6.055J / 2.038J The Art of Approximation in Science and Engineering
Spring 2008

Chapter 4
Proportional reasoning

4.1 Animal jump heights

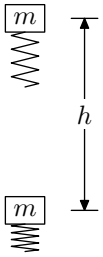
We next proportional reasoning to understand how high animals jump, as a function of their size. Do kangaroos jump higher than fleas? We study a jump from standing (or from rest, for animals that do not stand); a running jump depends on different physics. This problem looks underspecified. The height depends on how much muscle an animal has, how efficient the muscles are, what the animal's shape is, and much else. The first subsection introduces a simple model of jumping, and the second refines the model to consider physical effects neglected in the crude approximations.

4.1.1 Simple model

We want to determine only how jump height varies with body mass. Even this problem looks difficult; the height still depends on muscle efficiency, and so on. Let's see how far we get by just plowing along, and using symbols for the unknown quantities. Maybe all the unknowns cancel.

We want an equation for the height h in the form $h \sim m^\beta$, where m is the animal's mass and β is the so-called scaling exponent.

Jumping requires energy, which must be provided by muscles. This first, simplest model equates the required energy to the energy supplied by the animal's muscles.



The required energy is the easier estimation: An animal of mass m jumping to a height h requires an energy $E_{\text{jump}} \propto mh$. Because all animals feel the same gravity, this relation does not contain the gravitational acceleration g. You could include it in the equation, but it would just carry through the equations like unused baggage on a trip.

The available energy is the harder estimation. To find it, divide and conquer. It is the product of the muscle mass and of the energy per mass (the energy density) stored in muscle.

To approximate the muscle mass, assume that a fixed fraction of an animals mass is muscle, i.e. that this fraction is the same for all animals. If α is the fraction, then

$$m_{\text{muscle}} \sim \alpha m$$

or, as a proportionality,

$$m_{\text{muscle}} \propto m,$$

where the last step uses the assumption that all animals have the same α.

For the energy per mass, assume again that all muscle tissues are the same: that they store the same energy per mass. If this energy per mass is $\mathcal{E}$, then the available energy is

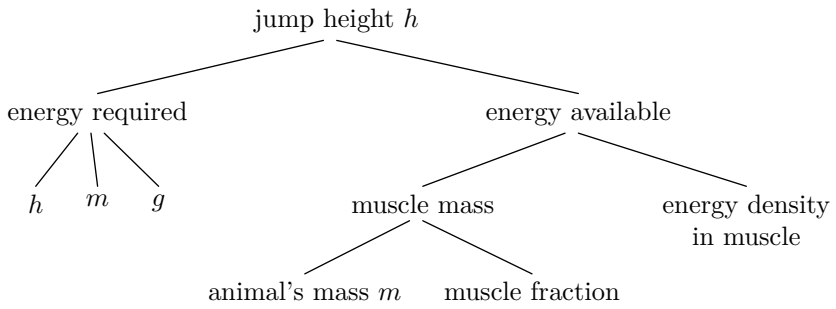
$$E_{\text{avail}} \sim \mathcal{E} m_{\text{muscle}}$$

or, as a proportionality,

$$E_{\text{avail}} \propto m_{\text{muscle}},$$

where this last step uses the assumption that all muscle has the same energy density $\mathcal{E}$.

Here is a tree that summarizes this model:



Now finish propagating toward the root. The available energy is

$$E_{\text{avail}} \propto m.$$

So an animal with three times the mass of another animal can store roughly three times the energy in its muscles, according to this simple model.

Now compare the available and required energies to find how the jump height as a function of mass. The available energy is

$$E_{\text{avail}} \propto m$$

and the required energy is

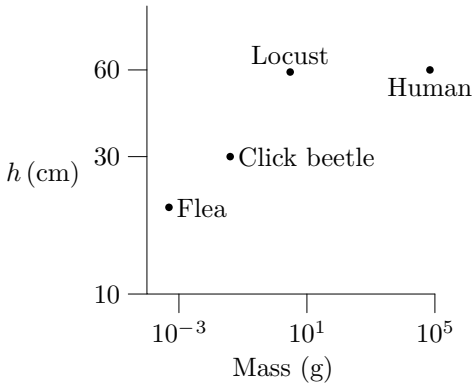
$$E_{\text{required}} \propto mh.$$

Equate these energies, which is an application of conservation of energy. Then $mh \propto m$ or

$$h \propto m^0.$$

In other words, all animals jump to the same height.

The result, that all animals jump to the same height, seems surprising. Our intuition tells us that people should be able to jump higher than locusts. The graph shows jump heights for animals of various sizes and shapes [source: *Scaling: Why Animal Size is So Important* [1], page 178]. Here is the data:



Animal	*Mass* (g)	*Height* (cm)
Flea	$5 \cdot 10^{-4}$	20
Click beetle	$4 \cdot 10^{-2}$	30
Locust	3	59
Human	$7 \cdot 10^{4}$	60

The height varies almost not at all when compared to variation in mass, so our result is roughly correct! The mass varies more than eight orders of magnitude (a factor of 10^8), yet the jump height varies only by a factor of 3. The predicted scaling of constant h ($h \propto 1$) is surprisingly accurate. The largest error shows up at the light end; fleas and beetles do not jump as high as larger animals.