Massachusetts Institute of Technology
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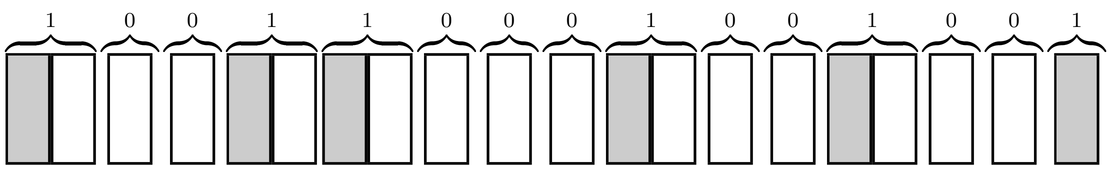
Solutions to Problem Set 7

Problem 1. There are 20 books arranged in a row on a shelf.

(a) Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected and 15-bit sequences with exactly 6 ones.

Solution. There is a bijection from 15-bit sequences with exactly six 1's to valid book selections: given such a sequence, map each zero to a non-chosen book, each of the first five 1's to a chosen book followed by a non-chosen book, and the last 1 to a chosen book. For example, here is a configuration of books and the corresponding binary sequence:



Selected books are darkened. Notice that the first fives ones are mapped to a chosen book *and* an non-chosen book in order to ensure that the binary sequence maps to a valid selection of books. ∎

(b) How many ways are there to select 6 books so that no two adjacent books are selected?

Solution. By the Bijection Rule, this is equal to the number of 15-bit sequences with exactly 6 ones, which is equal to

$$\frac{15!}{6!\,9!} = \binom{15}{6}$$

by the Bookkeeper Rule. ∎

Problem 2. Answer the following questions and provide brief justifications. Not every problem can be solved with a cute formula; you may have to fall back on case analysis, explicit enumeration, or ad hoc methods.

You may leave factorials and binomial coefficients in your answers.

(a) In how many different ways can the letters in the name of the popular 1980's band $BANANARAMA$ be arranged?

Solution. There are 5 A's, 2 N's, 1 B, 1 R, and 1 M. Therefore, the number of arrangements is

$$\frac{10!}{5!\ 2!\ 1!\ 1!\ 1!}$$

by the Bookkeeper Rule. ■

(b) How many different paths are there from point $(0, 0, 0)$ to point $(12, 24, 36)$ if every step increments one coordinate and leaves the other two unchanged?

Solution. There is a bijection between the set of all such paths and the set of strings containing 12 X's, 24 Y's, and 36 Z's. In particular, we obtain a path by working through a string from left to right. An X corresponds to a step that increments the first coordinate, a Y increments the second coordinate, and a Z increments the third. The number of such strings is:

$$\frac{72!}{12!\ 24!\ 36!}$$

Therefore, this is also the number of paths. ■

(c) In how many different ways can $2n$ students be paired up?

Solution. Pair up students by the following procedure. Line up the students and pair the first and second, the third and fourth, the fifth and sixth, etc. The students can be lined up in $(2n)!$ ways. However, this overcounts by a factor of 2^n, because we would get the same pairing if the first and second students were swapped, the third and fourth were swapped, etc. Furthermore, we are still overcounting by a factor of $n!$, because we would get the same pairing even if pairs of students were permuted, e.g. the first and second were swapped with the ninth and tenth. Therefore, the number of pairings is:

$$\frac{(2n)!}{2^n \cdot n!}$$

■

(d) How many different solutions over the natural numbers are there to the following equation?

$$x_1 + x_2 + x_3 + \ldots + x_8 = 100$$

A solution is a specification of the value of each variable x_i. Two solutions are different if different values are specified for some variable x_i.

Solution. There is a bijection between sequences containing 100 zeros and 7 ones. Specifically, the 7 ones divide the zeros into 8 segments. Let x_i be the number of zeros in the i-th segment. Therefore, the number of solutions is:

$$\binom{100+7}{7}$$

■

(e) In how many different ways can one choose n out of $2n$ objects, given that n of the $2n$ objects are identical and the other n are all unique?

Solution. We can select n objects as follows. First, take a subset of the unique objects. Then take however many identical elements are needed to bring the total to n. The first step can be done in 2^n ways, and the second can be done in only 1 way. Therefore, there are 2^n ways to choose n objects. ■

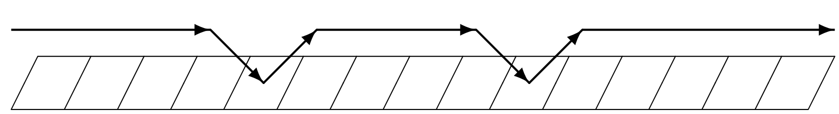
(f) How many undirected graphs are there with vertices $v_1, v_2, \ldots, v_n$ if self-loops are permitted?

Solution. There are $\binom{n}{2} + n$ potential edges, each of which may or may not appear in a given graph. Therefore, the number of graphs is:

$$2^{\binom{n}{2}+n}$$

■

(g) There are 15 sidewalk squares in a row. Suppose that a ball can be thrown so that it bounces on 0, 1, 2, or 3 distinct sidewalk squares. Assume that the ball always moves from left to right. How many different throws are possible? As an example, a two-bounce throw is illustrated below.



Solution.

$$\binom{15}{0} + \binom{15}{1} + \binom{15}{2} + \binom{15}{3}$$

■

(h) The working days in the next year can be numbered 1, 2, 3, ..., 300. I'd like to avoid as many as possible.

- On even-numbered days, I'll say I'm sick.
- On days that are a multiple of 3, I'll say I was stuck in traffic.
- On days that are a multiple of 5, I'll refuse to come out from under the blankets.

In total, how many work days will I *avoid* in the coming year?

Solution. Let D_2 be the set of even-numbered days, D_3 be the days that are a multiple of 3, and D_5 be days that are a multiple of 5. The set of days I can avoid is $D_2 \cup D_3 \cup D_5$. By the Inclusion-Exclusion Rule, the size of this set is:

$$\begin{aligned}
|D_2 \cup D_3 \cup D_5| &= |D_2| + |D_3| + |D_5| \\
&\quad - |D_2 \cap D_3| - |D_2 \cap D_5| - |D_3 \cap D_5| \\
&\quad + |D_2 \cap D_3 \cap D_5| \\
&= \frac{300}{2} + \frac{300}{3} + \frac{300}{5} - \frac{300}{2 \cdot 3} - \frac{300}{2 \cdot 5} - \frac{300}{3 \cdot 5} + \frac{300}{2 \cdot 3 \cdot 5} \\
&= 220
\end{aligned}$$

■

Problem 3. Use the pigeonhole principle to solve the following problems.

(a) Prove that among any n^2+1 points within an $n \times n$ square there must exist two points whose distance is at most $\sqrt{2}$.

Solution. Partition the $n \times n$ into n^2 unit squares. Each of the $n^2 + 1$ points lies in one of these n^2 unit squares. (If a point lies on the boundary between squares, assign it to a square arbitrarily.) Therefore, by the pigeonhole principle, there exist two points in the same unit square. And the distance between those two points can be at most $\sqrt{2}$. ■

(b) Jellybeans of 6 different flavors are stored in 5 jars. There are 11 jellybeans of each flavor. Prove that some jar contains at least three jellybeans of one flavor and also at least three jellybeans of some other flavor.

Solution. We use the pigeonhole principle twice. Since there are 11 beans of a given flavor and there are only 5 jars, some jar must get at least $\rceil 11/5 \lceil = 3$ beans of that flavor. Since there are 6 flavors and only 5 jars, some jar must get two pairs of same-flavored beans. ■

(c) Prove that among every set of 30 integers, there exist two whose difference *or* sum is a multiple of 51.

Solution. Regard the 30 integers as pigeons. Regard the 26 sets $\{0\}$, $\{1, 50\}$, $\{2, 49\}$, ..., $\{25, 26\}$ as pigeonholes. Map integer n to the set containing $n \text{ rem } 51$. By the pigeonhole principle, some two pigeons (integers a and b) are mapped to the same pigeonhole. Thus, either:

- $a \text{ rem } 51 = b \text{ rem } 51$ and so the difference of a and b is a multiple of 51.
- $a \text{ rem } 51 + b \text{ rem } 51$ is either 0 or 51 and so the sum of a and b is a multiple of 51.

■

Problem 4. Suppose you have seven dice— each a different color of the rainbow; otherwise the dice are standard, with six faces numbered 1 to 6. A *roll* is a sequence specifying a value for each die in rainbow (ROYGBIV) order. For example, one roll is $(3, 1, 6, 1, 4, 5, 2)$ indicating that the red die showed a 3, the orange die showed 1, the yellow 6, the green 1, the blue 4, the indigo 5, and the violet 2.

For the problems below, describe a bijection between the specified set of rolls and another set that is easily counted using the Product, Generalized Product, and similar rules. Then write a simple numerical expression for the size of the set of rolls. You do not need to prove that the correspondence between sets you describe is a bijection, and you do not need to simplify the expression you come up with.

For example, let A be the set of rolls where 4 dice come up showing the same number, and the other 3 dice also come up the same, but with a different number. Let R to be the set of seven rainbow colors and S be the set $\{1, \ldots, 6\}$ of dice values.

Define $B ::= S_2 \times \{3, 4\} \times R_3$, where S_2 is the set of size 2 subsets of S, and R_3 is the set of size 3 subsets of R. Then define a bijection from A to B by mapping a roll in A to the sequence in B whose first element is the set of two numbers that came up, whose second element is the number of times the smaller of the two numbers came up in the roll, and whose third element is the set of colors of the three matching dice.

For example, the roll

$$(4, 4, 2, 2, 4, 2, 4) \in A$$

maps to the triple

$$(\{2, 4\}, 3, \{\text{yellow,green,indigo}\}) \in B.$$

Now by the bijection Rule $|A| = |B|$, and by the Product rule,

$$|B| = \binom{6}{2} \cdot 2 \cdot \binom{7}{3}.$$

(a) For how many rolls is the value on every die different?

Solution. None, by the Pigeonhole Principle. ■

(b) For how many rolls do two dice have the value 6 and the remaining five dice all have different values?

Example: $(6, 2, 6, 1, 3, 4, 5)$ is a roll of this type, but $(1, 1, 2, 6, 3, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

Solution. As in the example, map a roll into an element of $B ::= R_2 \times P_5$ where P_5 is the set of permutations of $\{1, \ldots, 5\}$. A roll maps to the pair whose first element is the set of colors of the two dice with value 6, and whose second element is the sequence of values of the remaining dice (in rainbow order). So $(6, 2, 6, 1, 3, 4, 5)$ above maps to $(\{\text{red,yellow}\}, (2, 1, 3, 4, 5))$. By the Product rule,

$$|B| = \binom{7}{2} \cdot 5!.$$

■

(c) For how many rolls do two dice have the same value and the remaining five dice all have different values?

Example: $(4, 2, 4, 1, 3, 6, 5)$ is a roll of this type, but $(1, 1, 2, 6, 1, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

Solution. Map a roll into a triple whose first element is in S, indicating the value of the pair of matching dice, whose second element is set of colors of the two matching dice, and whose third element is the sequence of the remaining five dice values (in rainbow order).

So $(4, 2, 4, 1, 3, 6, 5)$ above maps to $(4, \{\text{red,yellow}\}, (2, 1, 3, 6, 5))$. Notice that the number of choices for the third element of a triple is the number of permutations of the remaining five values, namely, $5!$. This mapping is a bijection, so the number of such rolls equals the number of such triples. By the Generalized Product rule, the number of such triples is

$$6 \cdot \binom{7}{2} \cdot 5!.$$

Alternatively, we can define a map from this rolls in this part to the rolls in part (b), by replacing the value of the duplicated values with 6's and replacing any 6 in the remaining values by the value of the duplicated pair. So the roll $(4, 2, 4, 1, 3, 6, 5)$ would map to the role $(6, 2, 6, 1, 3, 4, 5)$. Now a type b role, r, is mapped to by exactly the rolls obtainable from r by exchanging occurrences of 6's and i's, for $i = 1, \ldots, 6$. So this map is 6-to-1, and by the Division Rule, the number of rolls here is 6 times the number of rolls in part (b).

■

(d) For how many rolls do two dice have one value, two different dice have a second value, and the remaining three dice a third value?

Example: $(6, 1, 2, 1, 2, 6, 6)$ is a roll of this type, but $(4, 4, 4, 4, 1, 3, 5)$ and $(5, 5, 5, 6, 6, 1, 2)$ are not.

Solution. Map a roll of this kind into a 4-tuple whose first element is the set of two numbers of the two pairs of matching dice, whose second element is the set of two colors of the pair of matching dice with the smaller number, whose third element is the set of two colors of the larger of the matching pairs, and whose fourth element is the value of the remaining three dice. For example, the roll $(6, 1, 2, 1, 2, 6, 6)$ maps to the triple $(\{1, 2\}, \{\text{orange,green}\}, \{\text{yellow,blue}\}, 6)$.

There are $\binom{6}{2}$ possible first elements of a triple, $\binom{7}{2}$ second elements, $\binom{5}{2}$ third elements since the second set of two colors must be different from the first two, and 4 ways to choose the value of the three dice since their value must differ from the values of the two pairs. So by the Generalized Product rule, there are

$$\binom{6}{2} \cdot \binom{7}{2} \cdot \binom{5}{2} \cdot 4$$

possible rolls of this kind. ■

Problem 5. A *derangement* is a permutation $(x_1, x_2, \ldots, x_n)$ of the set $\{1, 2, \ldots, n\}$ such that $x_i \neq i$ for all i. For example, $(2, 3, 4, 5, 1)$ is a derangement, but $(2, 1, 3, 5, 4)$ is not because 3 appears in the third position. The objective of this problem is to count derangements.

It turns out to be easier to start by counting the permutations that are *not* derangements. Let S_i be the set of all permutations $(x_1, x_2, \ldots, x_n)$ that are not derangements because $x_i = i$. So the set of non-derangements is

$$\bigcup_{i=1}^{n} S_i.$$

(a) What is $|S_i|$?

Solution. There is a bijection between permutations of $\{1, 2, \ldots, n\}$ with i in the i-th position and unrestricted permutations of $\{1, 2, \ldots, n\} - i$. Therefore, $|S_i| = (n-1)!$. ■

(b) What is $|S_i \cap S_j|$ where $i \neq j$?

Solution. The set $S_i \cap S_j$ consists of all permutations with i in the i-th position and j in the j-th position. Thus, there is a bijection with permutations of $\{1, 2, \ldots, n\} - \{i, j\}$, and so $|S_i \cap S_j| = (n-2)!$. ■

(c) What is $|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}|$ where $i_1, i_2, \ldots, i_k$ are all distinct?

Solution. By the same argument, $(n-k)!$. ■

(d) Use the inclusion-exclusion formula to express the number of non-derangements in terms of sizes of possible intersections of the sets $S_1, \ldots, S_n$.

Solution.

$$\sum_i |S_i| - \sum_{i,j} |S_i \cap S_j| + \sum_{i,j,k} |S_i \cap S_j \cap S_k| - \cdots \pm |S_1 \cap S_2 \cap \cdots \cap S_n|$$

In each summation, the subscripts are distinct elements of $\{1, \ldots, n\}$. ■

(e) How many terms in the expression in part (d) have the form $|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_k}|$?

Solution. There is one such term for each k-element subset of the n-element set $\{1, 2, \ldots, n\}$. Therefore, there are $\binom{n}{k}$ such terms. ■

(f) Combine your answers to the preceding parts to prove the number of non-derangements is:

$$n!\left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots \pm \frac{1}{n!}\right).$$

Conclude that the number of derangements is

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots\pm\frac{1}{n!}\right).$$

Solution. By Inclusion-Exclusion, the number of non-derangements is

$$\begin{aligned}
&\sum_i |S_i| - \sum_{i,j} |S_i \cap S_j| + \sum_{i,j,k} |S_i \cap S_j \cap S_k| - \cdots \pm |S_1 \cap S_2 \cap \cdots \cap S_n| \\
&= \binom{n}{1}\cdot(n-1)! - \binom{n}{2}\cdot(n-2)! + \binom{n}{3}\cdot(n-3)! - \cdots \pm \binom{n}{n}\cdot 0! \\
&= n!\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\cdots\pm\frac{1}{n!}\right) \qquad (1)
\end{aligned}$$

Since there are $n!$ permutation, the number of derangements is $n!$ minus expression (1). ■

(g) As n goes to infinity, the number of derangements approaches a constant fraction of all permutations. What is that constant? *Hint:*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Solution. $1/e$ ■