6.172
Performance
Engineering
of Software
Systems

LECTURE 2

Bentley Rules for Optimizing Work

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Work

Definition.
The work of a program (on a given input) is the sum total of all the operations executed by the program.

Optimizing Work

- Algorithm design can produce dramatic reductions in the amount of work it takes to solve a problem, as when a $\Theta(n \lg n)$–time sort replaces a $\Theta(n^2)$–time sort.
- Reducing the work of a program does not auto-matically reduce its running time, however, due to the complex nature of computer hardware:
  - instruction-level parallelism (ILP),
  - caching,
  - vectorization,
  - speculation and branch prediction,
  - etc.
- Nevertheless, reducing the work serves as a good heuristic for reducing overall running time.

“BENTLEY” OPTIMIZATION RULES

New "Bentley" Rules

- Most of Bentley's original rules dealt with work, but some dealt with the vagaries of computer architecture three and a half decades ago.
- We have created a new set of Bentley rules dealing only with work.
- We shall discuss architecture-dependent optimizations in subsequent lectures.

New Bentley Rules

Data structures

- Packing and encoding
- Augmentation
- Precomputation
- Compile-time initialization
- Caching
- Lazy evaluation
- Sparsity

Loops

- Hoisting
- Sentinels
- Loop unrolling
- Loop fusion
- Eliminating wasted iterations

Logic

- Constant folding and propagation
- Common-subexpression elimination
- Algebraic identities
- Short-circuiting
- Ordering tests
- Creating a fast path
- Combining tests

Functions

- Inlining
- Tail-recursion elimination
- Coarsening recursion

DATA STRUCTURES

Packing and Encoding

The idea of packing is to store more than one data value in a machine word. The related idea of encoding is to convert data values into a representation requiring fewer bits.

Example: Encoding dates

- The string "`September 11, 2018`" can be stored in 18 bytes — more than two double (64-bit) words — which must moved whenever a date is manipulated.
- Assuming that we only store years between 4096 B.C.E. and 4096 C.E., there are about $365.25 \times 8192 \approx 3\,\text{M}$ dates, which can be encoded in $\lceil \lg(3 \times 10^6) \rceil = 22$ bits, easily fitting in a single (32-bit) word.
- But determining the month of a date takes more work than with the string representation.

Packing and Encoding (2)

Example: Packing dates

- Instead, let us pack the three fields into a word:

```
typedef struct {
   int year: 13;
   int month: 4;
   int day: 5;
} date_t;
```

- This packed representation still only takes 22 bits, but the individual fields can be extracted much more quickly than if we had encoded the 3 M dates as sequential integers.

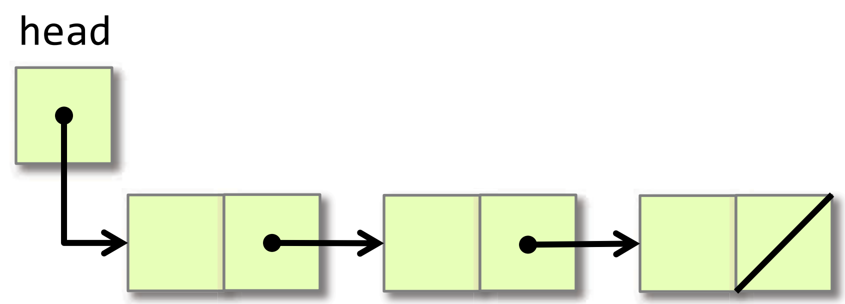
Sometimes unpacking and decoding are the optimization, depending on whether more work is involved moving the data or operating on it.

Augmentation

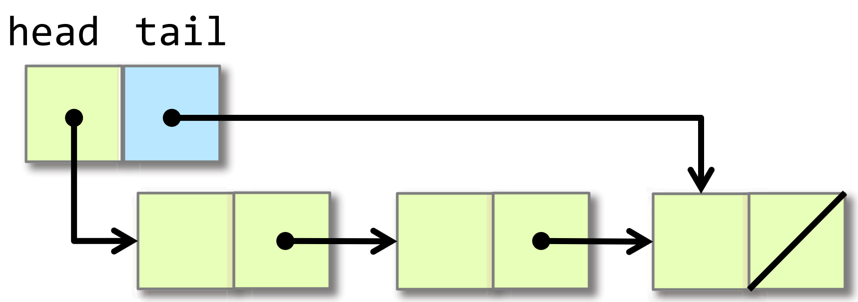
The idea of data-structure augmentation is to add information to a data structure to make common operations do less work.

Example: Appending singly linked lists

- Appending one list to another requires walking the length of the first list to set its null pointer to the start of the second.



- Augmenting the list with a tail pointer allows appending to operate in constant time.



Precomputation

The idea of precomputation is to perform calculations in advance so as to avoid doing them at "mission-critical" times.

Example: Binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Computing the "choose" function by implementing this formula can be expensive (lots of multiplications), and watch out for integer overflow for even modest values of n and k.

Idea: Precompute the table of coefficients when initializing, and perform table look-up at runtime.

Pascal's Triangle

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

1	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
1	2	1	0	0	0	0	0	0
1	3	3	1	0	0	0	0	0
1	4	6	4	1	0	0	0	0
1	5	10	10	5	1	0	0	0
1	6	15	20	15	6	1	0	0
1	7	21	35	35	21	7	1	0
1	8	28	56	70	56	28	8	1

```
int choose(int n, int k) {
  if (n < k) return 0;
  if (n == 0) return 1;
  if (k == 0) return 1;
  return choose(n-1, k-1) + choose(n-1, k);
}
```

Precomputing Pascal

```
#define CHOOSE_SIZE 100
int choose[CHOOSE_SIZE][CHOOSE_SIZE];

void init_choose() {
  for (int n = 0; n < CHOOSE_SIZE; ++n) {
    choose[n][0] = 1;
    choose[n][n] = 1;
  }
 for (int n = 1; n < CHOOSE_SIZE; ++n) {
    choose[0][n] = 0;
    for (int k = 1; k < n; ++k) {
      choose[n][k] = choose[n-1][k-1] + choose[n-1][k];
      choose[k][n] = 0;
    }
  }
}
```

Now, whenever we need a binomial coefficient (less than 100), we can simply index the choose array.

Compile-Time Initialization

The idea of compile-time initialization is to store the values of constants during compilation, saving work at execution time.

Example

```
int choose[10][10] = {
  {  1,   0,   0,   0,   0,   0,   0,   0,   0,   0, },
  {  1,   1,   0,   0,   0,   0,   0,   0,   0,   0, },
  {  1,   2,   1,   0,   0,   0,   0,   0,   0,   0, },
  {  1,   3,   3,   1,   0,   0,   0,   0,   0,   0, },
  {  1,   4,   6,   4,   1,   0,   0,   0,   0,   0, },
  {  1,   5,  10,  10,   5,   1,   0,   0,   0,   0, },
  {  1,   6,  15,  20,  15,   6,   1,   0,   0,   0, },
  {  1,   7,  21,  35,  35,  21,   7,   1,   0,   0, },
  {  1,   8,  28,  56,  70,  56,  28,   8,   1,   0, },
  {  1,   9,  36,  84, 126, 126,  84,  36,   9,   1, },
};
```

Compile-Time Initialization (2)

Idea: Create large static tables by metaprogramming.

```
int main(int argc, const char *argv[]) {
  init_choose();
  printf("int choose[10][10] = {\n");
  for (int a = 0; a < 10; ++a) {
    printf("  {");
    for (int b = 0; b < 10; ++b) {
      printf("%3d, ", choose[a][b]);
    }
    printf("},\n");
  }
  printf("};\n");
}
```

Caching

The idea of caching is to store results that have been accessed recently so that the program need not compute them again.

```
inline double hypotenuse(double A, double B) {
  return sqrt(A*A + B*B);
}
```

```
double cached_A = 0.0;
double cached_B = 0.0;
double cached_h = 0.0;

inline double hypotenuse(double A, double B) {
  if (A == cached_A && B == cached_B) {
    return cached_h;
  }
  cached_A = A;
  cached_B = B;
  cached_h = sqrt(A*A + B*B);
  return cached_h;
}
```

About 30% faster if cache is hit 2/3 of the time.

Sparsity

The idea of exploiting sparsity is to avoid storing and computing on zeroes. *"The fastest way to compute is not to compute at all."*

Example: Matrix-vector multiplication

$$y = \begin{pmatrix} 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 5 & 9 \\ 0 & 0 & 0 & 2 & 0 & 6 \\ 5 & 0 & 0 & 3 & 0 & 0 \\ 5 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 9 & 7 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 2 \\ 8 \\ 5 \\ 7 \end{pmatrix}$$

Dense matrix-vector multiplication performs $n^2 = 36$ scalar multiplies, but only 14 entries are nonzero.

Sparsity

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Example: Matrix-vector multiplication

$$
y = \begin{pmatrix}
3 & & & & 1 & \\
 & 4 & 1 & & 5 & 9 \\
 & & & 2 & & 6 \\
5 & & & 3 & & \\
5 & & & & 8 & \\
 & & & 9 & 7 &
\end{pmatrix}
\begin{pmatrix} 1 \\ 4 \\ 2 \\ 8 \\ 5 \\ 7 \end{pmatrix}
$$

Dense matrix-vector multiplication performs $n^2 = 36$ scalar multiplies, but only 14 entries are nonzero.

Sparsity (2)

Compressed Sparse Row (CSR)

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
rows:	0	2	6	8	10	11	14							

cols:	0	4	1	2	4	5	3	5	0	3	0	4	3	4
vals:	3	1	4	1	5	9	2	6	5	3	5	8	9	7

$$
\begin{array}{c}
0\\1\\2\\3\\4\\5
\end{array}
\begin{pmatrix}
3 & 0 & 0 & 0 & 1 & 0\\
0 & 4 & 1 & 0 & 5 & 9\\
0 & 0 & 0 & 2 & 0 & 6\\
5 & 0 & 0 & 3 & 0 & 0\\
0 & 0 & 0 & 0 & 5 & 0\\
0 & 0 & 0 & 8 & 9 & 7
\end{pmatrix}
$$

0 1 2 3 4 5

$n = 6$
$nnz = 14$

Storage is $O(n+nnz)$ instead of n^2

Sparsity (3)

CSR matrix-vector multiplication

```
typedef struct {
  int n, nnz;
  int *rows;      // length n
  int *cols;      // length nnz
  double *vals;   // length nnz
} sparse_matrix_t;

void spmv(sparse_matrix_t *A, double *x, double *y) {
  for (int i = 0; i < A->n; i++) {
    y[i] = 0;
    for (int k = A->rows[i]; k < A->rows[i+1]; k++) {
      int j = A->cols[k];
      y[i] += A->vals[k] * x[j];
    }
  }
}
```

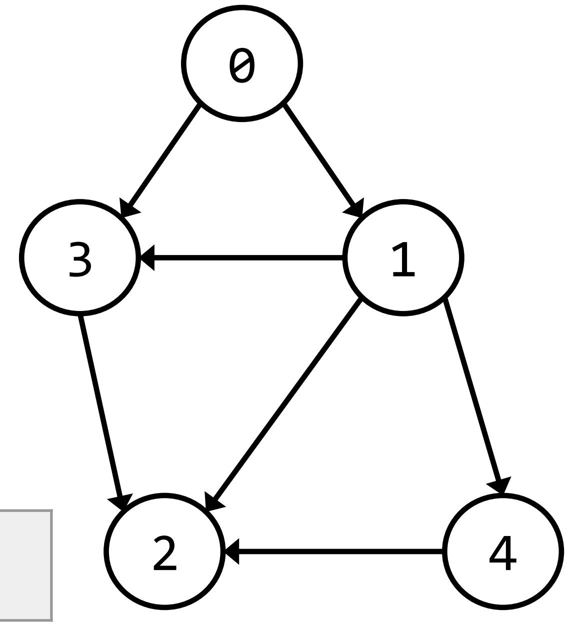
Number of scalar multiplications = nnz, which is potentially much less than n^2.

Sparsity (4)

Storing a static sparse graph

Vertex IDs	0	1	2	3	4	
Offsets	0	2	5	5	6	7

Edges	1	3	2	3	4	2	2



Can run many graph algorithms efficiently on this representation, e.g., breadth-first search, PageRank

Can store edge weights with an additional array or interleaved with Edges

LOGIC

Constant Folding and Propagation

The idea of constant folding and propagation is to evaluate constant expressions and substitute the result into further expressions, all during compilation.

```
#include <math.h>

void orrery() {
  const double radius = 6371000.0;
  const double diameter = 2 * radius;
  const double circumference = M_PI * diameter;
  const double cross_area = M_PI * radius * radius;
  const double surface_area = circumference * diameter;
  const double volume = 4 * M_PI * radius * radius * radius / 3;
  // ...
}
```

With a sufficiently high optimization level, all the expressions are evaluated at compile-time.

Common-Subexpression Elimination

The idea of common-subexpression elimination is to avoid computing the same expression multiple times by evaluating the expression once and storing the result for later use.

```
a = b + c;
b = a - d;
c = b + c;
d = a - d;
```

```
a = b + c;
b = a - d;
c = b + c;
d = b;
```

The third line cannot be replaced by `c = a`, because the value of `b` changes in the second line.

Algebraic Identities

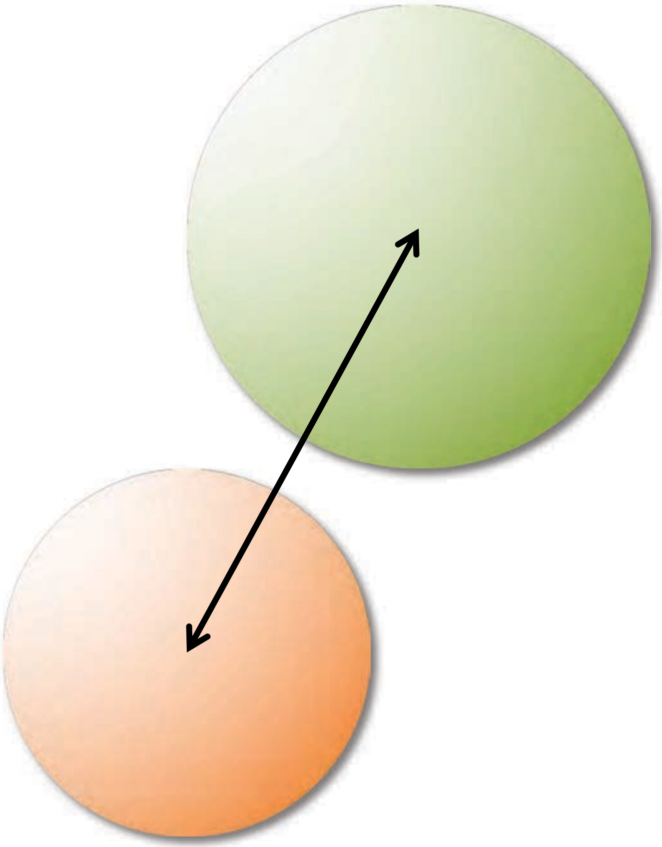
The idea of exploiting algebraic identities is to replace expensive algebraic expressions with algebraic equivalents that require less work.

```c
#include <stdbool.h>
#include <math.h>

typedef struct {
  double x;   // x-coordinate
  double y;   // y-coordinate
  double z;   // z-coordinate
  double r;   // radius of ball
} ball_t;

double square(double x) {
  return x*x;
}

bool collides(ball_t *b1, ball_t *b2) {
  double d = sqrt(square(b1->x - b2->x)
                + square(b1->y - b2->y)
                + square(b1->z - b2->z));
  return d <= b1->r + b2->r;
}
```



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} ball_t;

double square(double x) {
  return x*x;
}

bool collides(ball_t *b1, ball_t *b2) {
  double d = sqrt(square(b1->x - b2->x)
              + square(b1->y - b2->y)
              + square(b1->z - b2->z));
  return d <= b1->r + b2->r;
}
```

```c
bool collides(ball_t *b1, ball_t *b2) {
  double dsquared = square(b1->x - b2->x)
                  + square(b1->y - b2->y)
                  + square(b1->z - b2->z);
  return dsquared <= square(b1->r + b2->r);
}
```

$\sqrt{u} \leq v$ exactly when $u \leq v^2$.

Short-Circuiting

When performing a series of tests, the idea of short-circuiting is to stop evaluating as soon as you know the answer.

```
#include <stdbool.h>
// All elements of A are nonnegative
bool sum_exceeds(int *A, int n, int limit) {
  int sum = 0;
  for (int i = 0; i < n; i++) {
    sum += A[i];
  }
  return sum > limit;
}
```

```
#include <stdbool.h>
// All elements of A are nonnegative
bool sum_exceeds(int *A, int n, int limit) {
  int sum = 0;
  for (int i = 0; i < n; i++) {
    sum += A[i];
    if (sum > limit) {
      return true;
    }
  }
  return false;
}
```

Note that && and || are short-circuiting logical operators, and & and | are not.

Ordering Tests

Consider code that executes a sequence of logical tests. The idea of ordering tests is to perform those that are more often "successful" — a particular alternative is selected by the test — before tests that are rarely successful. Similarly, inexpensive tests should precede expensive ones.

```c
#include <stdbool.h>
bool is_whitespace(char c) {
  if (c == '\r' || c == '\t' || c == ' ' || c == '\n') {
    return true;
  }
  return false;
}
```

```c
#include <stdbool.h>
bool is_whitespace(char c) {
  if (c == ' ' || c == '\n' || c == '\t' || c == '\r') {
    return true;
  }
  return false;
}
```

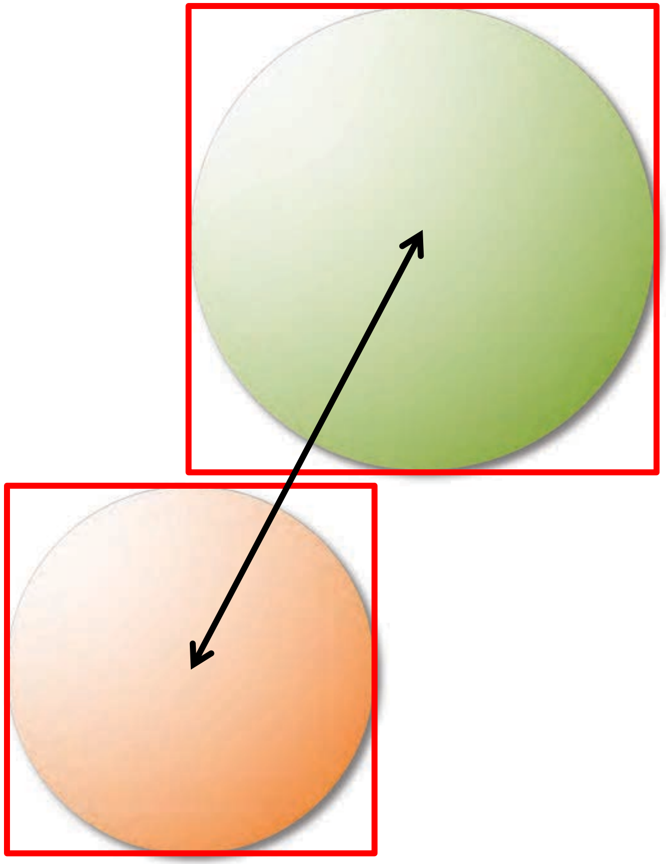
Creating a Fast Path

```c
#include <stdbool.h>
#include <math.h>

typedef struct {
  double x;   // x-coordinate
  double y;   // y-coordinate
  double z;   // z-coordinate
  double r;   // radius of ball
} ball_t;

double square(double x) {
  return x*x;
}

bool collides(ball_t *b1, ball_t *b2) {
  double dsquared = square(b1->x - b2->x)
                  + square(b1->y - b2->y)
                  + square(b1->z - b2->z);
  return dsquared <= square(b1->r + b2->r);
}
```



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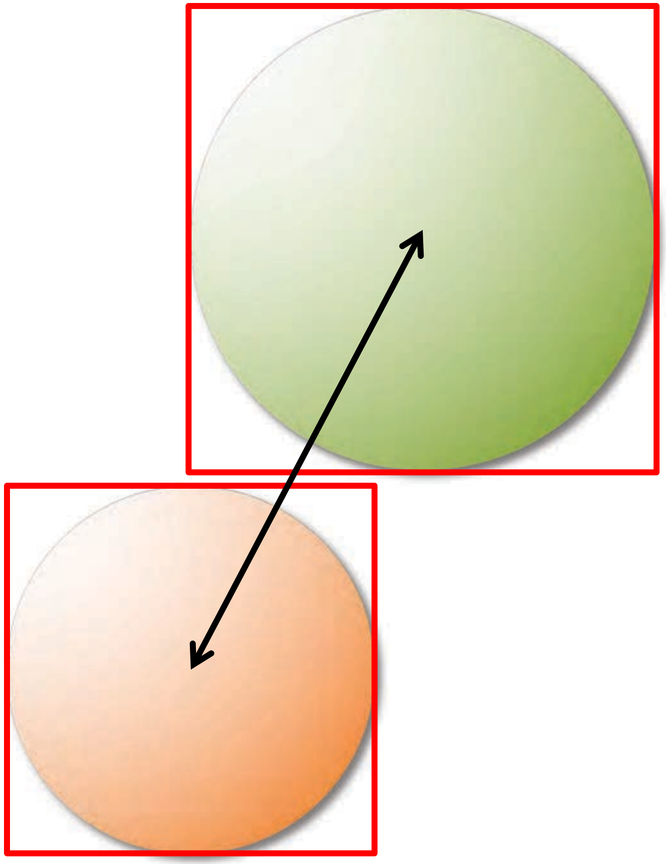
  double x;   // x-coordinate
  double y;   // y-coordinate
  double z;   // z-coordinate
  double r;   // radius of ball
} ball_t;

double square(double x) {
  return x*x;
}

bool collides(ball_t *b1, ball_t *b2) {
  if ((abs(b1->x – b2->x) > (b1->r + b2->r)) ||
      (abs(b1->y – b2->y) > (b1->r + b2->r)) ||
      (abs(b1->z – b2->z) > (b1->r + b2->r)))
    return false;
  double dsquared = square(b1->x - b2->x)
                  + square(b1->y - b2->y)
                  + square(b1->z - b2->z);
  return dsquared <= square(b1->r + b2->r);
}
```



Combining Tests

The idea of combining tests is to replace a sequence of tests with one test or switch.

Full adder

a	b	c	carry	sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

```c
void full_add (int a,
               int b,
               int c,
               int *sum,
               int *carry) {
  if (a == 0) {
    if (b == 0) {
      if (c == 0) {
        *sum = 0;
        *carry = 0;
      } else {
        *sum = 1;
        *carry = 0;
      }
    } else {
      if (c == 0) {
        *sum = 1;
        *carry = 0;
      } else {
        *sum = 0;
        *carry = 1;
      }
    }
  } else {
    if (b == 0) {
      if (c == 0) {
        *sum = 1;
        *carry = 0;
      } else {
        *sum = 0;
        *carry = 1;
      }
    } else {
      if (c == 0) {
        *sum = 0;
        *carry = 1;
      } else {
        *sum = 1;
        *carry = 1;
      }
    }
  }
}
```

Combining Tests (2)

The idea of combining tests is to replace a sequence of tests with one test or switch.

Full adder

a	b	c	carry	sum
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

For this example, table look-up is even better!

```
void full_add (int a,
               int b,
               int c,
               int *sum,
               int *carry) {
  int test = ((a == 1) << 2)
           | ((b == 1) << 1)
           | (c == 1);
  switch(test) {
    case 0:
      *sum = 0;
      *carry = 0;
      break;
    case 1:
      *sum = 1;
      *carry = 0;
      break;
    case 2:
      *sum = 1;
      *carry = 0;
      break;
    case 3:
      *sum = 0;
      *carry = 1;
      break;
    case 4:
      *sum = 1;
      *carry = 0;
      break;
    case 5:
      *sum = 0;
      *carry = 1;
      break;
    case 6:
      *sum = 0;
      *carry = 1;
      break;
    case 7:
      *sum = 1;
      *carry = 1;
      break;
  }
}
```

LOOPS

Hoisting

The goal of hoisting — also called loop-invariant code motion — is to avoid recomputing loop-invariant code each time through the body of a loop.

```
#include <math.h>

void scale(double *X, double *Y, int N) {
  for (int i = 0; i < N; i++) {
    Y[i] = X[i] * exp(sqrt(M_PI/2));
  }
}
```

```
#include <math.h>

void scale(double *X, double *Y, int N) {
  double factor = exp(sqrt(M_PI/2));
  for (int i = 0; i < N; i++) {
    Y[i] = X[i] * factor;
  }
}
```

Sentinels

Sentinels are special dummy values placed in a data structure to simplify the logic of boundary conditions, and in particular, the handling of loop-exit tests.

```
#include <stdint.h>
#include <stdbool.h>

bool overflow(int64_t *A, size_
// All elements of A are nonneg
  int64_t sum = 0;
  for ( size_t i = 0; i < n; ++
    sum += A[i];
    if ( sum < A[i] ) return tr
  }
  return false;
}
```

```
#include <stdint.h>
#include <stdbool.h>

// Assumes that A[n] and A[n+1] exist and
// can be clobbered
bool overflow(int64_t *A, size_t n) {
// All elements of A are nonnegative
  A[n] = INT64_MAX;
  A[n+1] = 1;   // or any positive number
  size_t i = 0;
  int64_t sum = A[0];
  while ( sum >= A[i] ) {
    sum += A[++i];
  }
  if (i < n) return true;
  return false;
}
```

Loop Unrolling

Loop unrolling attempts to save work by combining several consecutive iterations of a loop into a single iteration, thereby reducing the total number of iterations of the loop and, consequently, the number of times that the instructions that control the loop must be executed.

- Full loop unrolling: All iterations are unrolled.
- Partial loop unrolling: Several, but not all, of the iterations are unrolled.

Full Loop Unrolling

```
int sum = 0;
for (int i = 0; i < 10; i++) {
  sum += A[i];
}
```

```
int sum = 0;
sum += A[0];
sum += A[1];
sum += A[2];
sum += A[3];
sum += A[4];
sum += A[5];
sum += A[6];
sum += A[7];
sum += A[8];
sum += A[9];
```

Partial Loop Unrolling

```
int sum = 0;
for (int i = 0; i < n; ++i) {
  sum += A[i];
}
```

```
int sum = 0;
int j;
for (j = 0; j < n-3; j += 4) {
  sum += A[j];
  sum += A[j+1];
  sum += A[j+2];
  sum += A[j+3];
}
for (int i = j; i < n; ++i) {
  sum += A[i];
}
```

Benefits of loop unrolling

- Lower number of instructions in loop control code
- Enables more compiler optimizations

Unrolling too much can cause poor use of instruction cache

Loop Fusion

The idea of loop fusion — also called jamming — is to combine multiple loops over the same index range into a single loop body, thereby saving the overhead of loop control.

```
for (int i = 0; i < n; ++i) {
  C[i] = (A[i] <= B[i]) ? A[i] : B[i];
}

for (int i = 0; i < n; ++i) {
  D[i] = (A[i] <= B[i]) ? B[i] : A[i];
}
```

```
for (int i = 0; i < n; ++i) {
  C[i] = (A[i] <= B[i]) ? A[i] : B[i];
  D[i] = (A[i] <= B[i]) ? B[i] : A[i];
}
```

Eliminating Wasted Iterations

The idea of eliminating wasted iterations is to modify loop bounds to avoid executing loop iterations over essentially empty loop bodies.

```
for (int i = 0; i < n; ++i) {
  for (int j = 0; j < n; ++j) {
    if (i > j) {
      int temp = A[i][j];
      A[i][j] = A[j][i];
      A[j][i] = temp;
    }
  }
}
```

```
for (int i = 1; i < n; ++i) {
  for (int j = 0; j < i; ++j) {
      int temp = A[i][j];
      A[i][j] = A[j][i];
      A[j][i] = temp;
  }
}
```

FUNCTIONS

Inlining

The idea of inlining is to avoid the overhead of a function call by replacing a call to the function with the body of the function itself.

```
double square(double x) {
  return x*x;
}

double sum_of_squares(double *A, int n) {
  double sum = 0.0;
  for (int i = 0; i < n; ++i) {
    sum += square(A[i]);
  }
  return sum;
}
```

```
double sum_of_squares(double *A, int n) {
  double sum = 0.0;
  for (int i = 0; i < n; ++i) {
    double temp = A[i];
    sum += temp*temp;
  }
  return sum;
}
```

Inlining (2)

The idea of inlining is to avoid the overhead of a function call by replacing a call to the function with the body of the function itself.

```
double square(double x) {
  return x*x;
}

double sum_of_squares(double *A, int n) {
  double sum = 0.0;
  for (int i = 0; i < n; ++i) {
    sum += square(A[i]);
  }
  return sum;
}
```

```
static inline double square(double x) {
  return x*x;
}

double sum_of_squares(double *A, int n) {
  double sum = 0.0;
  for (int i = 0; i < n; ++i) {
    sum += square(A[i]);
  }
  return sum;
}
```

Inlined functions can be just as efficient as macros, and they are better structured.

Tail-Recursion Elimination

The idea of tail-recursion elimination is to replace a recursive call that occurs as the last step of a function with a branch, saving function-call overhead.

```
void quicksort(int *A, int n) {
  if (n > 1) {
    int r = partition(A, n);
    quicksort (A, r);
    quicksort (A + r + 1, n - r - 1);
  }
}
```

```
void quicksort(int *A, int n) {
  while (n > 1) {
    int r = partition(A, n);
    quicksort (A, r);
    A += r + 1;
    n -= r + 1;
  }
}
```

Coarsening Recursion

The idea of coarsening recursion is to increase the size of the base case and handle it with more efficient code that avoids function-call overhead.

```
void quicksort(int *A, int n) {
  while (n > 1) {
    int r = partition(A, n);
    quicksort (A, r);
    A += r + 1;
    n -= r + 1;
  }
}
```

```
#define THRESHOLD 10
void quicksort(int *A, int n) {
  while (n > THRESHOLD) {
    int r = partition(A, n);
    quicksort (A, r);
    A += r + 1;
    n -= r + 1;
  }
  // insertion sort for small arrays
  for (int j = 1; j < n; ++j) {
    int key = A[j];
    int i = j - 1;
    while (i >= 0 && A[i] > key) {
      A[i+1] = A[i];
      --i;
    }
    A[i+1] = key;
  }
}
```

SUMMARY

New Bentley Rules

Data structures

- Packing and encoding
- Augmentation
- Precomputation
- Compile-time initialization
- Caching
- Lazy evaluation
- Sparsity

Loops

- Hoisting
- Sentinels
- Loop unrolling
- Loop fusion
- Eliminating wasted iterations

Logic

- Constant folding and propagation
- Common-subexpression elimination
- Algebraic identities
- Short-circuiting
- Ordering tests
- Creating a fast path
- Combining tests

Functions

- Inlining
- Tail-recursion elimination
- Coarsening recursion

Closing Advice

- Avoid premature optimization. First get correct working code. Then optimize, preserving correctness by regression testing.
- Reducing the work of a program does not necessarily decrease its running time, but it is a good heuristic.
- The compiler automates many low-level optimizations.
- To tell if the compiler is actually performing a particular optimization, look at the assembly code.

If you find interesting examples of work optimization, please let us know!