In-Class Problems Week 10, Mon.

Problem 1.
Recall that for functions f, g on $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|. \tag{1}$$

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is O() of the other, indicate the *smallest nonnegative integer*, c, and for that smallest c, the *smallest corresponding nonnegative integer* n_0 ensuring that condition (1) applies.

(a) $f(n) = n^2, g(n) = 3n$.

(b) $f(n) = (3n - 7)/(n + 4), g(n) = 4$

(c) $f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n$

Problem 2.

(a) Indicate which of the following asymptotic relations below on the set of nonnegative real-valued functions are equivalence relations (**E**), strict partial orders (**S**), weak partial orders (**W**), or *none* of the above (**N**).

- $f \sim g$, the "asymptotically equal" relation.
- $f = o(g)$, the "little Oh" relation.
- $f = O(g)$, the "big Oh" relation.
- $f = \Theta(g)$, the "Theta" relation.
- $f = O(g)$ AND NOT$(g = O(f))$.

(b) Indicate the implications among the assertions in part (a). For example,

$$f = o(g) \text{ IMPLIES } f = O(g).$$

Problem 3.

False Claim.

$$2^n = O(1). \tag{2}$$

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

Bogus proof. The proof is by induction on n where the induction hypothesis, $P(n)$, is the assertion (2).

base case: $P(0)$ holds trivially.

inductive step: We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

$$2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,$$

which implies that $2^{n+1} = O(1)$. That is, $P(n+1)$ holds, which completes the proof of the inductive step.

We conclude by induction that $2^n = O(1)$ for all n. That is, the exponential function is bounded by a constant.

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Supplemental problems

Problem 4.

Assign true or false for each statement and prove it.

- $n^2 \sim n^2 + n$
- $3^n = O\left(2^n\right)$
- $n^{\sin(n\pi/2)+1} = o\left(n^2\right)$
- $n = \Theta\left(\dfrac{3n^3}{(n+1)(n-1)}\right)$

Problem 5.

Give an elementary proof (without appealing to Stirling's formula) that $\log(n!) = \Theta(n \log n)$.