 0$ for $n < 0$ and $n \geq 2$, $h[0] > 0$, and

$$S_{yy}(e^{j\omega}) = \frac{5}{4} - \cos(\omega).$$

(a) Sketch the pole-zero plot for $S_{yy}(z)$ and determine one possible impulse response $h[n]$ that is consistent with the information given.

(b) Show that your answer to part (a) is not unique by determining a second distinct $h[n]$ consistent with the information given.

(c) Suppose we also observe $w[n] = g[n] * x[n]$ where $g[n] = 0$ for $n < 0$ and $n \geq 2$, and $g[0] > 0$. If

$$S_{ww}(e^{j\omega}) = \frac{5}{4} + \cos(\omega)$$

and

$$S_{yw}(e^{j\omega}) = \frac{1}{4}e^{j\omega} - e^{-j\omega},$$

sketch pole-zero plots for $S_{ww}(z)$ and $S_{yw}(z)$, then determine one possible impulse response $h[n]$ that is consistent with the information given.

(d) Is your answer to (c) unique? If your answer is yes, explain. If your answer is no, construct a second distinct such $h[n]$.

Problem 7.7 (practice)
Let $x[n]$ be a discrete-time, zero-mean, wide-sense stationary Gaussian random process with unknown correlation function $R_{xx}[m]$.

(a) Define a new process $y[n]$ by the relation

$$y[n] = x[n]\, x[n-m].$$

Find the mean function and the covariance function of this new process.

(b) Define the time-average correlation function by

$$\hat{R}_{xx}[m; N] = \frac{1}{N} \sum_{n=-(N-1)/2}^{(N-1)/2} x[n]\, x[n-m]$$

where N is an odd integer. Find the mean and the variance of $\hat{R}_{xx}[m; N]$.

(c) Suppose that the process $x[n]$ has a bounded spectral density, i.e.,

$$S_{xx}(e^{j\omega}) \leq M < \infty, \qquad \forall \omega.$$

Show that $\hat{R}_{xx}[m; N]$ is a consistent estimator for $R_{xx}[m]$.

Problem 7.8

We have determined a number of convenient properties related to *linear* systems. In this problem, we consider a memoryless *nonlinear* system and its properties. Consider a system whose output $y[n]$ is related to its input $x[n]$ by

$$y[n] = x^2[n], \qquad \forall n.$$

(a) If $x[n]$ is strict-sense stationary, must $y[n]$ be strict-sense stationary? Prove, or give a counterexample.

(b) If $x[n]$ is wide-sense stationary, must $y[n]$ be wide-sense stationary? Prove, or give a counterexample.

(c) If $x[n]$ is a Gaussian random process, must $y[n]$ be a Gaussian random process? Prove, or give a counterexample.

(d) (optional) If $x[n]$ has independent increments, must $y[n]$ have independent increments? Prove, or give a counterexample.

(e) (optional) If $x[n]$ is a Markov process, must $y[n]$ be a Markov process? Prove, or give a counterexample.

(f) Can $R_{yy}[n, m]$ be found in terms of the first and second moments of $x[n]$?

(g) Find an expression for $R_{yy}[n, m]$ when $x[n]$ is a zero-mean, stationary Gaussian process with a given $R_{xx}[n]$.