Problem Set 9

This problem set is due **at 9:00pm** on **Wednesday, May 9, 2012**.

Both exercises and problems should be solved, but *only the problems* should be turned in. Exercises are intended to help you master the course material. Even though you should not turn in the exercise solutions, you are responsible for material covered by the exercises.

Mark the top of the first page of your solution with your name, the course number, the problem number, your recitation section, the date, and the names of any students with whom you collaborated. The homework template (LATEX) is available on the course website.

You will often be called upon to "give an algorithm" to solve a certain problem. Your write-up should take the form of a short essay. A topic paragraph should summarize the problem you are solving and what your results are. The body of the essay should provide the following:

1. A description of the algorithm in English and, if helpful, pseudo-code.

2. At least one worked example or diagram to show more precisely how your algorithm works.

3. A proof (or indication) of the correctness of the algorithm.

4. An analysis of the running time of the algorithm.

Remember, your goal is to communicate. Full credit will be given only to correct solutions *that are described clearly*. Convoluted and opaque descriptions will receive lower marks.

Exercise 9-1. Do Exercise 35.1-4 on page 1111 of CLRS.

Exercise 9-2. Do Exercise 35.1-5 on page 1111 of CLRS.

Exercise 9-3. Do Exercise 35.3-4 on page 1122 of CLRS.

Exercise 9-4. Do Exercise 35.3-5 on page 1122 of CLRS.

Exercise 9-5. Do Exercise 35.5-4 on page 1133 of CLRS.

Exercise 9-6. Do Exercise 35.5-5 on page 1133 of CLRS.

Exercise 9-7. Do Exercise 16.3-7 on page 436 of CLRS.

Exercise 9-8. Do Exercise 16.3-8 on page 436 of CLRS.

Exercise 9-9. Do Exercise 16.3-9 on page 437 of CLRS.

Problem 9-1. Plumbing Problem

The Mario brothers are hired to outfit the Stata Center with a plumbing system. There is a set of n rooms in the building that ideally would all have plumbing. However, due to a reduced budget compounded by a fear of leaks, the administrators ask the brothers to provide the minimum piping needed to service at least the rooms in a mission-critical subset S of all rooms.

The Mario brothers have asked you to help find the piping of minimum overall length that serves all rooms in S, and may serve some other rooms in the building as well. Keep in mind that, for engineering reasons, pipes can only be joined inside rooms.

(a) Luigi, one of the Mario brothers, assumes that any pair of rooms can be connected by pipes and suggests finding the Minimum Pipe Spanning Tree (MPST) for the rooms in the mission-critical set S. Let G be a complete graph where vertices represent rooms in the building and the weight of each edge is the length of the pipes required to connect the corresponding rooms. The MPST for S is simply the Minimum Spanning Tree for the subgraph of G induced by S.

Provide an example of a room graph where finding the MPST for S would not give the piping of minimum overall length.

(b) Luigi has taken a closer look at the blueprint, which shows the distance $d(x, y)$ for each pair of rooms x, y in the building. He notes that the distances between any two rooms satisfy the triangle inequality. So, for any three rooms x,y and z, $d(x, z) \leq d(x, y) + d(y, z)$.

Assume any pair of rooms x,y can be connected using a pipe of length $d(x, y)$. Devise an efficient polynomial-time approximation algorithm for the plumbing problem with an approximation ratio $\alpha \leq 2$. **Hint:** Consider building an MPST on S.

(c) Mario notes that due to the topology of the building, pipes can only run between some pairs of rooms and not others. What's more, Alyssa P. Hacker sets up a few strategic wormholes, shortcuts through spacetime, so that the pipe-based space-time displacements no longer satisfy the triangle inequality.

You are provided with an updated blueprint of possible direct pipe connections and effective pipe lengths for each of them. Devise a polynomial-time approximation algorithm with approximation ratio $\alpha \leq 2$, using what you learned in part (b).

Problem 9-2. Huffman Coding

(a) You are working with a Huffman encoding scheme over n symbols. How long could a codeword be in the worst case? Provide an example set of frequencies $f_1, f_2, \ldots f_n$ (as a closed formula of n) that would give rise to this longest codeword.

(b) Prove that if some character occurs with frequency more than 2/5, then there is guaranteed to be a codeword of length 1.

(c) Prove that if all characters occur with frequency less than 1/3, then there is guaranteed to be no codeword of length 1.

6.046J / 18.410J Design and Analysis of Algorithms
Spring 2012