

Solutions to In-Class Problems Week 5, Fri.

Problem 1. Figures 1–4 show different pictures of planar graphs.

(a) For each picture, describe its faces by listing the cycles that form their boundaries.

Solution. Figs 1 & 2: $abda, bcdb, abcda$. Fig 3: $abcdea, adea, abda, bcdb$. Fig 4: $abcda, abdea, abda, adea$. ■

(b) Which of the pictured graphs are isomorphic? Which pictures represent the same (abstract) *planar drawing*?

Solution. Figs 1 & 2 have the same face boundaries, so are different pictures of the *same* planar drawing. Figs 3 & 4 both have four faces, but they are different, for example, Fig 3 has a face with 5 edges, but the longest boundary in Fig 4 has 4 edges. ■

Problem 2. Prove that in a drawing of a connected planar graph

(a) every edge is traversed exactly twice by the face boundaries.

Solution. By induction on the number, n , of edges in a planar drawing. Base case $n = 0$ holds vacuously. The inductive case works because rule (1) adds one new edge which is traversed twice by the face in which it is drawn, and all other faces are unchanged, and rule (2) divides the boundary of one face into two parts, leaving the number of edge traversals on them unchanged, adds a traversal of a new edge to each part, and leaves all other faces unchanged. ■

(b) if the graph has 3 or more vertices, then every face has length at least 3 (more precisely, every face traverses edges at least 3 times).

Solution. By induction on the number, n , of edges in a planar drawing. A connected graph with 3 vertices must have at least 2 edges, so we begin with base case $n = 2$. In this case the graph must be a 2 edge line graph; if its successive vertices are a, b, c , then it has a unique planar drawing with one face with boundary $abcba$ which traverses each edge twice and so has length 4.

The inductive case works because rule (1) adds 2 to the length of one face and leaves the others unchanged, and an edge added by rule (2) can only connect nonadjacent edges on a face boundary. This means the face across which the edge is drawn must have had length at least 4 and its boundary splits into into two paths of length at least 2. These paths each becomes the boundary of a new face by making one additional traversal of the new edge, ensuring that both new faces have length at least 3. All other faces are unchanged by rule (2). ■

Problem 3. Suppose we consider planar graphs that may *not* be connected. So in addition to the parameters v, e, f of Euler's formula, we also have the number, c , of connected components in the graph. In this case a face may have a boundary consisting of several unconnected cycles, as in Figure 5.

(a) Use the additional parameter, c , to state a generalized version of Euler's Formula that applies to possibly unconnected planar graphs.

Solution.

$$v - e + f - c = 1.$$

■

(b) Drawings of unconnected graphs use two additional rules: rule (3): add a new vertex of degree 0, and rule (4) add an edge between vertices on two different connected components. Using these rules, explain how to modify the inductive proof of Euler's formula to prove the generalized version.

Solution. By induction on the sum $n = e + v$. The base case is $n = 1$, which means $v = 1, e = 0, f = 1$ (just the outer face), and $c = 1$, and $1 - 0 + 1 - 1 = 1$ as required. The inductive case for rules (1) and (2) is the same since they do not change c . Rule (3) increase c and v by 1 and leaves f and e unchanged, so $v - e + f - c$ is unchanged. Rule (4) increases e by 1 and decreases c by 1 and leaves v and f unchanged, so $v - e + f - c$ is again unchanged. ■



