The RLC Circuit. Transient Response

Series *RLC* circuit

The circuit shown on Figure 1 is called the series *RLC* circuit. We will analyze this circuit in order to determine its transient characteristics once the switch *S* is closed.

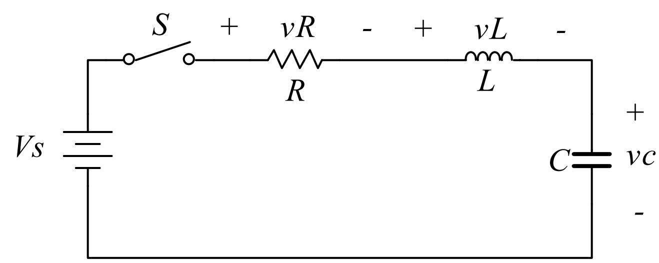


Figure 1

The equation that describes the response of the system is obtained by applying KVL around the mesh

$$vR + vL + vc = Vs \tag{1.1}$$

The current flowing in the circuit is

$$i = C\frac{dvc}{dt} \tag{1.2}$$

And thus the voltages *vR* and *vL* are given by

$$vR = iR = RC\frac{dvc}{dt} \tag{1.3}$$

$$vL = L\frac{di}{dt} = LC\frac{d^2vc}{dt^2} \tag{1.4}$$

Substituting Equations (1.3) and (1.4) into Equation (1.1) we obtain

$$\frac{d^2vc}{dt^2} + \frac{R}{L}\frac{dvc}{dt} + \frac{1}{LC}vc = \frac{1}{LC}Vs \tag{1.5}$$

The solution to equation (1.5) is the linear combination of the homogeneous and the particular solution $vc = vc_p + vc_h$

The particular solution is

$$vc_p = Vs \tag{1.6}$$

And the homogeneous solution satisfies the equation

$$\frac{d^2 vc_h}{dt^2} + \frac{R}{L}\frac{dvc_h}{dt} + \frac{1}{LC} vc_h = 0 \tag{1.7}$$

Assuming a homogeneous solution is of the form Ae^{st} and by substituting into Equation (1.7) we obtain the characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{1.8}$$

By defining

$$\alpha = \frac{R}{2L}: \quad \text{Damping rate} \tag{1.9}$$

And

$$\omega_o = \frac{1}{\sqrt{LC}}: \quad \text{Natural frequency} \tag{1.10}$$

The characteristic equation becomes

$$s^2 + 2\alpha s + \omega_o^2 = 0 \tag{1.11}$$

The roots of the characteristic equation are

$$s1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \tag{1.12}$$

$$s2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \tag{1.13}$$

And the homogeneous solution becomes

$$vc_h = A_1 e^{s1t} + A_2 e^{s2t} \tag{1.14}$$

The total solution now becomes

$$vc = Vs + A_1 e^{s1t} + A_2 e^{s2t} \tag{1.15}$$

The parameters A1 and A2 are constants and can be determined by the application of the initial conditions of the system $vc(t=0)$ and $\frac{dvc(t=0)}{dt}$.

The value of the term $\sqrt{\alpha^2-\omega_o^2}$ determines the behavior of the response. Three types of responses are possible:

1. $\alpha=\omega_o$ then $s1$ and $s2$ are equal and real numbers: no oscillatory behavior
   Critically Damped System

2. $\alpha>\omega_o$. Here $s1$ and $s2$ are real numbers but are unequal: no oscillatory behavior
   Over Damped System
   $vc=Vs+A_1e^{s1t}+A_2e^{s2t}$

3. $\alpha<\omega_o$. $\sqrt{\alpha^2-\omega_o^2}=j\sqrt{\omega_o^2-\alpha^2}$ In this case the roots s1 and s2 are complex numbers: $s1=-\alpha+j\sqrt{\omega_o^2-\alpha^2}$, $s2=-\alpha-j\sqrt{\omega_o^2-\alpha^2}$. System exhibits oscillatory behavior
   Under Damped System

Important observations for the series RLC circuit.

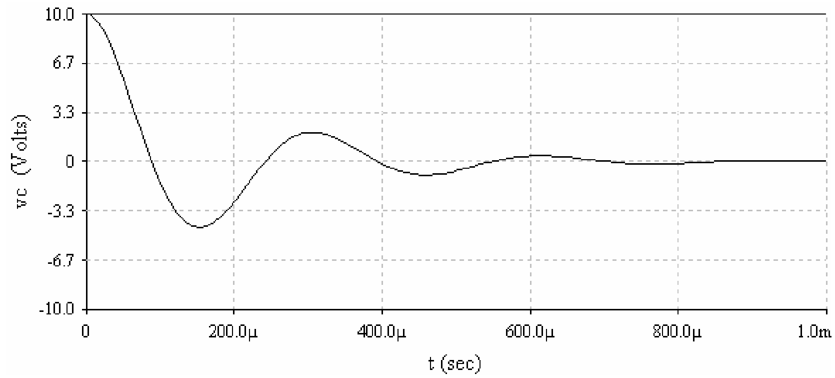
- As the resistance increases the value of α increases and the system is driven towards an over damped response.

- The frequency $\omega_o=\frac{1}{\sqrt{LC}}$ (rad/sec) is called the natural frequency of the system or the resonant frequency.

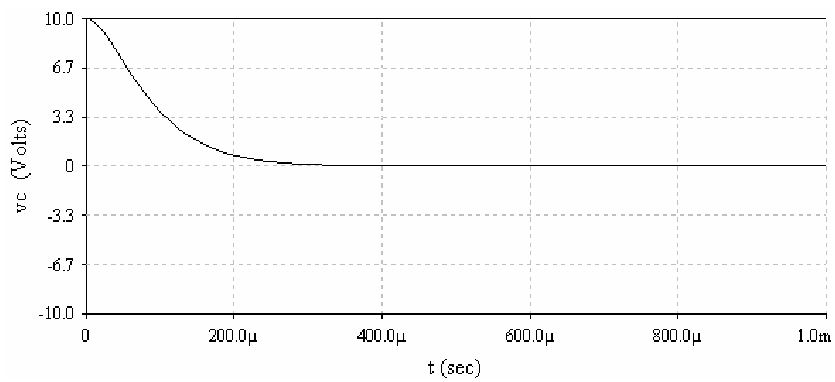
- The parameter $\alpha=\frac{R}{2L}$ is called the damping rate and its value in relation to ω_o determines the behavior of the response
  - $\alpha=\omega_o$: Critically Damped
  - $\alpha>\omega_o$: Over Damped
  - $\alpha<\omega_o$: Under Damped

- The quantity $\sqrt{\frac{L}{C}}$ has units of resistance

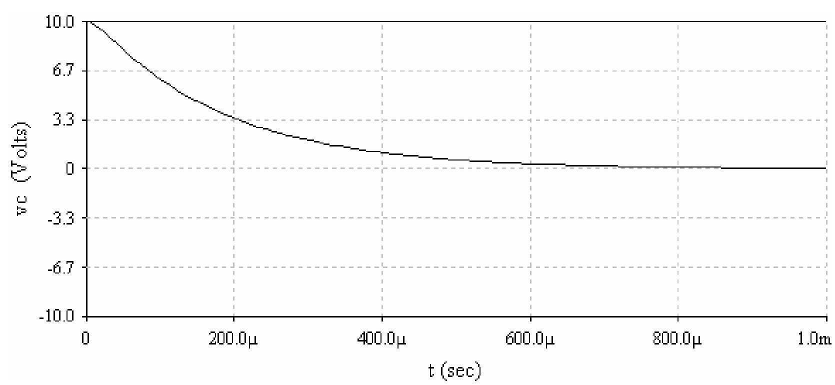
Figure 2 shows the response of the series RLC circuit with L=47mH, C=47nF and for three different values of R corresponding to the under damped, critically damped and over damped case. We will construct this circuit in the laboratory and examine its behavior in more detail.



(a) Under Damped. R=500Ω



(b) Critically Damped. R=2000 Ω



(c) Over Damped. R=4000 Ω

Figure 2

The *LC* circuit.

In the limit $R \to 0$ the *RLC* circuit reduces to the lossless *LC* circuit shown on Figure 3.

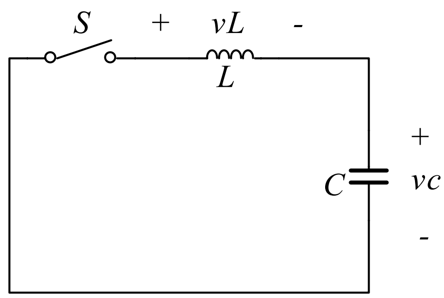


Figure 3

The equation that describes the response of this circuit is

$$\frac{d^2vc}{dt^2} + \frac{1}{LC}vc = 0 \tag{1.16}$$

Assuming a solution of the form Ae^{st} the characteristic equation is

$$s^2 + \omega_o^2 = 0 \tag{1.17}$$

Where $\omega_o = \dfrac{1}{\sqrt{LC}}$

The two roots are

$$s1 = +j\omega_o \tag{1.18}$$

$$s2 = -j\omega_o \tag{1.19}$$

And the solution is a linear combination of $A1e^{s1t}$ and $A2e^{s2t}$

$$vc(t) = A1e^{j\omega_o t} + A2e^{-j\omega_o t} \tag{1.20}$$

By using Euler's relation Equation (1.20) may also be written as

$$vc(t) = B1\cos(\omega_o t) + B2\sin(\omega_o t) \tag{1.21}$$

The constants *A1, A2* or *B1*, *B2* are determined from the initial conditions of the system.

For $vc(t=0)=Vo$ and for $\frac{dvc(t=0)}{dt}=0$ (no current flowing in the circuit initially) we have from Equation (1.20)

$$A1+A2=Vo \tag{1.22}$$

And

$$j\omega_o A1 - j\omega_o A2 = 0 \tag{1.23}$$

Which give

$$A1 = A2 = \frac{Vo}{2} \tag{1.24}$$

And the solution becomes

$$\begin{aligned} vc(t) &= \frac{Vo}{2}\left(e^{j\omega_o t}+e^{-j\omega_o t}\right) \\ &= Vo\cos(\omega_o t) \end{aligned} \tag{1.25}$$

The current flowing in the circuit is

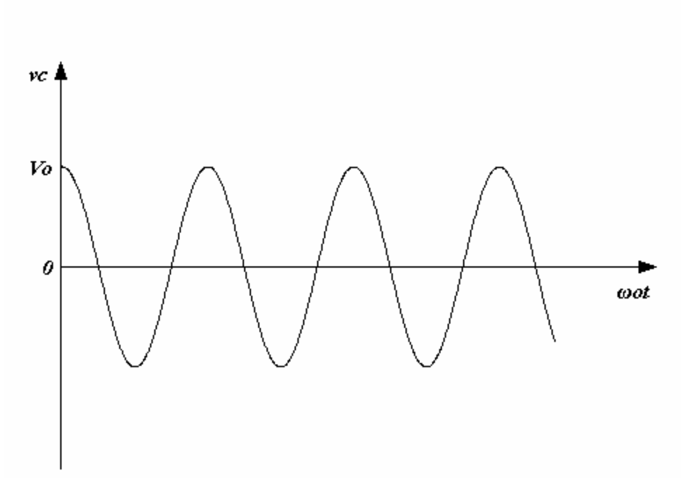
$$\begin{aligned} i &= C\frac{dvc}{dt} \\ &= -CVo\omega_o\sin(\omega_o t) \end{aligned} \tag{1.26}$$

And the voltage across the inductor is easily determined from KVL or from the element relation of the inductor $vL = L\frac{di}{dt}$

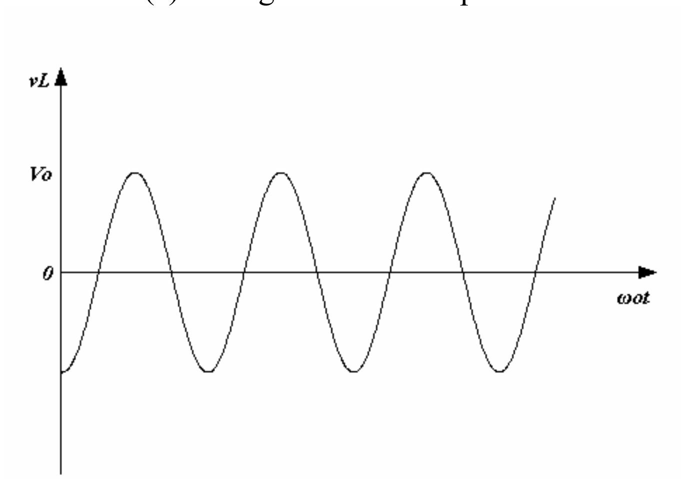
$$\begin{aligned} vL &= -vc \\ &= -Vo\cos(\omega_o t) \end{aligned} \tag{1.27}$$

Figure 4 shows the plots of $vc(t),\ vL(t),\ and\ i(t)$. Note the 180 degree phase difference between *vc(t)* and *vL(t)* and the 90 degree phase difference between *vL(t)* and *i(t)*.

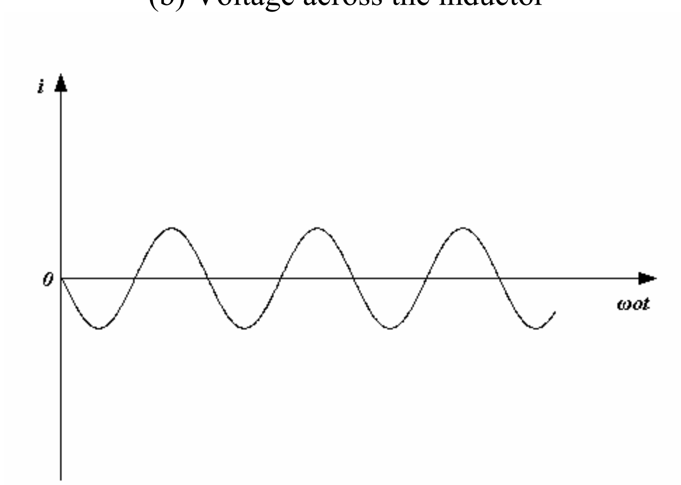
Figure 5 shows a plot of the energy in the capacitor and the inductor as a function of time. Note that the energy is exchanged between the capacitor and the inductor in this lossless system



(a) Voltage across the capacitor

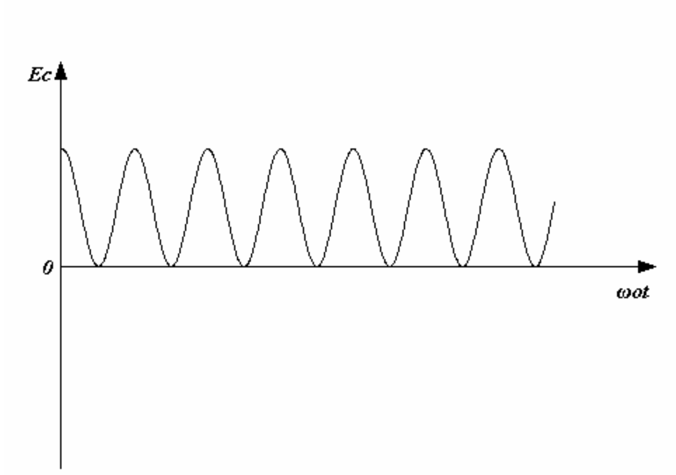


(b) Voltage across the inductor

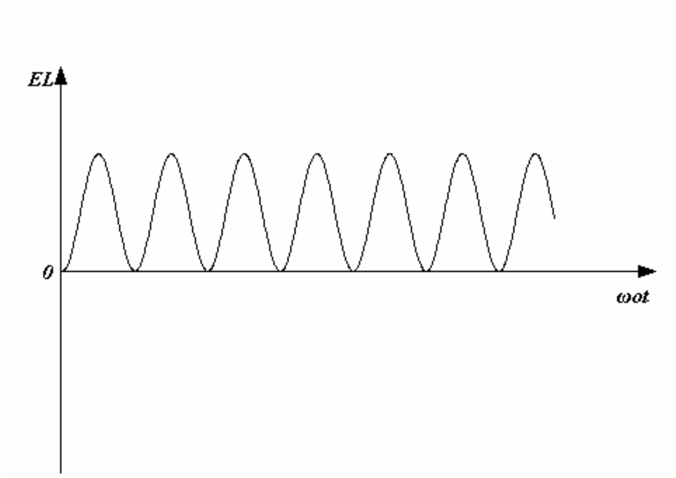


(c) Current flowing in the circuit

Figure 4



(a) Energy stored in the capacitor



(b) Energy stored in the inductor

Figure 5

Parallel *RLC* Circuit

The *RLC* circuit shown on Figure 6 is called the parallel *RLC* circuit. It is driven by the DC current source *Is* whose time evolution is shown on Figure 7.

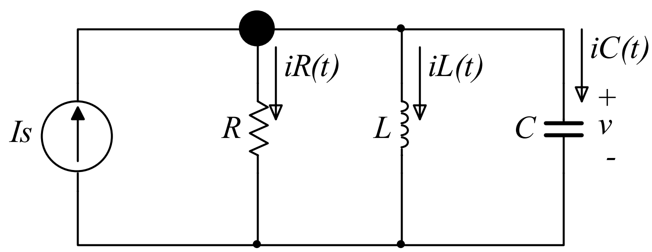


Figure 6

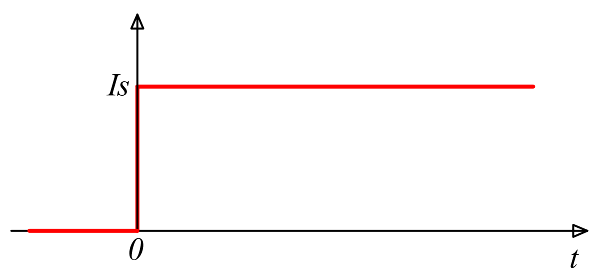


Figure 7

Our goal is to determine the current *iL(t)* and the voltage *v(t)* for *t>0*.

We proceed as follows:

1. Establish the initial conditions for the system
2. Determine the equation that describes the system characteristics
3. Solve the equation
4. Distinguish the operating characteristics as a function of the circuit element parameters.

Since the current *Is* was zero prior to *t=0* the initial conditions are:

$$\text{Initial Conditions:}\begin{cases} iL(t=0)=0 \\ v(t=0)=0 \end{cases} \tag{1.28}$$

By applying KCl at the indicated node we obtain

$$Is = iR + iL + iC \tag{1.29}$$

The voltage across the elements is given by

$$v = L\frac{d\,iL}{dt} \tag{1.30}$$

And the currents *iR* and *iC* are

$$iR = \frac{v}{R} = \frac{L}{R}\frac{d\,iL}{dt} \tag{1.31}$$

$$iC = C\frac{dv}{dt} = LC\frac{d^2 iL}{dt^2} \tag{1.32}$$

Combining Equations (1.29), (1.31), and (1.32) we obtain

$$\frac{d^2 iL}{dt^2} + \frac{1}{RC}\frac{d\,iL}{dt} + \frac{1}{LC}iL = \frac{1}{LC}Is \tag{1.33}$$

The solution to equation (1.33) is a superposition of the particular and the homogeneous solutions.

$$iL(t) = iL_p(t) + iL_h(t) \tag{1.34}$$

The particular solution is

$$iL_p(t) = Is \tag{1.35}$$

The homogeneous solution satisfies the equation

$$\frac{d^2 iL_h}{dt^2} + \frac{1}{RC}\frac{d\,iL_h}{dt} + \frac{1}{LC}iL_h = 0 \tag{1.36}$$

By assuming a solution of the form Ae^{st} we obtain the characteristic equation

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \tag{1.37}$$

Be defining the following parameters

$$\omega_o \equiv \frac{1}{\sqrt{LC}}: \text{ Resonant frequency} \tag{1.38}$$

And

$$\alpha = \frac{1}{2RC}: \text{ Damping rate} \tag{1.39}$$

The characteristic equation becomes

$$s^2 + 2\alpha\, s + \omega_o^2 = 0 \tag{1.40}$$

The two roots of this equation are

$$s1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \tag{1.41}$$

$$s2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2} \tag{1.42}$$

The homogeneous solution is a linear combination of e^{s1t} and e^{s2t}

$$iL_h(t) = A_1 e^{s1t} + A_2 e^{s2t} \tag{1.43}$$

And the general solution becomes

$$iL(t) = Is + A_1 e^{s1t} + A_2 e^{s2t} \tag{1.44}$$

The constants A_1 and A_2 may be determined by using the initial conditions.

Let's now proceed by looking at the physical significance of the parameters α and ω_o.

The form of the roots s1 and s2 depend on the values of α and ω_o. The following three cases are possible.

1. $\alpha = \omega_o$: **Critically Damped System.**
   s1 and *s2* are equal and real numbers: no oscillatory behavior

2. $\alpha > \omega_o$: **Over Damped System**
   Here *s1* and *s2* are real numbers but are unequal: no oscillatory behavior

3. $\alpha < \omega_o$: **Under Damped System**
   $\sqrt{\alpha^2 - \omega_o^2} = j\sqrt{\omega_o^2 - \alpha^2}$ In this case the roots s1 and s2 are complex numbers:
   $s1 = -\alpha + j\sqrt{\omega_o^2 - \alpha^2}$, $s2 = -\alpha - j\sqrt{\omega_o^2 - \alpha^2}$. System exhibits oscillatory behavior

Let's investigate the under damped case, $\alpha < \omega_o$, in more detail.

For $\alpha < \omega_o$, $\sqrt{\alpha^2 - \omega_o^2} = j\sqrt{\omega_o^2 - \alpha^2} \equiv j\omega_d$ the solution is

$$iL(t) = Is + \underbrace{e^{-\alpha t}}_{Decaying} \underbrace{\left(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}\right)}_{Oscillatory} \tag{1.45}$$

By using Euler's identity $e^{\pm j\omega_d t} = \cos\omega_d t \pm j\sin\omega_d t$, the solution becomes

$$iL(t) = Is + \underbrace{e^{-\alpha t}}_{Decaying} \underbrace{\left(K_1 \cos\omega_d t + K_2 \sin\omega_d t\right)}_{Oscillatory} \tag{1.46}$$

Now we can determine the constants K_1 and K_2 by applying the initial conditions

$$\begin{aligned} iL(t=0) = 0 &\Rightarrow Is + K_1 = 0 \\ &\Rightarrow \boxed{K_1 = -Is} \end{aligned} \tag{1.47}$$

$$\begin{aligned} \left.\frac{diL}{dt}\right|_{t=0} = 0 &\Rightarrow -\alpha K_1 + (0 + K_2\omega_d) = 0 \\ &\Rightarrow \boxed{K_2 = \frac{-\alpha}{\omega_d} Is} \end{aligned} \tag{1.48}$$

And the solution is

$$iL(t) = Is\left[1 - \underbrace{e^{-\alpha t}}_{Decaying}\underbrace{\left(\cos\omega_d t + \frac{\alpha}{\omega_d}\sin\omega_d t\right)}_{Oscillatory}\right] \tag{1.49}$$

By using the trigonometric identity $B_1\cos t + B_2\sin t = \sqrt{B_1^2 + B_2^2}\cos\left(t - \tan^{-1}\frac{B_2}{B_1}\right)$ the solution becomes

$$iL(t) = Is - Is\,\frac{\omega_o}{\omega_d}\,e^{-\alpha t}\cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right) \tag{1.50}$$

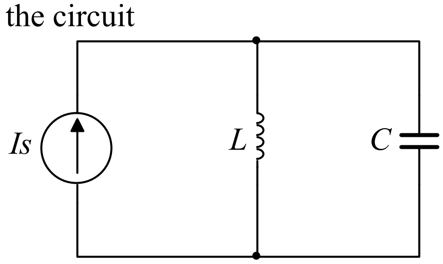
Recall that $\omega_d \equiv \sqrt{\omega_o^2 - \alpha^2}$ and thus ω_d is always smaller than ω_o

Let's now investigate the important limiting case:

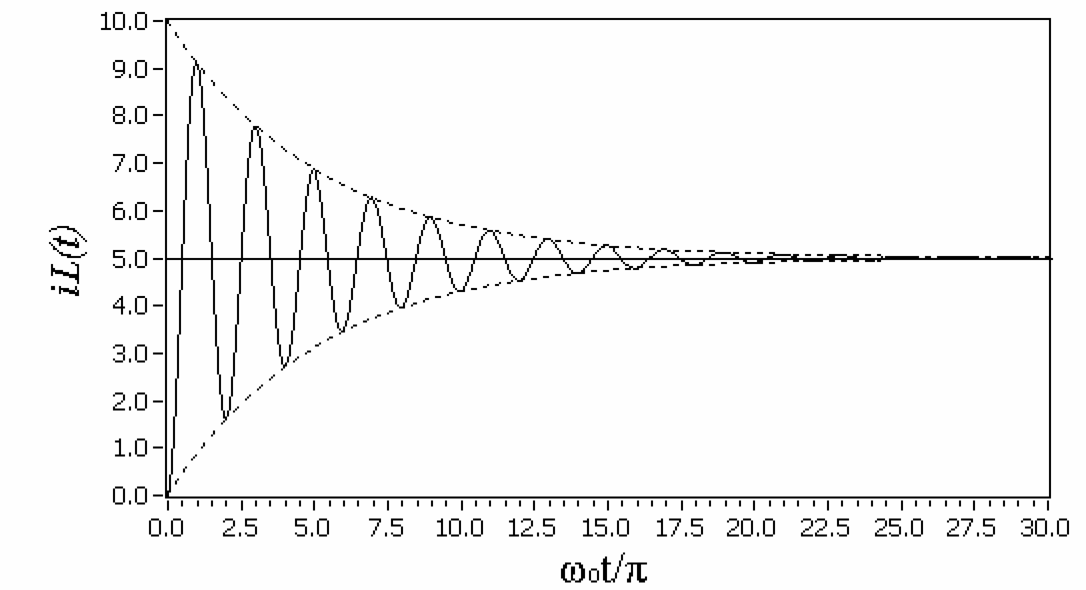
As $R \to \infty$, $\alpha << \omega_0$

$\omega_d \equiv \sqrt{\omega_o^2 - \alpha^2} \approx \omega_o$ and $\tan^{-1}\frac{\alpha}{\omega_o} \approx 0$, $e^{-\alpha t} \approx 1$

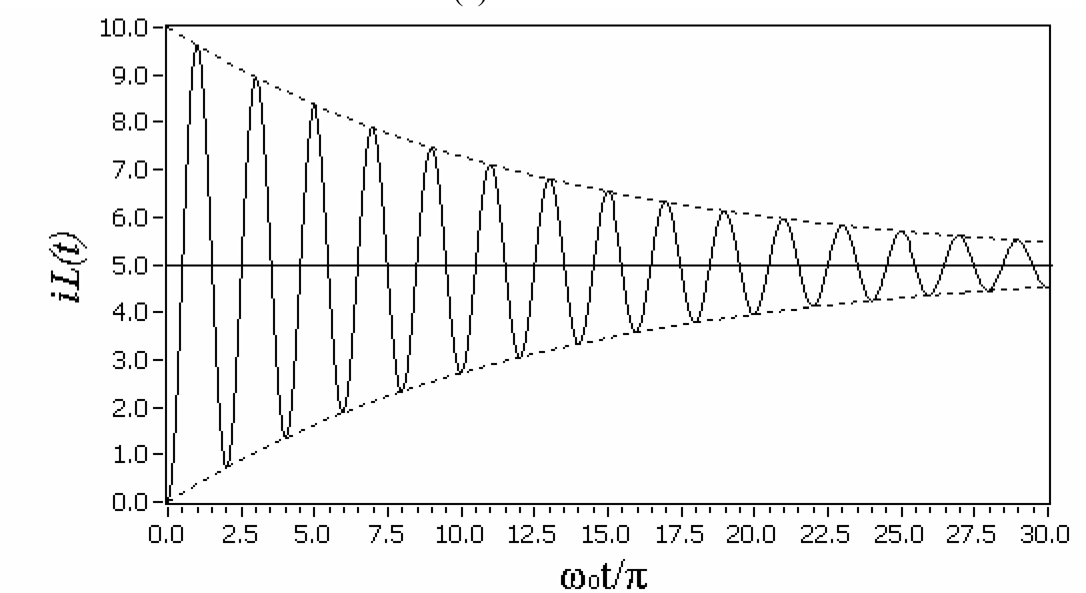
And the solution reduces to $iL(t) = Is - Is\cos\omega_o t$ which corresponds to the response of the circuit



The plot of *iL(t)* is shown on Figure 8 for *C=47nF, L=47mH, Is=5A* and for *R=20kΩ* and *8kΩ,* The dotted lines indicate the decaying characteristics of the response. For convenience and easy visualization the plot is presented in the normalized time $\omega_o t/\pi$. Note that the peak current through the inductor is greater than the supply current *Is*.

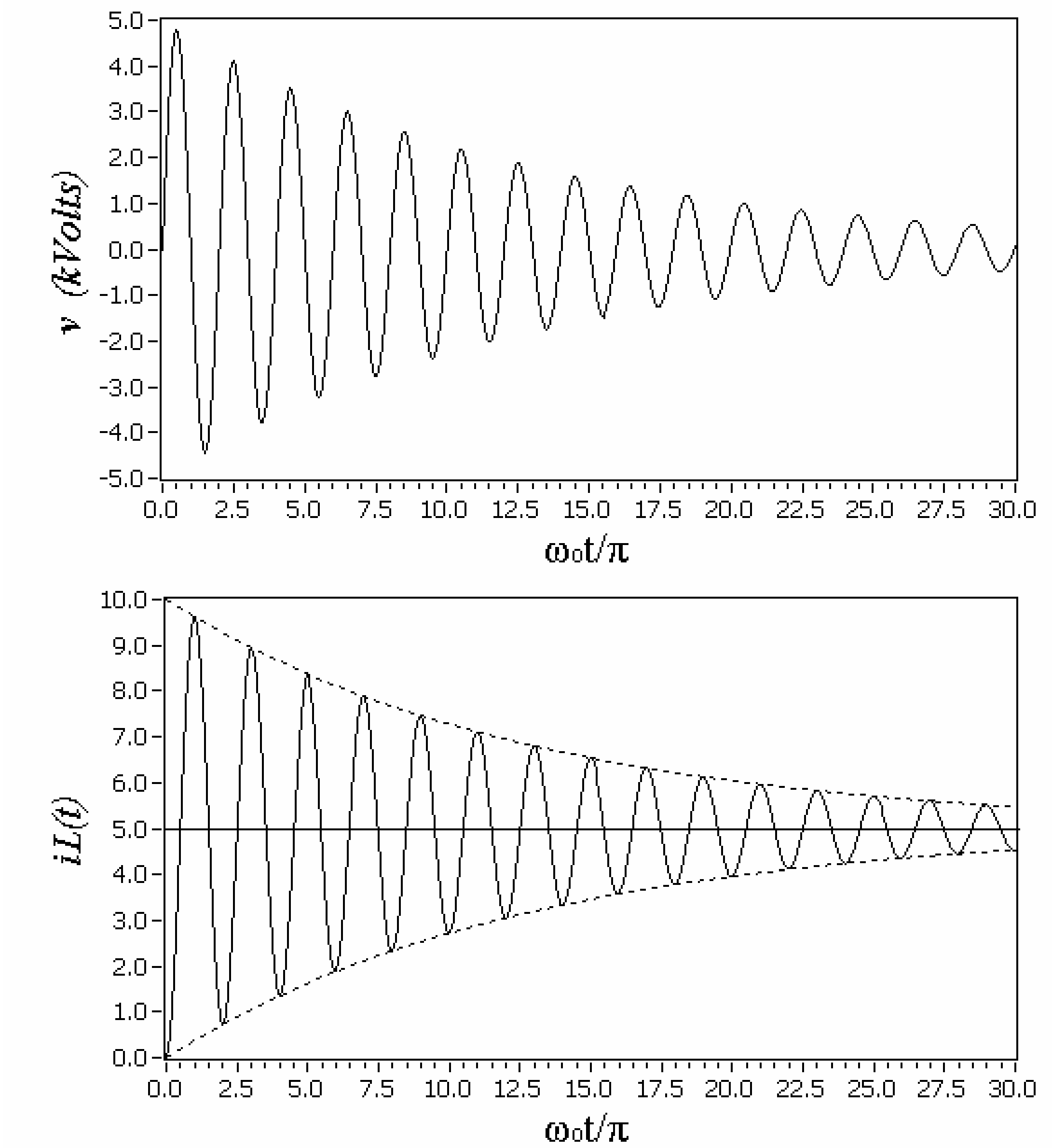


(a) For *R=8kΩ*

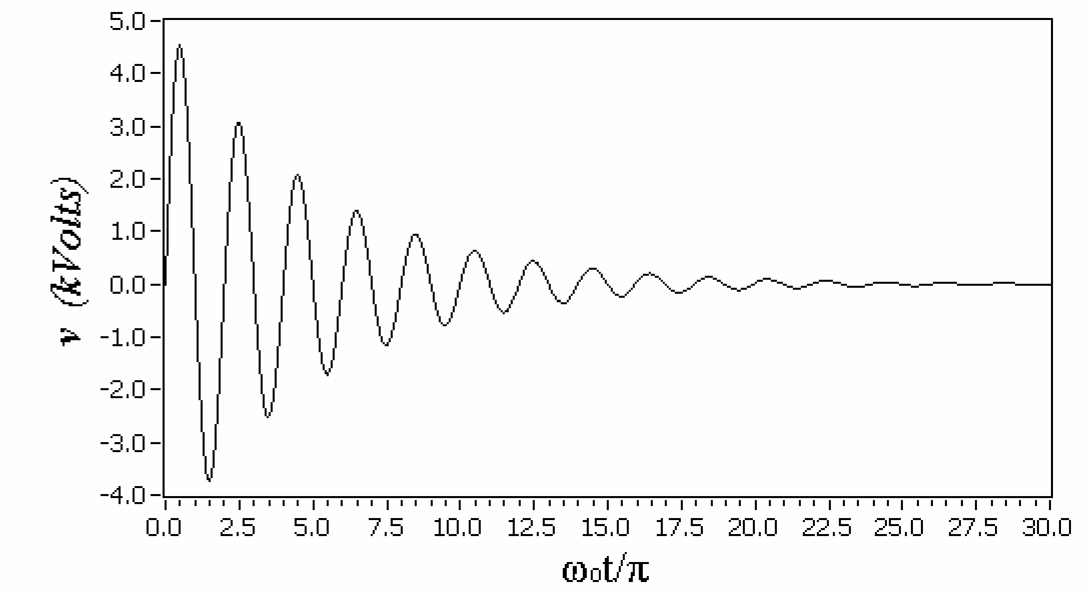


(b) For *R=20kΩ*

Figure 8.


v (kVolts)
5.0
4.0
3.0
2.0
1.0
0.0
-1.0
-2.0
-3.0
-4.0
-5.0
0.0 2.5 5.0 7.5 10.0 12.5 15.0 17.5 20.0 22.5 25.0 27.5 30.0
ω0t/π
iL(t)
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
0.0 2.5 5.0 7.5 10.0 12.5 15.0 17.5 20.0 22.5 25.0 27.5 30.0
ω0t/π

(a) $R=20k\Omega$



(b) *R=8kΩ*
Figure 9

The energy stored in the inductor and the capacitor is shown on Figure 10.

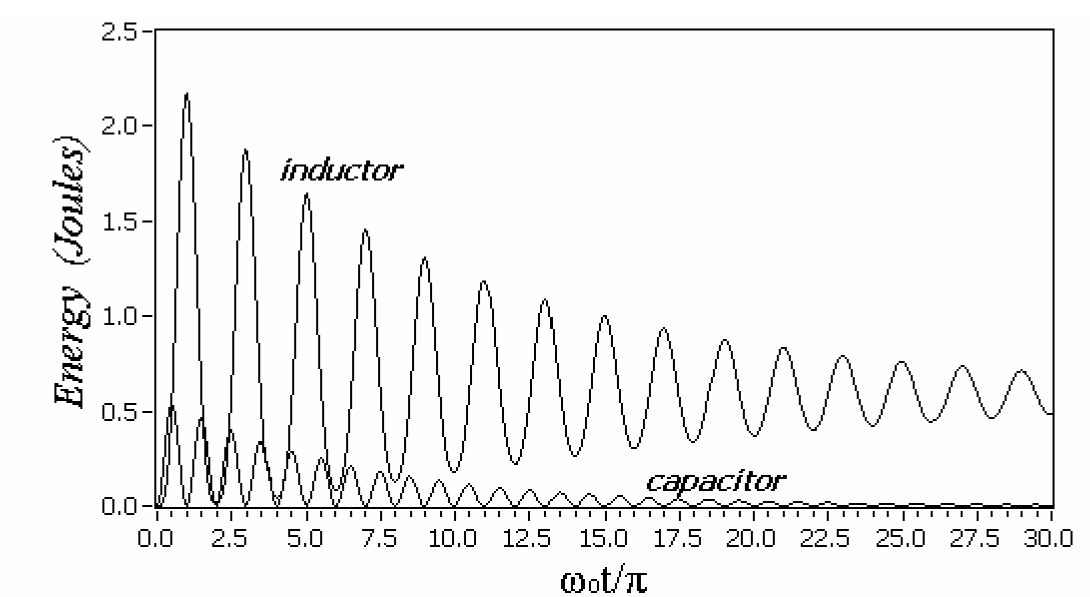


Figure 10. Energy as a function of time

Figure 11 shows the plot of the response corresponding to the case where $\alpha << \omega_0$. This shows the persistent oscillation for the current *iL(t)* with frequency ω_0.

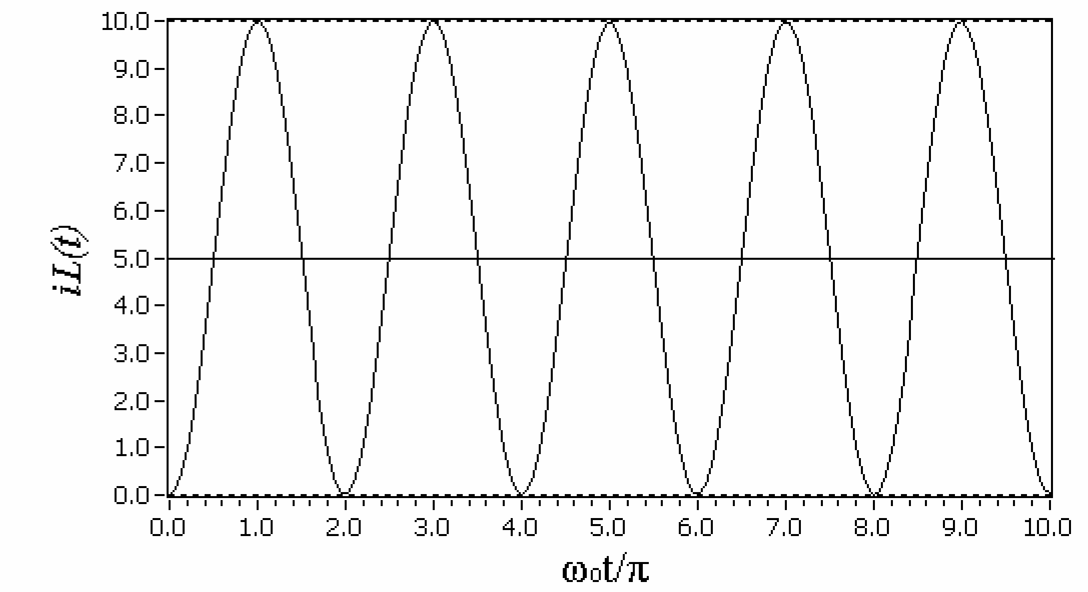

10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0.0
iL(t)
0.0 1.0 2.0 3.0 4.0 5.0 6.0 7.0 8.0 9.0 10.0
ω0t/π

Figure 11

The Critically Damped Response.

When $\alpha = \omega_o$ the two roots of the characteristic equation are equal *s1=s2=s.* And our assumed solution becomes

$$\begin{aligned} iL(t) &= A_1 e^{st} + A_2 e^{st} \\ &= A_3 e^{st} \end{aligned} \tag{1.51}$$

Now we have only one arbitrary constant. This is a problem for our second order system since our two initial conditions can not be satisfied.
The problem stems from an incorrect assumption for the solution for this special case. For $\alpha = \omega_o$ the differential equation of the homogeneous problem becomes

$$\frac{d^2 iL_h}{dt^2} + 2\alpha \frac{d\, iL_h}{dt} + \alpha^2 iL_h = 0 \tag{1.52}$$

The solution of this equation is[1]

$$iL(t) = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t} \tag{1.53}$$

Which is a linear combination of the exponential term and an exponential term multiplied by t.

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[1] The equation $\frac{d^2 i}{dt^2} + 2\alpha \frac{d\, i}{dt} + \alpha^2 i = 0$ may be rewritten as $\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0$, by defining $\xi = \frac{di}{dt} + \alpha i$ the equation becomes $\frac{d\xi}{dt} + \alpha\xi = 0$ whose solution is $\xi = K_1 e^{-\alpha t}$. Therefore $e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = K_1$ which may be written as $\frac{d}{dt}(e^{\alpha t} i) = K_1$. By integration we obtain the solution $i = K_1 t e^{-\alpha t} + K_2 e^{-\alpha t}$

Summary of RLC transient response

	Series	Parallel
ω_o	$\omega_o = \dfrac{1}{\sqrt{LC}}$	$\omega_o = \dfrac{1}{\sqrt{LC}}$
α	$\alpha = \dfrac{R}{2L}$	$\alpha = \dfrac{1}{2RC}$
Critically Damped	$\alpha = \omega_o$ Response: $A_1 t e^{-\alpha t} + A_2 e^{-\alpha t}$	
Under Damped	$\alpha < \omega_o$ Response: $\underbrace{e^{-\alpha t}}_{Decaying} \underbrace{\left(K_1 \cos \omega_d t + K_2 \sin \omega_d t\right)}_{Oscillatory}$ Where $\omega_d \equiv \sqrt{\omega_o^2 - \alpha^2}$	
Over Damped	$\alpha > \omega_o$ Response: $A_1 e^{s1t} + A_2 e^{s2t}$ Where $s1,2 = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2}$	

Problem

For the circuit below, the switch *S1* has been closed for a long time while switch *S2* is open. Now switch *S1* is opened and then at time *t=0* switch *S2* is closed.
Determine the current *i(t)* as indicated.