Massachusetts Institute of Technology

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Lecture 3: Continuous Dependence On Parameters[1]

Arguments based on continuity of functions are common in dynamical system analysis. They rarely apply to quantitative statements, instead being used mostly for proofs of existence of certain objects (equilibria, open or closed invariant set, etc.) Alternatively, continuity arguments can be used to show that certain qualitative conditions cannot be satisfied for a class of systems.

3.1 Uniqueness Of Solutions

In this section our main objective is to establish sufficient conditions under which solutions of ODE with given initial conditions are unique.

3.1.1 A counterexample

Continuity of the function $a: \mathbf{R}^n \mapsto \mathbf{R}^n$ on the right side of ODE

$$\dot{x}(t) = a(x(t)), \quad x(t_0) = \bar{x}_0 \tag{3.1}$$

does *not* guarantee uniqueness of solutions.

Example 3.1 The ODE

$$\dot{x}(t) = 3|x(t)|^{2/3}, \quad x(0) = 0$$

has solutions $x(t) \equiv 0$ and $x(t) \equiv t^3$ (actually, there are infinitely many solutions in this case).

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3.1.2 A general uniqueness theorem

The key issue for uniqueness of solutions turns out to be the maximal slope of $a = a(x)$: to guarantee uniqueness on time interval $T = [t_0, t_f]$, it is sufficient to require existence of a constant M such that

$$|a(\bar{x}_1) - a(\bar{x}_2)| \leq M|\bar{x}_1 - \bar{x}_2|$$

for all $\bar{x}_1, \bar{x}_2$ from a neigborhood of a solution $x : [t_0, t_f] \mapsto \mathbf{R}^n$ of (3.1). The proof of both existence and uniqueness is so simple in this case that we will formulate the statement for a much more general class of integral equations.

Theorem 3.1 *Let X be a subset of $\mathbf{R}^n$ containing a ball*

$$B_r(\bar{x}_0) = \{\bar{x} \in \mathbf{R}^n : \ |\bar{x} - \bar{x}_0| \leq r\}$$

of radius $r > 0$, and let $t_1 > t_0$ be real numbers. Assume that function $a : X \times [t_0, t_1] \times [t_0, t_1] \mapsto \mathbf{R}^n$ is such that there exist constants M, K satisfying

$$|a(\bar{x}_1, \tau, t) - a(\bar{x}_2, \tau, t)| \leq K|\bar{x}_1 - \bar{x}_2| \quad \forall \ \bar{x}_1, \bar{x}_2 \in B_r(\bar{x}_0), \ t_0 \leq \tau \leq t \leq t_1, \tag{3.2}$$

and

$$|a(\bar{x}, \tau, t)| \leq M \quad \forall \ \bar{x} \in B_r(\bar{x}_0), \ t_0 \leq \tau \leq t \leq t_1. \tag{3.3}$$

Then, for a sufficiently small $t_f > t_0$, there exists unique *function $x : [t_0, t_f] \mapsto X$ satisfying*

$$x(t) = \bar{x}_0 + \int_{t_0}^{t} a(x(\tau), \tau, t) d\tau \quad \forall \ t \in [t_0, t_f]. \tag{3.4}$$

A proof of the theorem is given in the next section. When a does not depend on the third argument, we have the standard ODE case

$$\dot{x}(t) = a(x(t), t).$$

In general, Theorem 3.1 covers a variety of nonlinear systems with an infinite dimensional state space, such as feedback interconnections of convolution operators and memoryless nonlinear transformations. For example, to prove well-posedness of a feedback system in which the forward loop is an LTI system with input v, output w, and transfer function

$$G(s) = \frac{e^{-s} - 1}{s},$$

and the feedback loop is defined by $v(t) = \sin(w(t))$, one can apply Theorem 3.1 with

$$a(\bar{x}, \tau, t) = \begin{cases} \sin(\bar{x}) + h(t), & t - 1 \leq \tau \leq t, \\ h(t), & \text{otherwise}, \end{cases}$$

where $h = h(t)$ is a given continuous function depending on the initial conditions.

3.1.3 Proof of Theorem 3.1.

First prove existence. Choose $t_f > t_1$ such that $t_f - t_0 \le r/M$ and $t_f - t_0 \le 1/(2K)$. Define functions $x_k : \ [t_0, t_f] \mapsto X$ by

$$x_0(t) \equiv \bar{x}_0, \quad x_{k+1}(t) = \bar{x}_0 + \int_{t_0}^{t} a(x_k(\tau), \tau, t) d\tau.$$

By (3.3) and by $t_f - t_0 \le r/M$ we have $x_k(t) \in B_r(\bar{x}_0)$ for all $t \in [t_0, t_f]$. Hence by (3.2) and by $t_f - t_0 \le 1/(2K)$ we have

$$\begin{aligned} |x_{k+1}(t) - x_k(t)| &\le \int_{t_0}^{t} |a(x_k(\tau), \tau, t) - a(x_{k-1}(\tau), \tau, t)| d\tau \\ &\le \int_{t_0}^{t} K|x_k(\tau) - x_{k-1}(\tau)| d\tau \\ &\le 0.5 \max_{t \in [t_0, t_f]} \{|x_k(t) - x_{k-1}(t)|\}. \end{aligned}$$

Therefore one can conclude that

$$\max_{t \in [t_0, t_f]} \{|x_{k+1}(t) - x_k(t)|\} \le 0.5 \max_{t \in [t_0, t_f]} \{|x_k(t) - x_{k-1}(t)|\}.$$

Hence $x_k(t)$ converges exponentially to a limit $x(t)$ which, due to continuity of a with respoect to the first argument, is the desired solution of (3.4).

Now let us prove uniqueness. Note that, due to $t_f - t_0 \le r/M$, all solutions of (3.4) must satisfy $x(t) \in D_r(\bar{x}_0)$ for $t \in [t_0, t_f]$. If x_a and x_b are two such solutions then

$$\begin{aligned} |x_a(t) - x_b(t)| &\le \int_{t_0}^{t} |a(x_a(\tau), \tau, t) - a(x_b(\tau), \tau, t)| d\tau \\ &\le \int_{t_0}^{t} K|x_a(\tau) - x_b(\tau)| d\tau \\ &\le 0.5 \max_{t \in [t_0, t_f]} \{|x_a(t) - x_b(t)|\}, \end{aligned}$$

which immediately implies

$$\max_{t \in [t_0, t_f]} \{|x_a(t) - x_b(t)|\} = 0.$$

The proof is complete now. Note that the same proof applies when (3.2),(3.3) are replaced by the weaker conditions

$$|a(\bar{x}_1, \tau, t) - a(\bar{x}_2, \tau, t)| \le K(\tau)|\bar{x}_1 - \bar{x}_2| \ \ \forall \ \bar{x}_1, \bar{x}_2 \in B_r(\bar{x}_0), \ t_0 \le \tau \le t \le t_1,$$

and

$$|a(\bar{x}, \tau, t)| \le m(t) \ \ \forall \ \bar{x} \in B_r(\bar{x}_0), \ t_0 \le \tau \le t \le t_1,$$

where the functions $K(\cdot)$ and $M(\cdot)$ are integrable over $[t_0, t_1]$.

3.2 Continuous Dependence On Parameters

In this section our main objective is to establish sufficient conditions under which solutions of ODE depend continuously on initial conditions and other parameters.

Consider the parameterized integral equation

$$x(t,q) = \bar{x}_0(q) + \int_{t_0}^{t} a(x(\tau,q),\tau,t,q)d\tau, \quad t \in [t_0,t_1], \tag{3.5}$$

where $q \in \mathbf{R}$ is a parameter. For every fixed value of q integral equation (3.5) has the form of (3.4).

Theorem 3.2 *Let $x^0 : \ [t_0,t_f] \mapsto \mathbf{R}^n$ be a solution of (3.5) with $q = q_0$. For some $d > 0$ let*

$$X^d = \{\bar{x} \in \mathbf{R}^n : \quad \exists\ t \in [t_0,t_f] : \ |\bar{x} - x^0(t)| < d\}$$

be the d-neigborhood of the solution. Assume that

(a) there exists $K \in \mathbf{R}$ such that

$$|a(\bar{x}_1,\tau,t,q)-a(\bar{x}_2,\tau,t,q)| \le K|\bar{x}_1-\bar{x}_2| \ \ \forall\, \bar{x}_1,\bar{x}_2 \in X^d,\ t_0 \le \tau \le t \le t_f,\ q \in (q_0-d,q_0+d); \tag{3.6}$$

(b) there exists $K \in \mathbf{R}$ such that

$$|a(\bar{x},\tau,t,q)| \le M \quad \forall\ \bar{x} \in X^d,\ t_0 \le \tau \le t \le t_f,\ q \in (q_0-d,q_0+d); \tag{3.7}$$

(c) for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\bar{x}_0(q_1) - \bar{x}_0(q_2)| \le \epsilon \ \ \forall\ q_1,q_2 \in (q_0-d,q_0+d) : \ |q_1-q_2| < \delta,$$

$$|a(\bar{x},\tau,t,q_1) - a(\bar{x},\tau,t,q_2)| \le \epsilon \ \ \forall\ q_1,q_2 \in (q_0-d,q_0+d) : \ |q_1-q_2| < \delta,\ \bar{x} \in X^d.$$

Then there exists $d_1 \in (0,d)$ such that the solution $x(t,q)$ of (3.5) is continuous on

$$\{(t,q)\} = [t_0,t_f] \times (q_0-d_1,q_0+d_1).$$

Condition (a) of Theorem 3.2 is the familiar Lipschitz continuity requirement of the dependence of $a = a(x,\tau,t,q)$ on x in a neigborhood of the trajectory of x^0. Condition (b) simply bounds a uniformly. Finally, condition (c) means continuous dependence of equations and initial conditions on parameter q.

The proof of Theorem 3.2 is similar to that of Theorem 3.1.

3.3 Implications of continuous dependence on parameters

This section contains some examples showing how the general continuous dependence of solutions on parameters allows one to derive qualitative statements about nonlinear systems.

3.3.1 Differential flow

Consider a time-invariant autonomous ODE

$$\dot{x}(t) = a(x(t)), \tag{3.8}$$

where $a : \mathbf{R}^n \mapsto \mathbf{R}^m$ is satisfies the Lipschitz constraint

$$|a(\bar{x}_1) - a(\bar{x}_2)| \leq M|\bar{x}_1 - \bar{x}_2| \tag{3.9}$$

on every bounded subset of $\mathbf{R}^n$. According to Theorem 3.1, this implies existence and uniqueness of a maximal solution $x : (t_-, t_+) \mapsto \mathbf{R}^n$ of (3.8) subject to given initial conditions $x(t_0) = \bar{x}_0$ (by this definition, $t_- < t_0 < t_+$, and it is possible that $t_- = -\infty$ and/or $t_+ = \infty$). To specify the dependence of this solution on the initial conditions, we will write $x(t) = x(t, t_0, \bar{x}_0)$. Due to the time-invariance of (3.8), this notation can be further simplified to $x(t) = x(t - t_0, \bar{x}_0)$, where $x(t, \bar{x})$ means "the value $x(t)$ of the solution of (3.8) with initial conditions $x(0) = \bar{x}$". Remember that this definition makes sense only when uniqueness of solutions is guaranteed, and that $x(t, \bar{x})$ may by undefined when $|t|$ is large, in which case we will write $x(t, \bar{x}) = \infty$.

According to Theorem 3.2, $x : \Omega \mapsto \mathbf{R}^n$ is a *continuous* function defined on an open subset $\Omega \subset \mathbf{R} \times \mathbf{R}^n$. With $\bar{x}$ considered a parameter, $t \mapsto x(t, \bar{x})$ defines a family of smooth curves in $\mathbf{R}^n$. When t is fixed, $\bar{x} \mapsto x(t, \bar{x})$ defines a continuous map form an open subset of $\mathbf{R}^n$ and with values in $\mathbf{R}^n$. Note that $x(t_1, x(t_2, \bar{x})) = x(t_1 + t_2, \bar{x})$ whenever $x(t_2, \bar{x}) \neq \infty$. The function $x : \Omega \mapsto \mathbf{R}^n$ is sometimes called "differential flow" defined by (3.8).

3.3.2 Attractors of asymptotically stable equilibria

A poiint $\bar{x}_0 \in \mathbf{R}^n$ is called an *equilibrium* of (3.8) when $a(\bar{x}_0) = 0$, i.e. $x(t, \bar{x}_0) \equiv \bar{x}_0$ is a constant solution of (3.8).

Definition An equilibrium $\bar{x}_0$ of (3.8) is called *asymptotically stable* if the following two conditions are satisfied:

(a) there exists $d > 0$ such that $x(t, \bar{x}) \to \bar{x}_0$ as $t \to \infty$ for all $\bar{x}$ satisfying $|\bar{x}_0 - \bar{x}| < d$;

(b) for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x(t, \bar{x}) - \bar{x}_0| < \epsilon$ whenever $t \geq 0$ and $|\bar{x} - \bar{x}_0| < \delta$.

In other words, all solutions starting sufficiently close to an asymptotically stable equilibrium $\bar{x}_0$ converge to it as $t \to \infty$, and none of such solutions can escape far away before finally converging to $\bar{x}_0$.

Theorem 3.3 *Let $\bar{x}_0 \in \mathbf{R}^n$ be an asymptotically stable equilibrium of (3.8). The set $A = A(\bar{x}_0)$ of all $\bar{x} \in \mathbf{R}^n$ such that $x(t, \bar{x}) \to \bar{x}_0$ as $t \to \infty$ is an open subset of $\mathbf{R}^n$, and its boundary is invariant under the transformations $\bar{x} \mapsto x(t, \bar{x})$.*

The proof of the theorem follows easily from the continuity of $x(\cdot, \cdot)$.

3.3.3 Limit points of a trajectory

For a fixed $\bar{x}_0 \in \mathbf{R}^n$, the set of all possible limits $x(t_k, \bar{x}_0) \to \tilde{x}$ as $k \to \infty$, where the sequence $\{t_k\}$ also converges to infinity, is called the *limit set* of the "trajectory" $t \mapsto x(t, \bar{x}_0)$.

Theorem 3.4 *The limit set of a given trajectory is always closed and invariant under the transformations $\bar{x} \mapsto x(t, \bar{x})$.*