Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science
6.245: MULTIVARIABLE CONTROL SYSTEMS

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Problem Set 7 Solution [1]

Problem 7.1

(a) A state-space representation for this problem is given by

$$
\begin{aligned}
\dot{x} &= -ax + \frac{1}{\epsilon}w_2 + v \\
z_1 &= \epsilon v \\
z_2 &= x \\
y &= x + w_1
\end{aligned}
$$

By making the change of variable $u = \epsilon v$, the description becomes

$$
\begin{aligned}
\dot{x} &= -ax + \frac{1}{\epsilon}w_2 + \frac{1}{\epsilon}u \\
z_1 &= u \\
z_2 &= x \\
y &= x + w_1
\end{aligned}
$$

which fits the form of the simplified H_∞ setup with

$$A = -a \quad B_1 = 1/\epsilon \quad B_2 = 1 = 1/\epsilon \quad C_1 = 1 \quad C_2 = 1$$

By virtue of the conditions derived by the Generalized Parrot's Lemma, if a real number γ is an admissible value for the closed loop H_∞ norm for some stabilizing controller $C(s)$, then the stabilizing solutions P and Q of the respective Riccati equations corresponding to the the abstract H_2 optimization problems

$$\dot{x} = -Ax - B_1 w \quad \int_0^\infty \gamma^2 |w_1|^2 + \gamma^2 |C_2 x|^2 - |C_1 x|^2 \, dt \to \min$$

$$\dot{\Psi} = -A'\Psi - C_1' q \quad \int_0^\infty |q|^2 + |B_2'\Psi|^2 - \gamma^{-2} |B_1'\Psi|^2 \, dt \to \min$$

satisfy the conditions $P > 0$, $Q > 0$, and $P > Q^{-1}$.

The corresponding Hamiltonian matrices $\mathcal{H}_P$ and $\mathcal{H}_Q$ are given by

$$\mathcal{H}_P = \begin{bmatrix} -A & \gamma^{-2} B_1 B_1' \\ \gamma^2 C_2' C_2 - C_1' C_1 & A' \end{bmatrix} \quad \mathcal{H}_Q = \begin{bmatrix} -A' & C_1' C_1 \\ B_2 B_2' - \gamma^{-2} B_1 B_1' & A \end{bmatrix}$$

which, for this particular example, reduce to

$$\mathcal{H}_P = \begin{bmatrix} a & \gamma^{-2} \epsilon^{-2} \\ \gamma^2 - 1 & -a \end{bmatrix} \quad \mathcal{H}_Q = \begin{bmatrix} a & 1 \\ \epsilon^{-2}(1 - \gamma^{-2}) & -a \end{bmatrix}$$

For the eigenvalues of $\mathcal{H}_P$ and Hc_Q to not lie on the imaginary axis, γ must satisfy the condtion that $\gamma > 1/\sqrt{\epsilon^2 a^2 + 1}$, in which case

$$P = \epsilon^2 \gamma^2 (a + \sqrt{a^2 + \epsilon^{-2}(1 - \gamma^{-2})}) \quad Q = a + \sqrt{a^2 + \epsilon^{-2}(1 - \gamma^{-2})} \ .$$

Note that if $a \geq 0$, P_0 and Q_0 are both necessarily positive definite. If however $a < 0$, then we require additionally that $\gamma > 1$. Hence, we have the following lower bound for the admissible values of γ:

$$\gamma > \begin{cases} \frac{1}{\sqrt{\epsilon^2 a^2 + 1}} & a \geq 0 \\ 1 & a < 0 \end{cases} .$$

To satisfy the condition that $P > Q^{-1}$, some algebraic manipulation shows that an equivalent constraint is given by

$$\sqrt{a^2 + \epsilon^{-2} - 1\epsilon^{-2}\gamma - 2)} > 1 - \epsilon \gamma a.$$

If $a < 0$, then we may safely square boths sides of the inequality without having to worry about whether the inequality flips. If $a \geq 0$, the situation is a bit more

complicated since the right hand side could either be positive or negative. However, it should be clear that the for the *smallest* γ that makes the inequality true, the right-hand side is still positive, and, therefore, we can safely assume that, again, the inequality will not flip.

By squaring both sides of the above inequality and solving the resulting quadratic inequality, we find that a necessary condition on γ is that

$$\gamma > -a\epsilon + \sqrt{a^2\epsilon^2 + 2}$$

for all values of a. It is straightforward to show that this lower bound is greater than the lower bounds imposed by the strict positivity of P and Q. Hence

$$\gamma(a, \epsilon) = -a\epsilon + \sqrt{a^2\epsilon^2 + 2}.$$

Note that the closed loop H_∞ norm is an increasing function of $|a|$ when $a < 0$ which intuitively reflects the need to exert extra control to stabilize the unstable pole at $s = -a$.

(b) To find a suboptimal controller, we revert back to the Generalized Parrot's Lemma. The quadratic form of interest here is given by

$$\sigma(f, g, v) = \bar{\gamma}^2|w_1|^2 + \bar{\gamma}^2|w_2|^2 - |C_1x|^2 - |u|^2 - 2 \begin{bmatrix} x \\ x_f \end{bmatrix}' \begin{bmatrix} P & L \\ L' & I \end{bmatrix} \begin{bmatrix} Ax + B_1w_2 + B_2u \\ \Theta \end{bmatrix}$$

where

$$f = \begin{bmatrix} x \\ w_2 \end{bmatrix} \quad g = \begin{bmatrix} x_f \\ w_1 + C_2x = y \end{bmatrix} \quad v = \begin{bmatrix} \Theta \\ u \end{bmatrix} \quad \Theta = \dot{x}_f \quad LL' = P - Q^{-1}$$

(note that w_1 and w_2 are interchanged in comparison to what was seen in lecture since they are switched in the formulation of this problem). We which to find the matrix $K = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix}$ such that $\sigma(f, g, Kg) \geq 0$ for all f, g. As the structure of K indicates, this will be the system matrix corresponding to a suboptimal H_∞ controller which achieves closed-loop H_∞ norm strictly less than $\bar{\gamma}$ for any value of $\bar{\gamma} > \gamma(1,1) = \sqrt{4 - 2\sqrt{3}} \approx 0.7312$. The matrix K is given explicitly by

$$K = \arg\max_v \min_f \sigma(f, g, v).$$

If we make the substiution $w_1 = y - C_2 x$ and also notice that $\bar{\gamma}^2 - 1 + 2P = \bar{\gamma}^{-2} P^2$ by virtue of the Riccati equation for P, then we find that the terms which are either linear of quadratic in x and w_2 are given by

$$p^2 x^2 - 2pxw_2 + \bar{\gamma}^2 w_2^2 - 2x(pu + \bar{\gamma}^2 y - Lx_f + L\Theta) - 2w_2 Lx_f.$$

If we first minimize over w_2 for fixed x, we obtain

$$-2x(pu + \bar{\gamma}^2 y - L(1 - P\bar{\gamma}^{-2})x_f + L\Theta) - L\bar{\gamma}^{-2}x_f^2.$$

In order for the minimum over x to exist, the coefficient of the x term must be 0, which constrains Θ:

$$\Theta = (1 - P\bar{\gamma}^{-2})x_f - \frac{\bar{\gamma}^2}{L}y - \frac{P}{L}u.$$

If we plug this expression into the remaining terms for $\sigma(f, g, v)$, we find that the required maximation is

$$u^* = \arg\max_u \ -u^2 - 2L(1 - \frac{P}{L^2})x_f u$$

or namely

$$u^* = L(1 - \frac{P}{L^2})x_f.$$

Hence, a suboptimal controller which achieves closed-loop H_∞ norm less than $\bar{\gamma}$ is given by

$$
\begin{aligned}
\dot{x}_f &= \left(1 - \frac{P}{\bar{\gamma}^2} + P\left(1 - \frac{P}{L^2}\right)\right)x_f - \frac{\bar{\gamma}^2}{L}y \\
u &= -L\left(1 - \frac{P}{L^2}\right)x_f
\end{aligned}
$$

where

$$P = \bar{\gamma}^2(1 + \sqrt{2 - \bar{\gamma}^{-2}}) \quad Q = 1 + \sqrt{2 - \bar{\gamma}^{-2}} \quad L = \sqrt{P - Q^{-1}} \ .$$

Note that in the general case, we must substitute $v = \epsilon u$, which will augment our above expressions slightly. However, since $\epsilon = 1$ in this case, the above is an expression for a suboptimal controller.

(c) Using MATLAB to check the above calculations, performing `hinfsyn` on the standard original setup in terms of v for different values of the lumped parameter $a\epsilon$ yields an optimal cost which exactly matches the expression given for the maximal lower bound given in part (a) for at least up to several decimal places. The MATLAB file `ps7_1_1.m` takes in values of a and ϵ to show this. To verify that the controller of part (b) yields an appropriate suboptimal controller, see the MATLAB file `ps7_1_2.m` which takes in a value of δ and computes the H_∞ norm of the resulting closed-loop transfer function. I chose values of $\delta = 0.01$, 0.1, and 1, which correspond to values of $\bar{\gamma} = 0.7421$, 0.8321, and 1.7321. After finding the corresponding closed-loop transfer function and evaluating the H_∞ norm, I found that the respective closed-loop norms were given by 0.7418, 0.8069, and 0.9069 (!). So, indeed, this suboptimal controller yields a closed loop norm to within the desired δ accuracy. In fact, it seems that the $\gamma(1,1) + \delta$ bound is far from tight as δ becomes large, which is great!

Problem 2

(a) A simple state-space description for $G(s)$ is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & -2a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w \quad , \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Because $B = C'$ in this case, we find that the observability grammian W_o and controllability Grammian W_c are the same, and are given by

$$W_o = W_c = \frac{1}{a} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}.$$

The singular value decomposition of the Hankel operator is closely related to a singular value decomposition of $(W_o W_c)^{\frac{1}{2}} = W_o$, which is given by

$$\begin{bmatrix} 0.8219 & 0.5696 \\ -0.5696 & 0.8219 \end{bmatrix} \begin{bmatrix} \dfrac{0.7310}{a} & 0 \\ 0 & \dfrac{0.019}{a} \end{bmatrix} \begin{bmatrix} 0.8219 & 0.5696 \\ -0.5696 & 0.8219 \end{bmatrix}^T$$

where T denotes transposition. If we denote the columns of the third matrix as w_k and the corresponding singular values as σ_k then, according to the lecture notes, a selection of the corresponding vectors $u_k(t)$ and $v_k(t)$ are given by

$$\begin{aligned} u_k(t) &= Ce^{At}W_c^{\frac{1}{2}}w_k\sigma_k^{-1} \ (t > 0) \\ v_k(t) &= \sigma_k^{-2}B'e^{-A't}W_oW_c^{\frac{1}{2}}w_k \ (t < 0) \end{aligned}$$

For the particular example here, we find that

$$v_1(t) = \sqrt{\tfrac{a}{0.7310}}(0.8219e^{-at} + 0.5696e^{-2at}) \quad v_2(t) = \sqrt{\tfrac{a}{0.0190}}(0.5696e^{-at} + 0.8219e^{-2at})$$
$$u_1(t) = v_1(-t) \qquad\qquad\qquad\qquad\qquad u_2(t) = v_2(-t)$$

Note that the singular vectors are scalar functions in this example, which follows from the fact that the system is single-input single-output. Because the vectors $v_k(t)$ are linear operators which act upon the input of they system, the singular vectors $v_k(t)$ are generally vector functions of time of length m, where m is the number of inputs. Likewise, the singular vectors $u_k(t)$ are generally vector functions of time of length p, where p is the number of outputs.

(b) This problem was set-up in lecture (and in the lecture 9 notes) as an application of the (stronger) Generalized Parrot's Lemma to the following quadratic form:

$$\sigma(f, g, v) = \gamma^2|w|^2 - |Cx - y_r|^2 - 2 \begin{bmatrix} x \\ x_r \end{bmatrix}' p \begin{bmatrix} Ax + Bw \\ \Theta \end{bmatrix}$$

where

$$f = x \quad g = \begin{bmatrix} x_r \\ w \end{bmatrix} \quad v = \begin{bmatrix} \Theta \\ y_r \end{bmatrix} \quad \Theta = \dot{x}_r \ .$$

As was proven in lecture, we may take the matrix p to be

$$p = \begin{bmatrix} W_o & W_o - \gamma^2 W_c^{-1} \\ W_o - \gamma^2 W_c^{-1} & W_o - \gamma^2 W_c^{-1} \end{bmatrix} .$$

(Actually, in lecture, the off-diagonal elements of the above matrix were taken to be the negative of how they appear now, but both forms are acceptable). Additionally, it was shown that condition (a) of the stronger version of Parrot's Lemma can be satisfied if the equation

$$W_o Bw + A'(W_o - \gamma^2 W_c^{-1})x_r + (W_o - \gamma^2 W_c^{-1})\Theta - C'y_r = 0$$

has a solution (Θ, y_r) for every (w, x_r). Now, since the value of γ is equal to σ_2, the matrix $W_o - \gamma^2 W_c^{-1}$ is singular. Hence, to solve the above equation, it is sufficient to set one of the parameters of $\Theta = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}$ to 0. Now the problem is reduced to solving the equation

$$\begin{bmatrix} \rho & -C' \end{bmatrix} \begin{bmatrix} \theta_1 \\ y_r \end{bmatrix} = -A'(W_o - \gamma^2 W_c^{-1})x_r - W_o Bw.$$

Performing the calculations in MATLAB yields the state-space description

$$\begin{bmatrix} \dot{x}_{1r} \\ \dot{x}_{2r} \end{bmatrix} = \begin{bmatrix} -1.2574a & -0.8713a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} - \begin{bmatrix} 1.6502 \\ 0 \end{bmatrix} w$$

$$y_r = \begin{bmatrix} -1.1140 & -0.7720 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} - 0.0190w.$$

Note that the uncontrollable second mode is removed when we compute the transfer function:

$$\hat{G}(s) = \frac{1.8383}{s + 1.2574a} + 0.019.$$

Because the constant term does not affect the Hankel norm, it is unnecessary in the above expression. Note also in the above analysis that we could have set θ_2 to any linear function of the x_r and w. In this case, the resulting state-space description would have an anticausal pole that we would manually need to remove by means of computing the transfer function and performing a partial fraction expansion. Choosing $\theta_2 = 0$ in this case is computationally easier since it removes this extra step.

Note further that, because of the non-invertibility of the ρ matrix, there is no further optimization to be performed once we find a solution for θ, y_r. Informally speaking, this relates to the fact that we must use all of our parameters to find a solution which makes the infimization over x feasible, leaving us no further degrees of freedom to optimize over.

(c) Using the commands `sysbal` and `hankmr` in MATLAB yields a reduced transfer function that is *exactly* equal to the answer in part (b)! On one hand, this is somewhat surprising since Hankel optimal reduced models are not unique. On the other hand, one may argue that it is not surprising since we chose a natural choice for the variable θ_2. So, if the person who wrote the algorithm for `hankmr` at the Mathworks is smart (and it seems these days that this is a BIG if!), the algorithm would tend to choose this natural selection of θ_2 as well.

Problem 7.3

If we approximate the integral as a Riemann sum, we find that an approximate expression for $G(s)$, which we will denote here as $G_a(s)$ is given by

$$G_a(s) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{(s-1)(s+1+\frac{k}{n})} = \frac{A}{s-1} - \frac{1}{n} \sum_{k=0}^{n-1} \frac{\frac{1}{2+\frac{k}{n}}}{s+1+\frac{k}{n}}$$

where n is a large number and $A = \frac{1}{n}\sum_{k=0}^{n-1} 1/(2 + \frac{k}{n})$. In order for the difference $G(s) - G_a(s)$ to be stable, the pole at $s = 1$ must be removed. If we write out the expression for $G(s)$ explicitly rather than in terms of an integral, we find that this is equivalent to making the difference

$$\frac{\text{Log}\left(\frac{s+2}{s+1}\right)}{s - 1} - \frac{A}{s - 1}$$

bounded as s approaches 1. But this can only happen if A is equal to the residue of $G(s)$ at $s = 1$ or, namely, $\ln(3/2)$. It is clear that since A is a finite sum of rational numbers, then A is not equal to $\ln(3/2)$. So, in order to make sure that the difference between $G(s)$ and $G_a(s)$ is stable, we must arbitrarily *force* the value of A to be $\ln(3/2)$. Note this will not increase the infinity norm of the error since as n grows unboundedly

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{2 + \frac{k}{n}} = \int_0^1 \frac{dx}{x + 2} = \ln(3/2)$$

which means that, for n large enough, A is relatively close to $\ln(3/2)$ anyway.

Now the procedure is as follows: we initially choose some very large value of n to make $||G(s) - G_a(s)||_\infty$ much smaller than the bound of interest, which in this case is 0.02. Once we do this, we may then perform Hankel model order reduction on the *stable* part of $G_a(s)$ (no hope of removing the unstable pole if we want the difference to be stable!), which we will refer to as $G_{as}(s)$. If we denote the Hankel reduced model as $\hat{G}_{as}(s)$, then the overall low-order approximation $\hat{G}(s)$ is given by $\ln(3/2)/(s - 1) + \hat{G}_{as}(s)$.

Function ps7_3_1.m takes in a parameter n and plots the magnitude of the frequency response of $G(j\omega) - G_a(j\omega)$ to check the infinity norm of the difference for different values of n. Once a suitably large value of n is found, the function ps7_3_2.m which takes in as arguments n and a parameter k (which is the desired order of the reduced model of the stable part) can be used. This function creates the transfer function $\hat{G}_{as}(s)$ and plots $|G(j\omega) - \hat{G}(j\omega)|$ in order to verify whether the reduced model satisfies the norm bound constraint $||G(s) - \hat{G}(s)||_\infty < 0.02$. The resulting transfer function $\hat{G}(s)$ is created and displayed.

I found that the stable part of the approximation could be reduced all the way to the first order system and still be *well* within the desired bound. Running the command ps7_3_2(100,1) yields the transfer function

$$\hat{G}(s) = \frac{-0.001399s^2 + 0.01058s + 0.9543}{s^2 + 0.3762s - 1.376}$$

which satisfies the norm bound constraint $||G(s) - \hat{G}(s)||_\infty < 0.0025$, almost a factor of ten better than our requirement!