Problem Set 1

Due: September 21

Reading: Notes for Week1 and Week 2

Problem 1. A real number r is called *sensible* if there exist positive integers a and b such that $\sqrt{a/b} = r$. For example, setting $a = 2$ and $b = 1$ shows that $\sqrt{2}$ is sensible. Prove that $\sqrt[3]{2}$ is not sensible. (Consider only positive real roots in this problem)

Problem 2. Translate the following sentence into a predicate formula:

> There is a student who has e-mailed exactly two other people in the class, besides possibly herself.

The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are equality and $E(x, y)$, meaning that "x has sent e-mail to y."

Problem 3. Express each of the following predicates and propositions in formal logic notation. The domain of discourse is the nonnegative integers, $\mathbb{N}$.

In addition to the propositional operators, variables and quantifiers, you may define predicates using addition, multiplication, and equality symbols, but no *constants* (like 0, 1, . . .). For example, the proposition "n is an even number" could be written

$$\exists m.\ (m + m = n).$$

(a) n is the sum of three perfect squares.

Since the constant 0 is not allowed to appear explicitly, the predicate "$x = 0$" can't be written directly, but note that it could be expressed in a simple way as:

$$x + x = x.$$

Then the predicate $x > y$ could be expressed

$$\exists w. \ (y + w = x) \wedge (w \neq 0).$$

Note that we've used "$w \neq 0$" in this formula, even though it's technically not allowed. But since "$w \neq 0$" is equivalent to the allowed formula "$\neg(w + w = w)$," we can use "$w \neq 0$" with the understanding that it abbreviates the real thing. And now that we've shown how to express "$x > y$", it's ok to use it too.

(b) $x > 1$.

(c) n is a prime number.

(d) n is a product of two distinct primes.

(e) There is no largest prime number.

(f) (Goldbach Conjecture) Every even natural number $n > 2$ can be expressed as the sum of two primes.

(g) (Bertrand's Postulate) If $n > 1$, then there is always at least one prime p such that $n < p < 2n$.

Problem 4. If a set, A, is finite, then $|A| < 2^{|A|} = |\mathcal{P}(A)|$, and so there is no surjection from set A to its powerset. Show that this is still true if A is infinite. *Hint:* Remember Russell's paradox and consider $\{x \in A \mid x \notin f(x)\}$ where f is such a surjection.

Problem 5. **(a)** Prove that

$$\exists z. \ [P(z) \wedge Q(z)] \longrightarrow [\exists x. \ P(x) \wedge \exists y. \ Q(y)] \tag{1}$$

is valid. (Use the proof in the subsection on Validity in Week 2 Notes as a guide to writing your own proof here.)

(b) Prove that the converse of (1) is not valid by describing a counter model as in Week 2 Notes.

Problem 6. **(a)** Give an example where the following result fails:

False Theorem. *For sets A, B, C, and D, let*

$$L ::= (A \cup C) \times (B \cup D),$$
$$R ::= (A \times B) \cup (C \times D).$$

Then $L = R$.

(b) Identify the mistake in the following proof of the False Theorem.

Bogus proof. Since L and R are both sets of pairs, it's sufficient to prove that $(x, y) \in L \longleftrightarrow (x, y) \in R$ for all x, y.

The proof will be a chain of iff implications:

$(x, y) \in L$	iff
$x \in A \cup C$ and $y \in B \cup D$,	iff
either $x \in A$ or $x \in C$, and either $y \in B$ or $y \in D$,	iff
($x \in A$ and $y \in B$) or else ($x \in C$ and $y \in D$),	iff
$(x, y) \in A \times B$, or $(x, y) \in C \times D$,	iff
$(x, y) \in (A \times B) \cup (C \times D) = R$.	

□

(c) Fix the proof to show that $R \subseteq L$.

Student's Solutions to Problem Set 1

Your name:	
Due date:	September 21
Submission date:	
Circle your TA:	David Jelani Sayan

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

[1] People other than course staff.

[2] Give citations to texts and material other than the Fall '02 course materials.