6.055J / 2.038J The Art of Approximation in Science and Engineering
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Large lenses warp and crack; one of the largest lenses made is 6 m. So there is no chance of detecting an angle of 10^{-9}.

Physicists therefore searched for another source of light bending. In the solar system, the largest mass is the sun. At the surface of the sun, the field strength is

$$\frac{Gm}{rc^2} \sim \frac{6.7\cdot 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}} \times 2.0\cdot 10^{30}\,\mathrm{kg}}{7.0\cdot 10^{8}\,\mathrm{m} \times 3.0\cdot 10^{8}\,\mathrm{m\,s^{-1}} \times 3.0\cdot 10^{8}\,\mathrm{m\,s^{-1}}} \sim 2.1\cdot 10^{-6} \approx 0.4''.$$

This angle, though small, is possible to detect: The required lens diameter is roughly

$$d \sim \lambda/\theta \sim \frac{0.5\cdot 10^{-6}\,\mathrm{m}}{2.1\cdot 10^{-6}} \sim 20\,\mathrm{cm}.$$

The eclipse expedition of 1919, led by Arthur Eddington of Cambridge, tried to measure exactly this effect.

For many years Einstein believed that his theory of gravity would predict the Newtonian value, which turns out to be 0.87 arcseconds for light just grazing the surface of the sun. The German mathematician, Soldner, derived the same result in 1803. Fortunately for Einstein's reputation, the eclipse expeditions that went to test his (and Soldner's) prediction got rained or clouded out. By the time an expedition got lucky with the weather (Eddington's in 1919), Einstein had invented a new theory of gravity, which predicted 1.75 arcseconds. The goal of Eddington's expedition was to decide between the Newtonian and general relativity values. The measurements are difficult, and the results were not accurate enough to decide which theory was right. But 1919 was the first year after the World War, in which Germany and Britain had fought each other almost to oblivion. A theory invented by a German, confirmed by an Englishman (from Newton's university, no less) – such a picture was comforting after the trauma of war, so the world press and scientific community saw what they wanted to: Einstein vindicated! A proper confirmation of Einstein's prediction came only with the advent of radio astronomy, which could measure small deflections accurately. I leave you with this puzzle: If the accuracy of a telescope is λ/d, how could radio telescopes be more accurate than optical ones, since radio waves have a longer wavelength than light has?!

7.6 Buckingham Pi theorem

The second step is in a dimensional analysis is to make dimensionless groups. That task is simpler by knowing in advance how many groups to look for. The Buckingham Pi theorem provides that number. I derive it with a series of examples.

Here is a possible beginning of the theorem statement: *The number of dimensionless groups is. . . .* Try it on the light-bending example. How many groups can the variables θ, G, m, r, and c produce? The possibilities include θ, θ^2, Gm/rc^2, $\theta Gm/rc^2$, and so on. The possibilities are infinite! Now apply the theorem statement to estimating the size of hydrogen, before including quantum mechanics in the list of variables. That list is a_0 (the size), $e^2/4\pi\epsilon_0$, and m_e. That list produces no dimensionless groups. So it seems that the number of groups would be zero – if no groups are possible – or infinity, if even one group is possible.

Here is an improved theorem statement taking account of the redundancy: *The number of **independent** dimensionless groups is....* To complete the statement, try a few examples:

1. Bending of light. The five quantities θ, G, m, r, and c produce two independent groups. A convenient choice for the two groups is θ and Gm/rc^2, but any other independent set is equally valid, even if not as intuitive.
2. Size of hydrogen without quantum mechanics. The three quantities a_0 (the size), $e^2/4\pi\epsilon_0$, and m_e produce zero groups.
3. Size of hydrogen with quantum mechanics. The four quantities a_0 (the size), $e^2/4\pi\epsilon_0$, m_e, and $\hbar$ produce one independent group.

These examples fit a simple pattern:

$$\text{no. of independent groups} = \text{no. of quantities} - 3.$$

The 3 is a bit distressing because it is a magic number with no explanation. It is also the number of basic dimensions: length, mass, and time. So perhaps the statement is

$$\text{no. of independent groups} = \text{no. of quantities} - \text{no. of dimensions}.$$

Test this statement with additional examples:

1. Period of a spring–mass system. The quantities are T (the period), k, m, and x_0 (the amplitude). These four quantities form one independent dimensionless group, which could be kT^2/m. This result is consistent with the proposed theorem.
2. Period of a spring–mass system (without x_0). Since the amplitude x_0 does not affect the period, the quantities could have been T (the period), k, and m. These three quantities form one independent dimensionless group, which again could be kT^2/m. This result is also consistent with the proposed theorem, since T, k, and m contain only two dimensions (mass and time).

The theorem is safe until we try to derive Newton's second law. The force F depends on mass m and acceleration a. Those three quantities contain three dimensions – mass, length, and time. Three minus three is zero, so the proposed theorem predicts zero independent dimensionless groups. Whereas $F = ma$ tells me that F/ma is a dimensionless group.

Var	*Dim*	What
F	MLT^{-2}	force
m	M	mass
a	LT^{-2}	acceleration

This problem can be fixed by adding one word. Look at the dimensions of F, m, and a. All the dimensions – M or MLT^{-2} or LT^{-2} – can be constructed from only *two* dimensions: M and LT^{-2}. The key idea is that the original set of three dimensions are not independent, whereas the pair M and LT^{-2} are independent. So:

$$\text{no. of independent groups} = \text{no. of quantities} - \text{no. of } \textit{independent} \text{ dimensions}.$$

And that statement is the Buckingham Pi theorem [9].