Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.061/6.690 Introduction to Power Systems

Problem Set 3 Solutions February 14, 2011

Problem 1: Voltage across the resistance is given by a voltage divider:

$$V_R = V_S \frac{R}{R + j\omega L + \frac{1}{j\omega C}}$$

The capacitance to maximize voltage across the resistance is the one that balances (cancels) inductor impedance.

This is:

$$C = \frac{1}{\omega X} \approx 265\mu F$$

The phasor diagram for voltages is, at resonance, shown in Figure 1.

The magnitude of the receiving end voltage is plotted in Figure 2

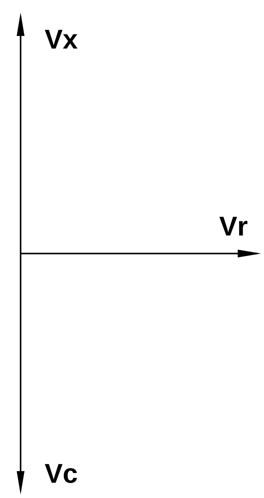

Figure 1: Voltage Phasors at Maximum Output Voltage

Finally, to get the sending end real and reactive power, see that current OUT of the source is

$$\underline{I}_s = V_s \left(\frac{1}{R + jX + \frac{1}{j\omega C}} + j\omega C \right)$$

Real and reactive power are simply $P + jQ = V_s I_s^*$, and these are plotted in Figure 3.

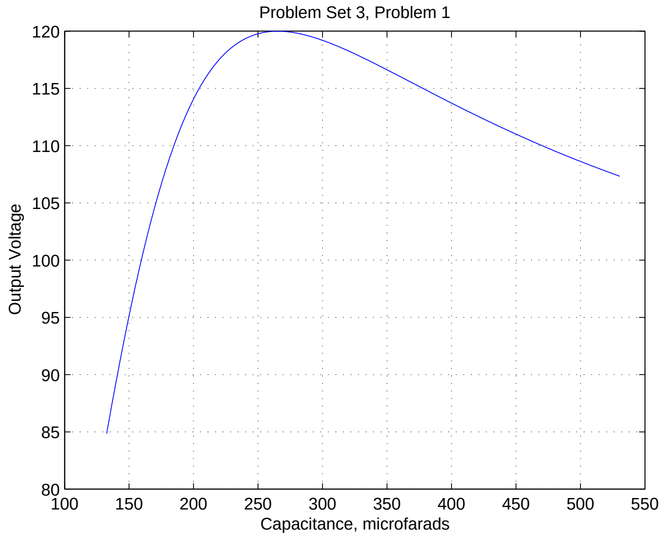

Figure 2: Voltage vs. Compensating Capacitor Value

Problem 2: Inductive reactance is $X = 2\pi \times 60 \times .02 \approx 7.54\Omega$, so receiving end voltage is

$$V_r = V_s \frac{R}{R + jX} = V_s \frac{R^2 - jXR}{R^2 + X^2} \approx 76.6 - j57.7\text{V}$$

A phasor diagram of this case is shown in Figure 4.

With the capacitor in place, the ratio of input to output voltages is:

$$V_r = V_s \frac{R || \frac{1}{j\omega C}}{R || \frac{1}{j\omega C} + j\omega L} = V_s \frac{1}{1 - \omega^2 LC + \frac{j\omega L}{R}}$$

To make the magnitude of output voltage equal to input voltage, it is necessary that:

$$\left(1 - \omega^2 LC\right)^2 + \left(\frac{\omega L}{R}\right)^2 = 1$$

Or noting $X = \omega L$ and $Y = \omega C$

$$(XY)^2 - 2XY + \left(\frac{X}{R}\right)^2 = 0$$

This is easily solved by:

$$Y = \frac{1}{X} \pm \sqrt{\left(\frac{1}{X}\right)^2 - \frac{1}{R^2}}$$

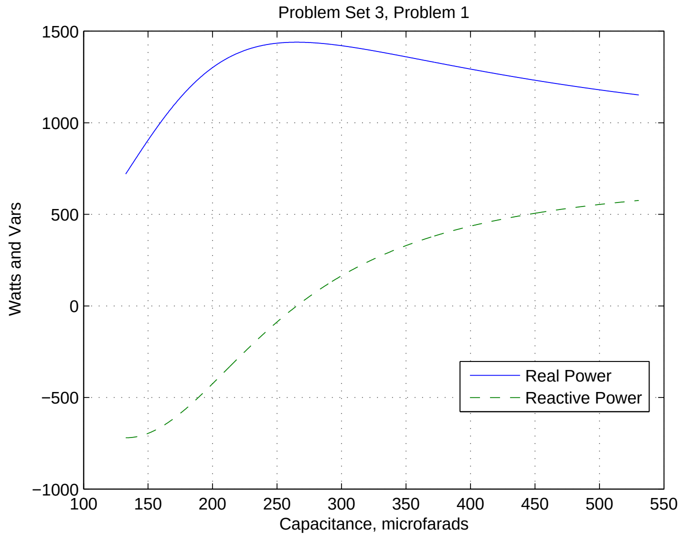

Figure 3: Real and Reactive Power Vs. Capacitor Value

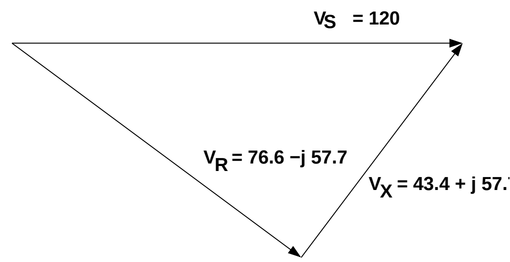

Figure 4: Phasor Diagram: Uncompensated

With $X = 7.54\Omega$ and $R = 10\Omega$, this evaluates to $Y = .0455$S, so that $C = \frac{.0455}{377} \approx 120\mu F$.

To construct the phasor diagram, start by assuming the output voltage is real ($V_r = 120$), Then the capacitance draws current $I_c = .0455j \times 120 \approx j \times 5.46A$. Current through the inductance is $I_x = 12 + j5.46$, and the voltage across the inductance is $V_x = -41 + j90.48$. Source voltage is $V_s = 78.8 + j90.48$, which has magnitude of 120 V (all of this is RMS). The resulting phasor diagram is shown in Figure 5.

Maximum voltage at the outupt is clearly achieved when $\omega^2 LC = 1$, when $C = 351.8\mu F$. Maximum output voltage is $V_r = V_s \frac{R}{\omega L} \approx 1.33 \times 120 \approx 159$V. A plot of relative output vs. input voltage is shown in Figure 6

Problem 3: Noting that the switch will have the same *average* voltage as the source since they are separated only by an inductance, and that the switch is shorted by itself a fraction D of the time, then $< V_{sw} >= V_C(1 - D) = V_S$, then the capacitor voltage must be:

$$V_C = \frac{V_s}{1 - D} = \frac{10}{1 - .5} = 20V$$

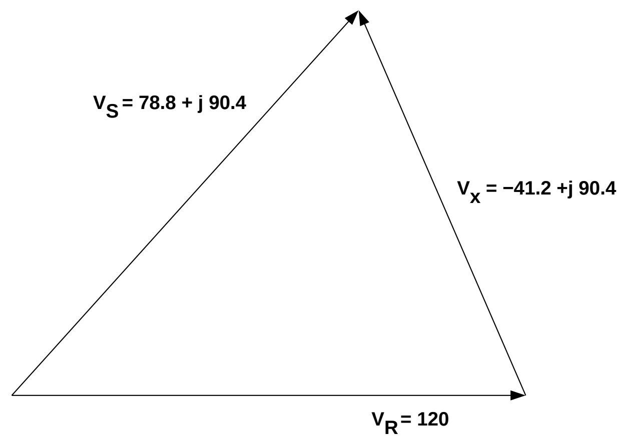

Figure 5: Phasor Diagram: compensated to equal voltage

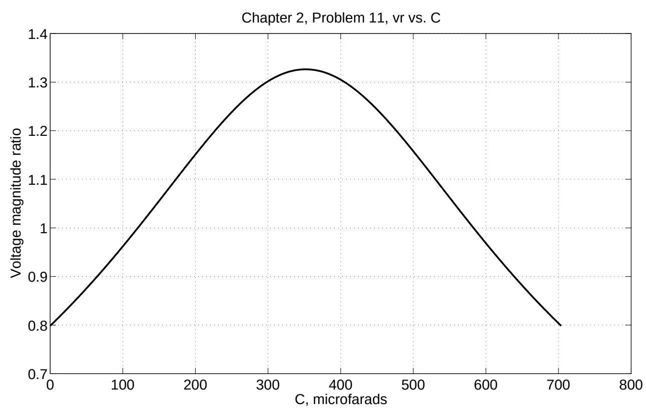

Figure 6: Voltage transfer ratio vs. Capacitance

Then, if V_C is about constant, so will be $L\frac{di}{dt}$. We can write expressions for the change in inductor current for both the ON and OFF times:

$$L\Delta I = DTV_S = (1-D)T(V_C - V_S)$$

Note these expressions are consistent with our initial calculation for V_C,

Plugging in numbers:

$$\Delta I = \frac{.5 \times 10^{-4} \times 10}{.01} \approx 0.05A$$

Getting capacitor ripple is slightly less obvious, but note that the capacitor is simply discharging during the ON period, and if the capacitor voltage is about constant, the discharge current will be $I_d = \frac{V_C}{R} = \frac{20}{40} = \frac{1}{2}A$ So in this case,

$$\Delta V_C = \frac{V_C}{R}\frac{DT}{C} = \frac{\frac{1}{2} \times \frac{1}{2} \times 10^{-4}}{50 \times 10^{-6}} \approx \frac{1}{2}V$$

A simulation of this is implemented by the MATLAB code appended. Voltage buildup is shown in Figure 7. A few cycles at the end of this are shown in Figure 8.

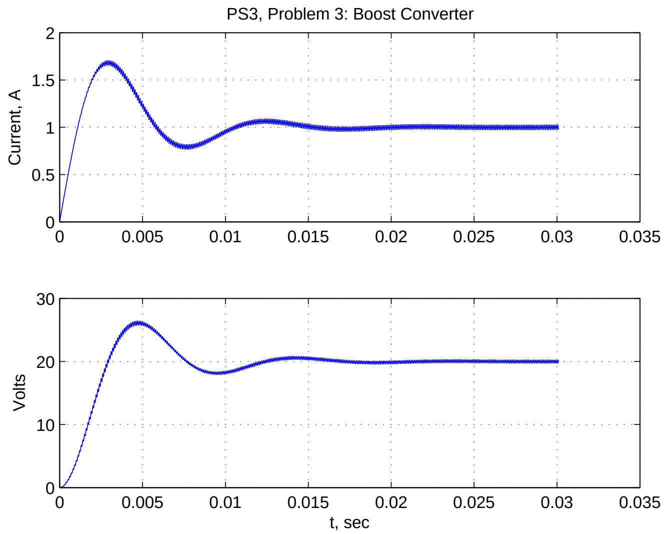

Figure 7: Buildup Transient for Boost Converter

Problem 4: With no compensating capacitor, the receiving end voltage is:

$$V_r = V_s \frac{R}{R + jX} = \frac{8000 \times 100}{100 + j12}$$

The magnitude of this is just about 7940 volts and the angle is about 6.8 degrees, current lagging. An approximate phasor diagram for this is shown in Figure 9.

To get the rest of the problem, see that the impedance of the receiving end capacitance and resistance is:

$$Z_r = R || \frac{1}{j\omega C_r} = \frac{R}{1 + j\omega RC}$$

The voltage transfer ratio is

$$\frac{V_r}{V_s} = \frac{Z_r}{Z_r + jX} = \frac{R}{R + j\omega L - \omega^2 RLC}$$

To make this have a magnitude of unity, we need

$$\left(1 - \omega^2 LC\right)^2 + \left(\omega \frac{L}{R}\right)^2 = 1$$

Using shorthand notation $Y = \omega C$, and $X = \omega L$, we have

$$(1 - YX)^2 + (\frac{X}{R})^2 = 1$$

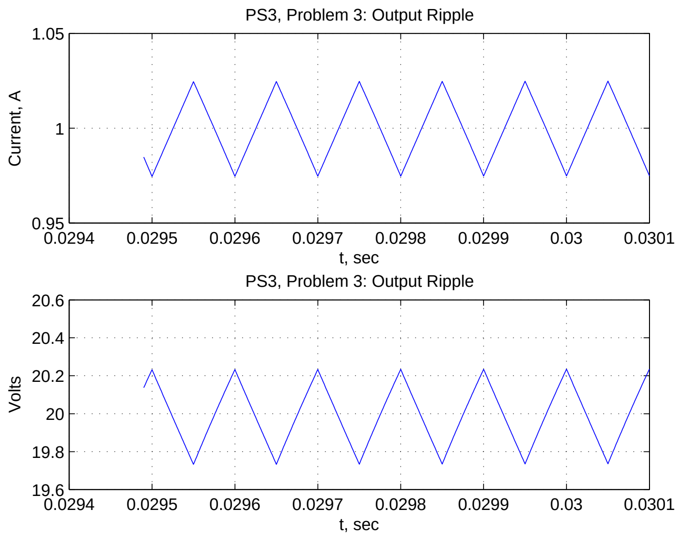

Figure 8: Ripple Current and Voltage for Boost Converter

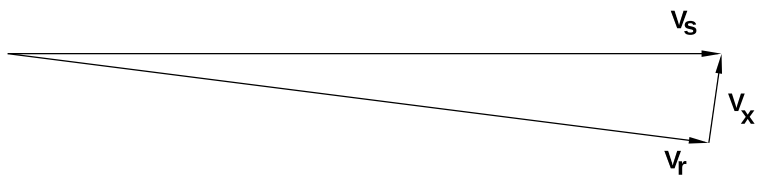

Figure 9: Phasor Diagram of Voltages

and carrying out the squares we find a solution to a quadratic:

$$Y = \frac{1}{X} \pm \sqrt{(\frac{1}{X})^2 - (\frac{1}{R})^2}$$

The minus sign yields the smallest capacitance:

$$Y = \frac{1}{12} - \sqrt{\frac{1}{12^2} - \frac{1}{100^2}} \approx .000602S$$

So $C_r = Y/377 \approx 1.6\mu F$.

The rest of the problem is worked in a straightforward way. Current through the line is:

$$I_l = \frac{V_s}{jX_l + Z_r}$$

Current through the sending end capacitance is

$$I_s = V_s j\omega C_s$$

and then $P + jQ = V_s(I_l + I_s)^*$

These are plotted for this case in Figures 9 and 11.

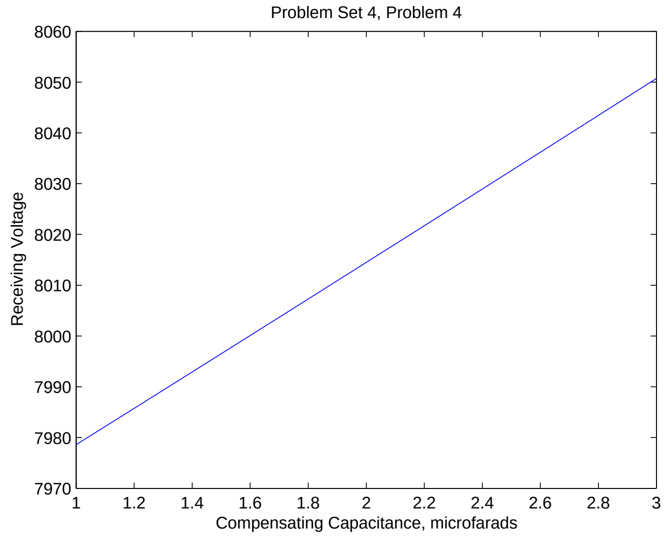

Figure 10: Receiving end voltage magnitude vs. compensating capacitor

Problem 5 This thing is most conveniently viewed by using the phasor diagram shown in Figure 12.

The voltage at the terminals of the current source is

$$\underline{V} = \underline{V_S} + jX\underline{I}$$

The phase angle betwen voltage V and current I is ψ. We know that and we know the system voltage V_S, but we don't know the terminal voltage V. To find that, we can invoke the 'law of cosines', something we should have learned in elementary triginometry. This involves only the magnitudes of the various phasors:

$$V_S^2 = V^2 + (XI)^2 = 2VXI\sin\psi$$

This can be solved for the magnitude of terminal voltage V, using the straightforward solution for a second order expression and is:

$$V = -XI\sin\psi \pm \sqrt{(XI\sin\psi)^2 + V_S^2 - (XI)^2}$$

Simplifying and noting that only the positive sign gives a reasonable answer:

$$V = \sqrt{V_S^2 - (XI\cos\psi)^2} - XI\sin\psi$$

This is evaluated by the script appended, and for power factor of 0.9 (current lagging), unity and 0.9 (current leading), respectively, terminal voltage is:

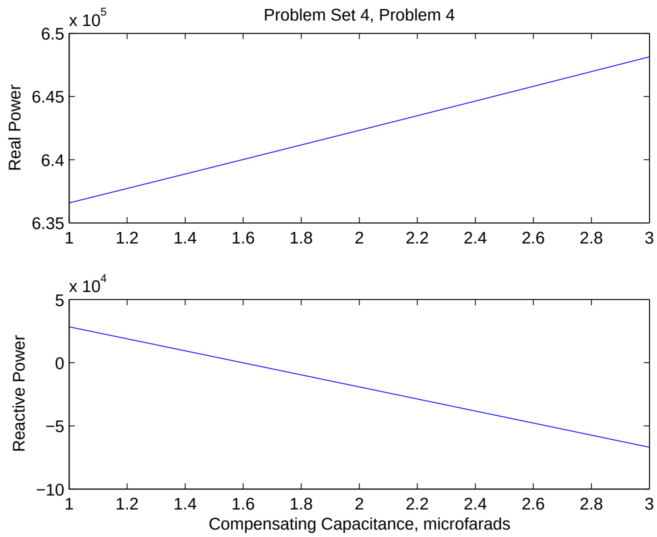

Figure 11: Real and reactive power vs. compensating capacitors

```
V1 = 1.0395e+04
V2 = 9.9594e+03
V3 = 9.5235e+03
```

Then, for varying current, voltages are as shown in Figure!13. To get the same plot vs. real power, note that $P = VI \cos \psi$ and do a cross-plot. The result is shown in Figure 14.

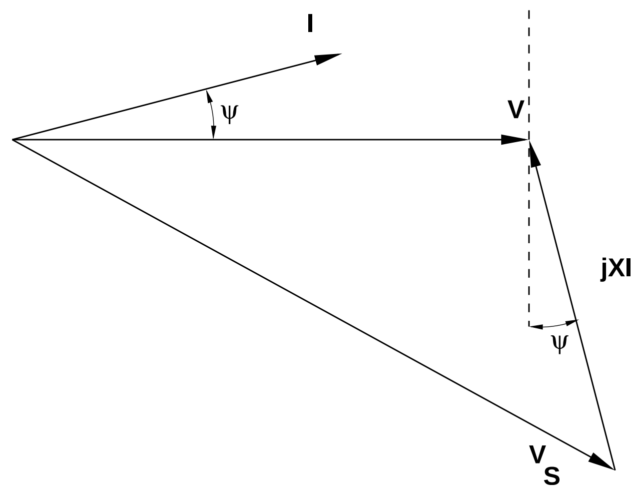

Figure 12: Phasor Diagram for Current Injection

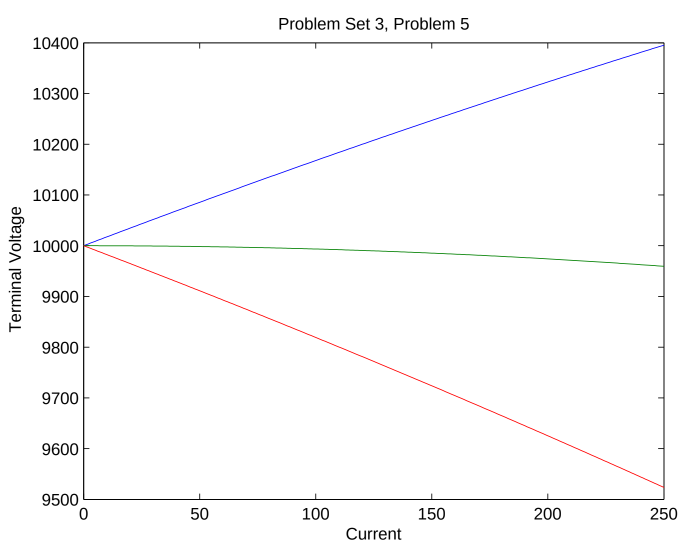

Figure 13: Voltage vs. Current

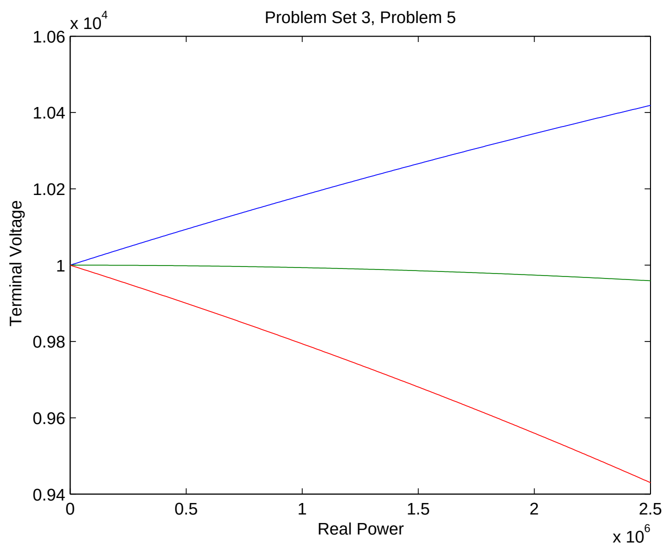

Figure 14: Voltage vs. Real Power

Scripts

```
% 6.061 Problem Set 3, Problem 1 (p2.9 of text)
% Parameters
R=10;                   % resistance
X=10;                   % reactance
Vs=120;                 % voltage
om = 120*pi;            % frequency
Cb = 1/(om*X)          % capacitance to balance
C = .5*Cb:.001*Cb:2*Cb; % capacitance
Cd = 1e6 .* C;          % in microfarads

Z = R + j*X + 1 ./ (j * om .* C); % series impedance
I = Vs ./ Z;            % current
Vr = R .* I;            % voltage
P = real(Vs .* conj(I));
Q = imag(Vs .* conj(I));

figure(1)
clf
plot(Cd, abs(Vr));
title('Problem Set 3, Problem 1')
ylabel('Output Voltage')
xlabel('Capacitance, microfarads')
grid on

figure(2)
clf
plot(Cd, P, Cd, Q, '--')
title('Problem Set 3, Problem 1')
ylabel('Watts and Vars')
xlabel('Capacitance, microfarads')
legend('Real Power', 'Reactive Power')
grid on
--------------
% Chapter 2, Problem 11
Vs = 120;
L = .02;
om = 2*pi*60;
R = 10;
Xl = om*L;
Yc = 1/Xl - sqrt((1/Xl)^2 - 1/R^2);
C = Yc/om;
Zc = 1/(j*Yc);
Zl = j*Xl;
```

```
Zo = 1/(1/R + j*Yc);
vr = Zo/(Zl + Zo);

% Construction of phasor diagram
Vr = Vs * abs(vr);
Ir = Vr/R;
Ic = Vr/Zc;
Ix = Ir + Ic;
Vx = Zl*Ix;
V_s = Vr + Vx;

fprintf('Chapter 2, Problem 11\n')
fprintf('Value of Capacitance = %g microfarads\n', 1e6*C)
fprintf('Phasors: Receiving End Voltage = %g\n', Vr);
fprintf('Inductor Voltage = %g + j %g\n', real(Vx), imag(Vx))
fprintf('Sending Voltage = %g + j %g\n', real(V_s), imag(V_s))
fprintf('Check: |V_s| = %g\n', abs(V_s))
fprintf('Angle of V_s = %g radians = %g degrees\n', angle(V_s), (180/pi)*angle(V_
s))

% capacitance for maximum voltage:

Cm = 1/(om^2 * L);

Cc = 0:Cm/500:2*Cm;

vr = 1 ./ (1 - om^2 * L .* Cc + j*om*L/R);

figure(1)
plot(1e6 .* Cc, abs(vr))
title('Chapter 2, Problem 11, vr vs. C')
ylabel('Voltage magnitude ratio')
xlabel('C, microfarads')
---------------
% trivial boost converter model
global vs L C R
vs = 10;
f = 10e3;
alf = .5;
L = 10e-3;
C = 50e-6;
R = 40;
T=1/f;
Dton = alf/f;
Dtoff = (1-alf)/f;
```

```
dton = Dton/10;
dtoff = Dtoff/10;
v0 = 0;
i0 = 0;
t = [];
i = [];
v = [];
for n = 0:300;
[tc, S] = ode45('upon',[n*T n*T+Dton] , [i0 v0]');
t = [t tc'];
ic = S(:,1);
vc = S(:,2);
i = [i ic'];
v = [v vc'];
i0 = ic(length(tc));
v0 = vc(length(tc));
[tc, S] = ode45('upoff',[n*T+Dton (n+1)*T] , [i0 v0]);
ic = S(:,1);
vc = S(:,2);
t = [t tc'];
i = [i ic'];
v = [v vc'];
i0 = ic(length(tc));
v0 = vc(length(tc));
end
figure(1)
clf
subplot 211
plot(t, i)
title('PS3, Problem 3: Boost Converter')
ylabel('Current, A');
grid on
subplot 212
plot(t, v)
ylabel('Volts');
xlabel('t, sec');
N = length(t);
grid on

figure(2)
clf
subplot 211
plot(t(N-500:N), i(N-500:N))
title('PS3, Problem 3: Output Ripple')
ylabel('Current, A');
```

```
xlabel('t, sec');
grid on
subplot 212
plot(t(N-500:N), v(N-500:N))
title('PS3, Problem 3: Output Ripple')
ylabel('Volts');
xlabel('t, sec');
grid on

vr = max(v(N-500:N))-min(v(N-500:N))
ir = max(i(N-500:N))-min(i(N-500:N))
------------
function Sdot = up(t, S)
global vs L C R
     il = S(1);
     vc = S(2);
     vdot = (1/C) * ( - vc/R);
     idot = (1/L) * (vs);
     Sdot = [idot vdot]';
-----------
function Sdot = up(t, S)
global vs L C R
     il = S(1);
     vc = S(2);
     vdot = (1/C) * (il - vc/R);
     idot = (1/L) * (vs - vc);
     Sdot = [idot vdot]';
-----------
% 6.061 Problem Set 3, Problem 4

Vs = 8000;
om = 120*pi;
R = 100;
X = 12;

% first, with no capacitance at all
Vr = Vs*R/(R + j*X);

fprintf('Uncompensated V = %g\n', abs(Vr))
fprintf('Angle = %g radians = %g degrees\n',angle(Vr), (180/pi)*angle(Vr))

Yc = 1/X - sqrt(1/X^2 - 1/R^2);
Cc = Yc/om;

fprintf('Cap to Compensate = %g microfarads\n', 1e6*Cc)
```

```
Cd = 1:.001:3;                % in microfarads
C = 1e-6 .* Cd;               % in farads

Zr = R ./(1 + j*om*R .* C);  % receiving end impedance

Il = Vs ./ (Zr + j*X);        % line current
Ic = Vs *j*om .*C;            % sending end comp current
Vr = Zr .* Il;                % receiving end voltage

Ps = real(Vs .* conj(Il + Ic));
Qs = imag(Vs .* conj(Il + Ic));

figure(3)
plot(Cd, abs(Vr))
title('Problem Set 4, Problem 4')
ylabel('Receiving Voltage')
xlabel('Compensating Capacitance, microfarads')

figure(4)
clf
subplot 211
plot(Cd, Ps)
title('Problem Set 4, Problem 4')
ylabel('Real Power')
subplot 212
plot(Cd, Qs)
ylabel('Reactive Power')
xlabel('Compensating Capacitance, microfarads')
-------------
% problem set 3, problem 5

X = 4;
I = 250;
Vs = 10e3;

% part 1:
phi = acos(.9);
V1 = sqrt(Vs^2-(X*I*cos(phi))^2)+X*I*sin(phi)
V2 = sqrt(Vs^2-(X*I)^2)
V3 = sqrt(Vs^2-(X*I*cos(phi))^2)-X*I*sin(phi)

I = 0:1:250;
V1 = sqrt(Vs^2-(X*cos(phi) .* I) .^2)+X*sin(phi).* I;
V2 = sqrt(Vs^2-(X .*I) .^2);
```

```
V3 = sqrt(Vs^2-(X*cos(phi) .* I) .^2)-X*sin(phi) .*I;

figure(1)
plot(I, V1, I, V2, I, V3)
title('Problem Set 3, Problem 5')
ylabel('Terminal Voltage')
xlabel('Current')

I = 0:1:300;
V1 = sqrt(Vs^2-(X*cos(phi) .* I) .^2)+X*sin(phi).* I;
V2 = sqrt(Vs^2-(X .*I) .^2);
V3 = sqrt(Vs^2-(X*cos(phi) .* I) .^2)-X*sin(phi) .*I;

P1 = V1 .* cos(phi) .* I;
P2 = V2 .* I;
P3 = V3 .* cos(phi) .* I;

figure(2)
plot(P1, V1, P2, V2, P3, V3);
title('Problem Set 3, Problem 5')
ylabel('Terminal Voltage')
xlabel('Real Power')
axis([0 2.5e6 9400 10600])
% problem set 3, problem 5

X = 4;
I = 250;
Vs = 10e3;

% part 1:
phi = acos(.9);
V1 = sqrt(Vs^2+(X*I)^2 + 2*Vs*X*I*sin(phi))
V2 = sqrt(Vs^2+(X*I)^2)
V3 = sqrt(Vs^2+(X*I)^2 - 2*Vs*X*I*sin(phi))

I = 0:1:250;
V1 = sqrt(Vs^2+(X .*I) .^2 + 2*Vs*X*sin(phi) .* I);
V2 = sqrt(Vs^2+(X .*I) .^2);
V3 = sqrt(Vs^2+(X .*I) .^2 - 2*Vs*X*sin(phi) .* I);

figure(1)
plot(I, V1, I, V2, I, V3)
title('Problem Set 3, Problem 5')
ylabel('Terminal Voltage')
xlabel('Current')
```

```
I = 0:1:300;
V1 = sqrt(Vs^2+(X .*I) .^2 + 2*Vs*X*sin(phi) .* I);
V2 = sqrt(Vs^2+(X .*I) .^2);
V3 = sqrt(Vs^2+(X .*I) .^2 - 2*Vs*X*sin(phi) .* I);

P1 = V1 .* cos(phi) .* I;
P2 = V2 .* I;
P3 = V3 .* cos(phi) .* I;

figure(2)
plot(P1, V1, P2, V2, P3, V3);
title('Problem Set 3, Problem 5')
ylabel('Terminal Voltage')
xlabel('Real Power')
axis([0 2.5e6 9400 10600])
```