6.241 Spring 2011

Midterm Exam

3/16/2011

You have a total of three hours to complete this exam. These three hours can be chosen at your convenience.

Problem 1

Let $A \in \mathbb{C}^{n\times n}$, and $B \in \mathbb{C}^{m\times m}$. Show that $X(t) = e^{At}X(0)e^{Bt}$ is the solution to $\dot{X} = AX + XB$.

Problem 2

Given two non-zero vectors $v, w \in \mathbb{R}^n$. Does there exist a matrix A such that $v = Aw$ and

1. $\sigma_{\max}(A) = \sqrt{v^T v / w^T w}$?
2. $\|A\|_1 = \|v\|_\infty / \|w\|_\infty$?

Prove or disprove each case separately.

Problem 3

Let $\|A\| < 1$. Show that $\|(I - A)^{-1}\| \geq \dfrac{1}{1 + \|A\|}$.

Problem 4

Use the projection theorem to solve the problem:

$$\min x \in \mathbb{R}^n \{x^T Q x : Ax = b\},$$

where Q is a positive-definite $n \times n$ matrix, A is a $m \times n$ real matrix, with rank $m < n$, and b is a real m-dimensional vector. Is the solution unique?

Problem 5

Consider a single-input discrete-time LTI system, described by

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]$$

$$y[k] = x[k],$$

and the initial condition $x[0] = 0$. Given $M > 1$, what is the maximum value of $\|y[M]\|_2$ that can be attained with an input of "unit energy,", i.e., such that $u[0]^2 + u[1]^2 + \ldots + u[M-1]^2 = 1$? What is the input that attains such value? How would your answer change if you were to double M, i.e., $M \leftarrow 2M$?

Problem 6

Consider a physical system whose behavior is modeled, in continuous time, by the differential equation

$$\dot{x} = Ax + Bu.$$

Assume that you have two sensors. The first sensor yields measurements $y_1 = C_1 x$ for $t = 0, 1, 2, 3, \ldots$, and the second sensor yields measurements $y_2 = C_2 x$ for $t = 0, 2, 4, \ldots$. Assuming that $u(t) = u(\lfloor t \rfloor)$, for all $t \geq 0$, derive a discrete-time state-space model for the system, relating the inputs at times $(u(0), u(1), u(2), \ldots)$ to the outputs at times $(y(0), y(1), y(2), \ldots)$.

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