In-Class Problems Week 13, Fri.

Problem 1.
A herd of cows is stricken by an outbreak of *cold cow disease*. The disease lowers a cow's body temperature from normal levels, and a cow will die if its temperature goes below 90 degrees F. The disease epidemic is so intense that it lowered the average temperature of the herd to 85 degrees. Body temperatures as low as 70 degrees, **but no lower**, were actually found in the herd.

(a) Use Markov's Bound to prove that at most 3/4 of the cows could survive.

(b) Suppose there are 400 cows in the herd. Show that the bound from part (a) is the best possible by giving an example set of temperatures for the cows so that the average herd temperature is 85 and 3/4 of the cows will have a high enough temperature to survive.

(c) Notice that the results of part (b) are purely arithmetic facts about averages, not about probabilities. But you verified the claim in part (a) by applying Markov's bound on the deviation of a random variable. Justify this approach by regarding the temperature, T, of a cow as a random variable. Carefully specify the probability space on which T is defined: what are the outcomes? what are their probabilities? Explain the precise connection between properties of T and average herd temperature that justifies the application of Markov's Bound.

Problem 2.
A gambler plays 120 hands of draw poker, 60 hands of black jack, and 20 hands of stud poker per day. He wins a hand of draw poker with probability 1/6, a hand of black jack with probability 1/2, and a hand of stud poker with probability 1/5.

(a) What is the expected number of hands the gambler wins in a day?

(b) What would the Markov bound be on the probability that the gambler will win at least 108 hands on a given day?

(c) Assume that the outcomes of the card games are *pairwise*, but possibly *not* mutually, independent. What is the variance of the number of hands won per day? You may answer with a numerical expression that is not completely evaluated.

(d) What would the Chebyshev bound be on the probability that the gambler will win at least 108 hands on a given day? You may answer with a numerical expression that is not completely evaluated.

Problem 3.
The hat-check staff has had a long day serving at a party, and at the end of the party they simply return the n checked hats in a random way such that the probability that any particular person gets their own hat back is $1/n$.

Let X_i be the indicator variable for the ith person getting their own hat back. Let S_n be the total number of people who get their own hat back.

(a) What is the expected number of people who get their own hat back?

(b) Write a simple formula for $\text{Ex}[X_i \cdot X_j]$ for $i \neq j$.

Hint: What is $\Pr\left[X_j = 1 \mid X_i = 1\right]$?

(c) Explain why you cannot use the variance of sums formula to calculate $\text{Var}[S_n]$.

(d) Show that $\text{Ex}[(S_n)^2] = 2$. *Hint:* $(X_i)^2 = X_i$.

(e) What is the variance of S_n?

(f) Show that there is at most a 1% chance that more than 10 people get their own hat back.

Supplementary Problems

Problem 4.
Let K_n be the complete graph with n vertices. Each of the edges of the graph will be randomly assigned one of the colors red, green, or blue. The assignments of colors to edges are mutually independent, and the probability of an edge being assigned red is r, blue is b, and green is g (so $r + b + g = 1$).

A set of three vertices in the graph is called a *triangle*. A triangle is *monochromatic* if the three edges connecting the vertices are all the same color.

(a) Let m be the probability that any given triangle, T, is monochromatic. Write a simple formula for m in terms of r, b, and g.

(b) Let I_T be the indicator variable for whether T is monochromatic. Write simple formulas in terms of m, r, b, and g for $\text{Ex}[I_T]$ and $\text{Var}[I_T]$.

Let T and U be distinct triangles.

(c) What is the probability that T and U are both monochromatic if they do not share an edge?...if they do share an edge?

Now assume $r = b = g = \frac{1}{3}$**.**

(d) Show that I_T and I_U are independent random variables.

(e) Let M be the number of monochromatic triangles. Write simple formulas in terms of n and m for $\text{Ex}[M]$ and $\text{Var}[M]$.

(f) Let $\mu ::= \text{Ex}[M]$. Use Chebyshev's Bound to prove that

$$\Pr[|M - \mu| > \sqrt{\mu \log \mu}] \leq \frac{1}{\log \mu}.$$

(g) Conclude that

$$\lim_{n \to \infty} \Pr[|M - \mu| > \sqrt{\mu \log \mu}] = 0$$

Problem 5.
Let R be a positive integer valued random variable.

(a) If $\text{Ex}[R] = 2$, how large can $\text{Var}[R]$ be?

(b) How large can $\text{Ex}[1/R]$ be?

(c) If $R \leq 2$, that is, the only values of R are 1 and 2, how large can $\text{Var}[R]$ be?