Lectures on Dynamic Systems and Control

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Chapter 30

Minimality and Stability of Interconnected Systems

30.1 Introduction: Relating I/O and State-Space Properties

We have already seen in Chapter 25 that a minimal realization of a transfer matrix $H(s)$ is uniquely defined by $H(s)$, up to a similarity transformation. We therefore expect properties of a minimal realization to be tightly associated with properties of the transfer matrix. This expectation is reflected in some of the results described in Chapter 27.

Specifically, we claimed in Chapter 27 that the *poles* of $H(s)$ are precisely given — in both location and multiplicity — by the *eigenvalues* and associated Jordan structure of the matrix A in a minimal realization (A, B, C, D) of $H(s) = C(sI - A)^{-1}B + D$; this structure is in turn equivalent to the *zero* structure of the matrix $(sI - A)$, although we did not draw attention to this fact in Chapter 27. The general proof of the preceding result is beyond the scope of the tools that we have available, but it is not hard to prove in the special case of an $H(s)$ that is amenable to the Gilbert realization procedure of Chapter 25, as we show below. Before turning to this demonstration, we note the following important implication of the result:

- For a minimal system, BIBO stability is equivalent to asymptotic stability; the state-space model is asymptotically stable if and only if $H(s)$ has no unstable (i.e. right half plane) poles.

For the Gilbert realization to work, each entry of $H(s)$ is required to have poles of multiplicity 1 only. For such an $H(s)$, using the notation of Chapter 25 and the definitions of poles and their multiplicities from Chapter 27, it is quite straightforward to argue that $H(s)$ has r_i *poles* located at ρ_i, each of multiplicity 1. The A matrix of the corresponding Gilbert realization that we constructed (and hence the A matrix of any other minimal realization of this transfer function) evidently has r_i Jordan blocks of size 1 associated with the *eigenvalue* at ρ_i. Also, the matrix $(sI - A)$ for the Gilbert realization evidently has r_i *zeros* of multiplicity 1 at ρ_i.

Similarly, as noted in Chapter 27, the *zeros* of $H(s)$ are given — in both location and multiplicity — by the *generalized eigenvalues* and associated "Jordan-Kronecker" structure of the matrix pair $(\mathcal{E}, \mathcal{A})$ associated with the system matrix $s\mathcal{E} - \mathcal{A}$ of a minimal realization of $H(s)$, or equivalently by the *zero*

structure of the system matrix. We shall not attempt to prove anything on zeros beyond what has already been shown in Chapter 27.

30.2 Loss of Minimality in Interconnections

In this section we shall examine the conditions under which minimality is lost when minimal subsystems are interconnected in various configurations, such as the series connection in Fig. 30.1 below. The standard convention in interpreting such figures, where the individual subsystem blocks are labeled with their transfer functions, is to assume that each subsystem block contains a *minimal realization*, i.e. a *reachable and observable* realization, of the indicated transfer function. This is a reasonable convention, since the transfer function is inadequate to describe any unreachable and/or unobservable parts of the system; if such parts existed and were important to the problem, they would have to be described in some appropriate way.

We will denote the minimal realization of $H_i(s)$ by (A_i, B_i, C_i, D_i), and denote its associated input, state and output vectors by u_i, x_i, y_i respectively. When it simplifies some of the algebra, we shall feel free to assume that $D_i = 0$, as the presence of a direct feedthrough from input to output adds no essential difficulty and introduces no significant features in the problems that we consider, but often makes the algebra cumbersome. Note that our assumption of minimality on the subsystems ensures that the eigenvalues of A_i are precisely the poles of $H_i(s)$, both in location and in multiplicity.

Series Connection

Consider subsystems with transfer matrices $H_1(s)$ and $H_2(s)$ connected in series (or "cascaded") as shown in Fig. 30.1. The transfer function of the cascaded system is evidently $H(s) = H_2(s)H_1(s)$ (the
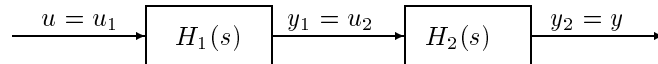

Figure 30.1: Two subsystems in series.

factors must be written in that order unless the subsystems are SISO!). The natural state vector for the cascaded system comprises x_1 and x_2, and the corresponding state-space description of the cascade is easily seen to be given (when $D_i = 0$) by the matrices

$$A = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} , \ B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} , \ C = \begin{pmatrix} 0 & C_2 \end{pmatrix} , \ D = 0 . \tag{30.1}$$

The structure of A shows that its eigenvalues, which are the natural frequencies of the cascade, are the eigenvalues of A_1 and A_2 taken together, i.e. the natural frequencies of the individual subsystems taken together.

The question of interest to us now is whether the cascaded system is minimal, i.e., is (A, B, C) a minimal realization of $H(s)$? It should be clear at this point that the cascade is minimal if and only if the number of poles of $H(s)$ is the sum of the number of poles in $H_1(s)$ and $H_2(s)$ (multiplicities

included). Otherwise the number of poles in $H(s)$ — and hence the number of state variables in a minimal realization of $H(s)$ — ends up being less than the number of state variables (and modes) in the cascaded system, signaling a loss of reachability and/or observability.

In the case of SISO subsystems, this condition for minimality can evidently be restated as requiring that no pole of $H_1(s)$, respectively $H_2(s)$, be canceled by a zero of $H_2(s)$, respectively $H_1(s)$. Furthermore, it is a straightforward exercise (which we leave you to carry out, using the controller or observer canonical forms for the subsystem realizations, the state-space description in (30.1) for the cascade, and the modal tests for reachability and observability) to show very explicitly that

- the cascade is *unreachable* if and only if a pole of $H_2(s)$ is canceled by a zero of $H_1(s)$;

- the cascade is *unobservable* if and only if a pole of $H_1(s)$ is canceled by a zero of $H_2(s)$.

(The demonstration of these results is worth working out in detail, and will make clear why we invested time in discussing canonical forms and modal tests.) These conditions make intuitive sense, in that the first kind of cancellation blocks access of the input to a system mode that is generated in the second subsystem, and the second kind of cancellation blocks access to the output for a system mode generated in the first subsystem.

Essentially the same interpretations in terms of pole-zero cancellations hold in the MIMO case, subject to certain rank conditions on the matrices. We shall content ourselves with demonstrating how the loss of observability is related to a pole of $H_1(s)$ being cancelled by a zero of $H_2(s)$. For this, note from the modal test and the structure of the model in (30.1) that observability of the cascade is lost iff, for some λ,

$$\begin{pmatrix} \lambda I - A_1 & 0 \\ -B_2 C_1 & \lambda I - A_2 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \ , \ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq 0 \ , \qquad (30.2)$$

Now we must have $v_1 \neq 0$, otherwise (30.2) shows (by the modal test) that the assumed observability of the second subsystem is contradicted. Hence v_1 is an eigenvector of the first subsystem. Also $C_1 v_1 \neq 0$, otherwise (again by the modal test!) the observability of the first subsystem is contradicted. Now rewriting the bottom two rows of (30.2), we get

$$\begin{pmatrix} \lambda I - A_2 & -B_2 \\ C_2 & 0 \end{pmatrix} \begin{pmatrix} v_2 \\ C_1 v_1 \end{pmatrix} = 0 \ . \qquad (30.3)$$

Thus the cascade is unobservable iff (30.3) holds for some eigenvalue and eigenvector pair (λ, v_1) of the first subsystem. From Chapter 27 we know that this equation is equivalent, in the case where $H_2(s)$ has full column rank, to the second subsystem having a transmission zero at λ, with input zero direction $C_1 v_1$ and state zero direction $v_2 (\neq 0)$. [If $H_2(s)$ does not have full column rank, then the loss of observability may be due to a mode of the first subsystem "hiding" in the *nullspace* of $H_2(s)$, rather than due to its being blocked by a transmission zero. Some exploration with diagonal $H_1(s)$ and $H_2(s)$ will show you what sorts of things can happen.]

Parallel Connection

A parallel connection of two subsystems is shown in Fig. 30.2. The transfer function of this system is $H(s) = H_1(s) + H_2(s)$. The natural state vector for the parallel system again comprises x_1 and x_2, and the corresponding state-space description of the combination is easily seen to be given by the matrices

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \ , \ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \ , \ C = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \ , \ D = D_1 + D_2 \ . \qquad (30.4)$$
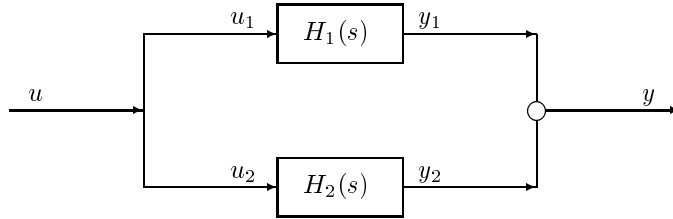
Figure 30.2: Two subsystems in parallel.

The structure of A shows that its eigenvalues, which are the natural frequencies of the parallel system, are the eigenvalues of A_1 and A_2 taken together, i.e. the natural frequencies of the individual subsystems taken together (just as in the case of cascaded subsystems).

It is easy in this case to state and prove the precise conditions under which reachability or observability is lost. We treat the case of observability below, and leave you to provide the dual statement and proof for reachability.

- **Claim:** The parallel combination loses observability if and only if:

  (i) A_1 and A_2 have a *common eigenvalue, and*

  (ii) some choice of associated right eigenvectors v_1 and v_2 satisfies $C_1v_1 + C_2v_2 = 0$ (this second condition is always satisfied in the single-output case if the first condition is satisfied).

  Proof: By the modal test, the parallel system is unobservable iff there is an eigenvector

  $$v = \left(\begin{array}{c} v_1 \\ v_2 \end{array} \right) \neq 0$$

  associated with some eigenvalue λ of A (so $Av = \lambda v$, $v \neq 0$) such that $Cv = C_1v_1 + C_2v_2 = 0$. If both $v_1 \neq 0$ and $v_2 \neq 0$, then we can conclude that λ is an eigenvalue of both A_1 and A_2, and the claim would be proved. To show $v_1 \neq 0$, note that $v_1 = 0$ would imply $C_2v_2 = 0$ which, together with the fact that $A_2v_2 = \lambda v_2$, would contradict the assumed observability of the second subsystem. Similarly, we must have $v_2 \neq 0$.

  In the single-output case, the fact that the quantities $C_1v_1 \neq 0$ and $C_2v_2 \neq 0$ are scalars means that we can always scale the eigenvectors so as to obtain $C_1v_1 + C_2v_2 = 0$. Hence all that is needed to induce unobservability in the single-output case is for the subsystems to have a common eigenvalue.

Feedback Connection

A feedback connection of two systems is shown in Fig. 30.3 We leave you to show that this feedback configuration is reachable from u if and only if the cascade configuration in Fig. 30.1 is reachable. (Hint: Feeding back the output of the cascade configuration does not affect whether it is reachable or not.) Similarly, argue that the feedback configuration in Fig. 30.3 is observable if and only if the cascade configuration in Fig 30.4 is observable.
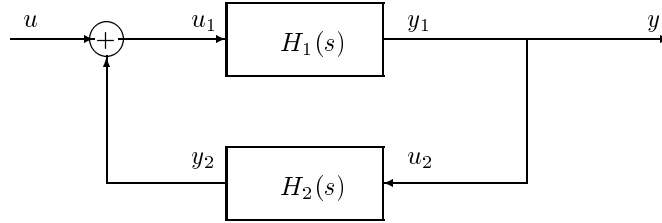
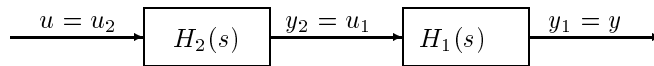Figure 30.3: Two subsystems in a feedback configuration.

Figure 30.4: A cascade configuration whose observability properties are equivalent to those of the feedback system in Fig. 20.3.

A state-space description of the feedback configuration (with $D_i = 0$) is easily seen to be given by

$$A = \begin{pmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{pmatrix} , \ B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} , \ C = \begin{pmatrix} 0 & C_2 \end{pmatrix} . \tag{30.5}$$

The eigenvalues of A are not evident by inspection, unlike in the case of the cascade and parallel connections, because feedback can shift eigenvalues from their open-loop locations. The characteristic polynomial of A, namely $a(s) = \det(sI - A)$, whose roots are the natural frequencies of the system, is easily shown (using various identities from Homework 1) to be

$$a(s) = a_1(s) a_2(s) \det \left(I - H_1(s) H_2(s) \right) . \tag{30.6}$$

If there is a pole-zero cancellation between $H_1(s)$ and $H_2(s)$, then this pole is unaffected by the feedback, and remains a natural frequency of the closed-loop system.

30.3   Stability of Interconnected Systems

The *composite* state-space description of an interconnected system is obtained by combining state-space realizations of the individual subsystems, using as state variables the union of the subsystem state variables. If a subsystem is specified by its transfer function, then we are obliged to use a minimal realization of this transfer function in constructing the composite description. Examples of such composite descriptions have already been seen in (30.1), (30.4) and (30.5). The interconnected

system is said to be *well-posed* precisely when its composite state-space description can be obtained (see Chapter 17).

Once a state-space description (A, B, C, D) of the interconnected system has been obtained, it is in principle straightforward to determine its natural frequencies and assess its asymptotic stability by examining the eigenvalues of A. However, if each subsystem has been specified via its transfer function, one might well ask if there is a way to determine the natural frequencies and evaluate stability *using transfer function computations alone, without* bothering to construct minimal realizations of each subsystem in order to obtain a composite realization of the interconnection.

A first thought might be to look at the poles of the transfer function between some input and output in the interconnected system. However, we know (and have again confirmed in the preceding section) that the poles of the transfer function between some input u and some output y will fail to show all the natural frequencies of the system if (and only if) some mode of the system is unreachable and/or unobservable with that input/output pair. Furthermore, the method we prescribe for determining natural frequencies through transfer function computations alone should be able to find natural frequencies even when no external inputs and outputs have been designated, because natural frequencies are well defined even when the system has no inputs or outputs.

In view of the above problem with "hidden" modes, a second thought might be to *not* limit ourselves to prespecified inputs and outputs of the interconnection. Instead, we could evaluate the transfer functions *from input signals added in at* **all** *subsystem entries, to output signals taken at* **all** *subsystem outputs.* This turns out to be the right idea, and we develop it in detail for the case of two subsystems interconnected in feedback.

Suppose we are given the feedback configuration in Fig. 30.5, and are asked to determine its natural frequencies. The first step is to add in inputs at each subsystem, as in Fig. 30.6.
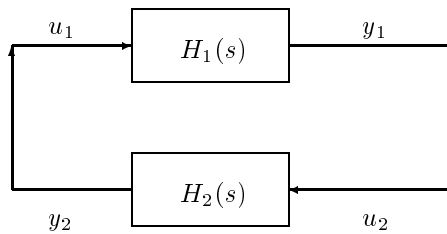

Figure 30.5: A feedback interconnection, with no prespecified external inputs or outputs.

Then examine the (four) transfer functions from ν_1 and ν_2 to y_1 and y_2, or equivalently the transfer matrix $oH(s)$ that relates

$$\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(in Chapter 17, $\mathbb{H}(s) = \mathcal{T}(H_1, H_2)(s)$). Instead of looking at the response at y_1 and y_2, we could alternatively compute the response at u_1 and u_2, or at u_1 and y_1, or at u_2 and y_1, because the response at y_1 and y_2 can be determined from these other responses, knowing ν_1 and ν_2. The choice is determined by convenience.

Letting (A_i, B_i, C_i) denote minimal realizations of $H_i(s)$ as before (and assuming for simplicity that the direct feedthrough term D_i is zero), we now have the following theorem, which provides the
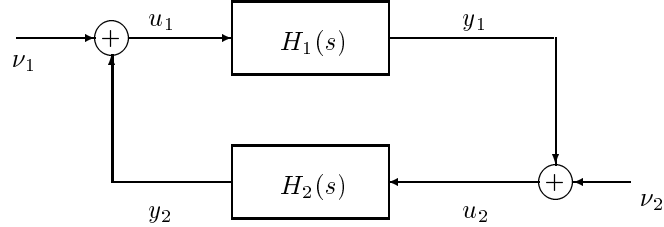
Figure 30.6: The same feedback interconnection, but with inputs added in at each subsystem.

basis for what we were seeking, namely a transfer function based approach to determining the natural frequencies of the interconnection.

Theorem 30.1 *The composite state-space description*

$$A = \begin{pmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{pmatrix} \ , \ B_d = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \ , \ C_d = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \tag{30.7}$$

for the system in Fig. 30.6 is a minimal realization of the transfer function $\mathbb{H}(s)$ from the external subsystem inputs ν_1 and ν_2 to the subsystem outputs y_1 and y_2, so its natural frequencies, i.e. the eigenvalues of A, are precisely the poles of $\mathbb{H}(s)$.

Proof. By inspection, a minimal (or equivalently, reachable and observable) realization of

$$H(s) = \begin{pmatrix} H_1(s) & 0 \\ 0 & H_2(s) \end{pmatrix} \ ,$$

which is the transfer matrix from u_1, u_2 to y_1, y_2, is given by

$$A_d = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \ , \ B_d = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \ , \ C_d = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \ .$$

Now output feedback around this realization will not destroy its reachability or observability, so

$$A_d + B_d \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} C_d \ , \ B_d \ , \ C_d \tag{30.8}$$

is a minimal realization of the system obtained by implementing the output feedback specified by the feedback gain matrix

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \ .$$

It is easy to check that the resulting system is precisely the one in Fig. 30.6, and the realization in (30.8) is precisely the composite description in (30.7), since

$$A_d + B_d \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} C_d \ = \ \begin{pmatrix} A_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{pmatrix} \ = \ A \ .$$

Now, for a minimal realization, the poles of the transfer function are equal to the natural frequencies of the system, so the poles of $\mathbb{H}(s)$ are precisely the eigenvalues of A.

Note that this *same* A matrix is obtained in the composite state-space descriptions of the systems in Fig. 30.3, Fig. 30.5 and Fig. 30.6, because these systems only differ in their specifications of inputs and outputs. For all these systems, we can determine the natural frequencies by determining the poles of $\mathbb{H}(s)$, and we can assess the asymptotic stability of these systems (i.e. the asymptotic stability of their composite realizations) by checking that the poles of $\mathbb{H}(s)$ are all in the left half plane, i.e. by checking BIBO stability from ν_1, ν_2 to y_1, y_2. (We leave you to construct examples that show the need to check all four of the transfer function entries in $\mathbb{H}(s)$, because a natural frequency can hide from any three of them — the fourth one is needed to flush such a natural frequency out.)

The same argument we used for the special feedback configuration above actually works for any well-posed interconnected system. We leave you to fashion a proof. Also, it should go without saying that everything we have done here in continuous-time holds for discrete-time systems too. You may find it profitable to revisit some of the examples of Chapter 17 with the new perspectives gained from this chapter.

1. Assume we have the configuration in Figure 17.4, with $P = \frac{s-1}{s+1}$ and $K = -\frac{1}{s-1}$. The transfer function relating r to y is

$$
\begin{aligned}
\frac{P}{1-PK} &= \frac{s-1}{s+1}\left(1 + \frac{1}{s+1}\right)^{-1} \\
&= \left(\frac{s-1}{s+1}\right)\left(\frac{s+1}{s+2}\right) \\
&= \frac{s-1}{s+2} \quad .
\end{aligned}
$$

Since the only pole of this transfer function is at $s = -2$, the input/output relation between r and y is stable. However, consider the transfer function from d to u, which is

$$
\begin{aligned}
\frac{K}{1-PK} &= \frac{1}{s-1}\left(\frac{1}{1 + \frac{1}{s+1}}\right) \\
&= \frac{s+1}{(s-1)(s+2)} \quad .
\end{aligned}
$$

This transfer function is unstable, which implies that the closed-loop system is externally unstable.

2. We leave you to show that the interconnected system in Figure 17.4 is externally stable if and only if the matrix

$$
\begin{bmatrix} (I-PK)^{-1}P & (I-PK)^{-1} \\ (I-KP)^{-1} & -(I-PK)^{-1}K \end{bmatrix}
\tag{30.9}
$$

has all its poles in the open left half plane.

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