Massachusetts Institute of Technology

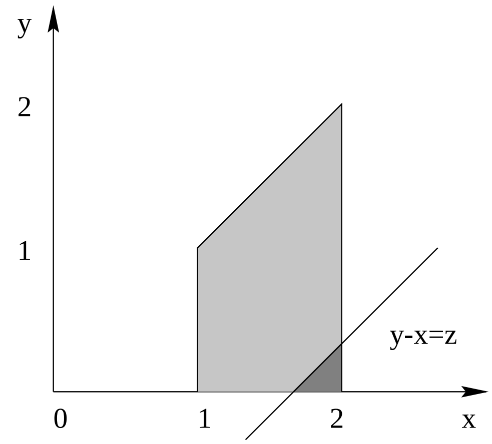
Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Problem Set 6: Solutions

1. Let us draw the region where $f_{X,Y}(x,y)$ is nonzero:



The joint PDF has to integrate to 1. From $\int_{x=1}^{x=2}\int_{y=0}^{y=x} ax\,dy\,dx = \frac{7}{3}a = 1$, we get $a = \frac{3}{7}$.

(b) $f_Y(y) = \int f_{X,Y}(x,y)\,dy = \begin{cases} \int_1^2 \frac{3}{7}x\,dx, & \text{if } 0 \le y \le 1, \\ \int_y^2 \frac{3}{7}x\,dx, & \text{if } 1 < y \le 2, \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{9}{14}, & \text{if } 0 \le y \le 1, \\ \frac{3}{14}(4-y^2), & \text{if } 1 < y \le 2, \\ 0, & \text{otherwise.} \end{cases}$

(c)

$$f_{X|Y}(x \mid \frac{3}{2}) \;=\; \frac{f_{X,Y}(x,\frac{3}{2})}{f_Y(\frac{3}{2})} \;=\; \frac{8}{7}x, \qquad \text{for } \tfrac{3}{2} \le x \le 2 \text{ and } 0 \text{ otherwise.}$$

Then,

$$\mathbf{E}\left[\frac{1}{X} \mid Y = \frac{3}{2}\right] \;=\; \int_{3/2}^{2} \frac{1}{x}\frac{8}{7}x\,dx \;=\; \frac{4}{7}.$$

(d) We use the technique of first finding the CDF and differentiating it to get the PDF.

$$\begin{aligned} F_Z(z) &= \mathbf{P}(Z \le z) \\ &= \mathbf{P}(Y - X \le z) \\ &= \begin{cases} 0, & \text{if } z < -2, \\ \displaystyle\int_{x=-z}^{x=2}\int_{y=0}^{y=x+z} \frac{3}{7}x\,dy\,dx \;= \frac{8}{7} + \frac{6}{7}z - \frac{1}{14}z^3, & \text{if } -2 \le z \le -1, \\ \displaystyle\int_{x=1}^{x=2}\int_{y=0}^{y=x+z} \frac{3}{7}x\,dy\,dx \;= 1 + \frac{9}{14}z, & \text{if } -1 < z \le 0, \\ 1, & \text{if } 0 < z. \end{cases} \end{aligned}$$

$$f_Z(z) \;=\; \frac{d}{dz}F_Z(z) \;=\; \begin{cases} \frac{6}{7} - \frac{3}{14}z^2, & \text{if } -2 \le z \le -1, \\ \frac{9}{14}, & \text{if } -1 < z \le 0, \\ 0, & \text{otherwise.} \end{cases}$$

2. The PDF of Z, $f_Z(z)$, can be readily computed using the convolution integral:

$$f_Z(z) \;=\; \int_{-\infty}^{\infty} f_X(t) f_Y(z-t)\,dt.$$

For $z \in [-1,0]$,

$$f_Z(z) \;=\; \int_{-1}^{z} \frac{1}{3}\cdot\frac{3}{4}(1-t^2)\,dt \;=\; \frac{1}{4}\left(z - \frac{z^3}{3} + \frac{2}{3}\right).$$

For $z \in [0,1]$,

$$f_Z(z) \;=\; \int_{z-1}^{z} \frac{1}{3}\cdot\frac{3}{4}(1-t^2)\,dt \;=\; \frac{1}{4}\left(1 - \frac{z^3}{3} + \frac{(z-1)^3}{3}\right).$$

For $z \in [1,2]$,

$$f_Z(z) \;=\; \int_{z-1}^{1} \frac{1}{3}\cdot\frac{3}{4}(1-t^2)\,dt + \int_{-1}^{z-2} \frac{2}{3}\cdot\frac{3}{4}(1-t^2)\,dt \;=\; \frac{1}{4}\left(z + \frac{(z-1)^3}{3} - \frac{2(z-2)^3}{3} - 1\right).$$

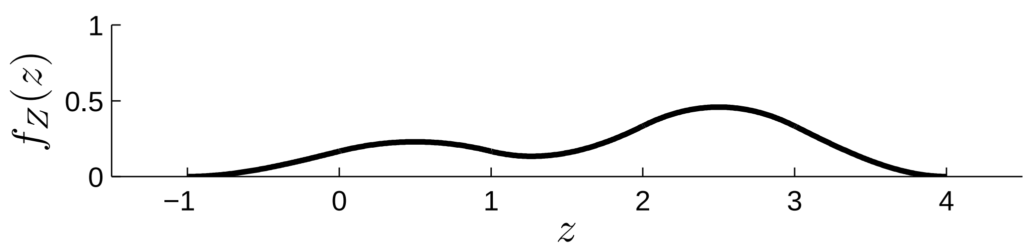
For $z \in [2,3]$,

$$f_Z(z) \;=\; \int_{z-3}^{z-2} \frac{2}{3}\cdot\frac{3}{4}(1-t^2)\,dt \;=\; \frac{1}{6}\left(3 + (z-3)^3 - (z-2)^3\right).$$

For $z \in [3,4]$,

$$f_Z(z) \;=\; \int_{z-3}^{1} \frac{2}{3}\cdot\frac{3}{4}(1-t^2)\,dt \;=\; \frac{1}{6}\left(11 - 3z + (z-3)^3\right).$$

A sketch of $f_Z(z)$ is provided below.



3. (a) X_1 and X_2 are negatively correlated. Intuitively, a large number of tosses that result in a 1 suggests a smaller number of tosses that result in a 2.

(b) Let A_t (respectively, B_t) be a Bernoulli random variable that is equal to 1 if and only if the tth toss resulted in 1 (respectively, 2). We have $\mathbf{E}[A_t B_t] = 0$ (since $A_t \neq 0$ implies $B_t = 0$) and

$$\mathbf{E}[A_t B_s] = \mathbf{E}[A_t]\mathbf{E}[B_s] = \frac{1}{k}\cdot\frac{1}{k} \qquad \text{for} \quad s \neq t.$$

Thus,

$$\begin{aligned}\mathbf{E}[X_1 X_2] &= \mathbf{E}\left[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)\right] \\ &= n\mathbf{E}\left[A_1(B_1 + \cdots + B_n)\right] \;=\; n(n-1)\cdot\frac{1}{k}\cdot\frac{1}{k}\end{aligned}$$

and

$$\begin{aligned}\operatorname{cov}(X_1, X_2) &= \mathbf{E}[X_1 X_2] - \mathbf{E}[X_1]E[X_2] \\ &= \frac{n(n-1)}{k^2} - \frac{n^2}{k^2} = -\frac{n}{k^2}.\end{aligned}$$

The covariance of X_1 and X_2 is negative as expected.

4. (a) If X takes a value x between -1 and 1, the conditional PDF of Y is uniform between -2 and 2. If X takes a value x between 1 and 2, the conditional PDF of Y is uniform between -1 and 1.

   Similarly, if Y takes a value y between -1 and 1, the conditional PDF of X is uniform between -1 and 2. If Y takes a value y between 1 and 2, or between -2 and -1, the conditional PDF of X is uniform between -1 and 1.

   (b) We have

$$\mathbf{E}[X \mid Y = y] = \begin{cases} 0, & \text{if } -2 \le y \le -1, \\ 1/2, & \text{if } -1 < y \le 1, \\ 0, & \text{if } 1 \le y \le 2, \end{cases}$$

   and

$$\operatorname{var}(X \mid Y = y) = \begin{cases} 1/3, & \text{if } -2 \le y \le -1, \\ 3/4, & \text{if } -1 < y \le 1, \\ 1/3, & \text{if } 1 \le y \le 2. \end{cases}$$

   It follows that $\mathbf{E}[X] = 3/10$ and $\operatorname{var}(X) = 193/300$.

   (c) By symmetry, we have $\mathbf{E}[Y \mid X] = 0$ and $\mathbf{E}[Y] = 0$. Furthermore, $\operatorname{var}(Y \mid X = x)$ is the variance of a uniform PDF (whose range depends on x), and

$$\operatorname{var}(Y \mid X = x) = \begin{cases} 4/3, & \text{if } -1 \le x \le 1, \\ 1/3, & \text{if } 1 < x \le 2. \end{cases}$$

   Using the law of total variance, we obtain

$$\operatorname{var}(Y) = \mathbf{E}[\operatorname{var}(Y \mid X)] = \frac{4}{5} \cdot \frac{4}{3} + \frac{1}{5} \cdot \frac{1}{3} = 17/15.$$

5. First let us write out the properties of all of our random variables. Let us also define K to be the number of members attending a meeting and B to be the Bernoulli random variable describing whether or not a member attends a meeting.

$$\begin{aligned} \mathbf{E}[N] &= \frac{1}{1-p}, & \operatorname{var}(N) &= \frac{p}{(1-p)^2}, \\ \mathbf{E}[M] &= \frac{1}{\lambda}, & \operatorname{var}(M) &= \frac{1}{\lambda^2}, \\ \mathbf{E}[B] &= q, & \operatorname{var}(B) &= q(1-q). \end{aligned}$$

   (a) Since $K = B_1 + B_2 + \cdots B_N$,

$$\begin{aligned} \mathbf{E}[K] &= \mathbf{E}[N] \cdot \mathbf{E}[B] = \frac{q}{1-p}, \\ \operatorname{var}(K) &= \mathbf{E}[N] \cdot \operatorname{var}(B) + (\mathbf{E}(B))^2 \cdot \operatorname{var}(N) = \frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}. \end{aligned}$$

(b) Let G be the total money brought to the meeting. Then $G = M_1 + M_2 + \cdots + M_K$.

$$\begin{aligned}
\mathbf{E}[G] &= \mathbf{E}[M] \cdot \mathbf{E}[K] = \frac{q}{\lambda(1-p)}, \\
\text{var}(G) &= \text{var}(M) \cdot \mathbf{E}[K] + (\mathbf{E}[M])^2 \text{var}(K) \\
&= \frac{q}{\lambda^2(1-p)} + \frac{1}{\lambda^2}\left(\frac{q(1-q)}{1-p} + \frac{pq^2}{(1-p)^2}\right).
\end{aligned}$$

G1†. (a) Let $X_1, X_2, \ldots X_n$ be independent, identically distributed (IID) random variables. We note that

$$\mathbf{E}[X_1 + \cdots + X_n \mid X_1 + \cdots + X_n = x_0] = x_0.$$

It follows from the linearity of expectations that

$$\begin{aligned}
x_0 &= \mathbf{E}[X_1 + \cdots + X_n \mid X_1 + \cdots + X_n = x_0] \\
&= \mathbf{E}[X_1 \mid X_1 + \cdots + X_n = x_0] + \cdots + \mathbf{E}[X_n \mid X_1 + \cdots + X_n = x_0]
\end{aligned}$$

Because the X_i's are identically distributed, we have the following relationship.

$$\mathbf{E}[X_i \mid X_1 + \cdots + X_n = x_0] = \mathbf{E}[X_j \mid X_1 + \cdots + X_n = x_0], \text{ for any } 1 \le i \le n, 1 \le j \le n.$$

Therefore,

$$\begin{aligned}
n\mathbf{E}[X_1 \mid X_1 + \cdots + X_n = x_0] &= x_0 \\
\mathbf{E}[X_1 \mid X_1 + \cdots + X_n = x_0] &= \frac{x_0}{n}.
\end{aligned}$$

(b) Note that we can rewrite $\mathbf{E}[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}]$ as follows:

$$\begin{aligned}
&\mathbf{E}[X_1 \mid S_n = s_n, S_{n+1} = s_{n+1}, \ldots, S_{2n} = s_{2n}] \\
=\ &\mathbf{E}[X_1 \mid S_n = s_n, X_{n+1} = s_{n+1} - s_n, X_{n+2} = s_{n+2} - s_{n+1}, \ldots, X_{2n} = s_{2n} - s_{2n-1}] \\
=\ &\mathbf{E}[X_1 \mid S_n = s_n],
\end{aligned}$$

where the last equality holds due to the fact that the X_i's are independent. We also note that

$$\mathbf{E}[X_1 + \cdots + X_n \mid S_n = s_n] = \mathbf{E}[S_n \mid S_n = s_n] = s_n.$$

It follows from the linearity of expectations that

$$\mathbf{E}[X_1 + \cdots + X_n \mid S_n = s_n] = \mathbf{E}[X_1 \mid S_n = s_n] + \cdots + \mathbf{E}[X_n \mid S_n = s_n].$$

Because the X_i's are identically distributed, we have the following relationship:

$$\mathbf{E}[X_i \mid S_n = s_n] = \mathbf{E}[X_j \mid S_n = s_n], \text{ for any } 1 \le i \le n, 1 \le j \le n.$$

Therefore,

$$\mathbf{E}[X_1 + \cdots + X_n \mid S_n = s_n] = n\mathbf{E}[X_1 \mid S_n = s_n] = s_n \Rightarrow \mathbf{E}[X_1 \mid S_n = s_n] = \frac{s_n}{n}.$$

†Required for 6.431; optional for 6.041