Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2006)

Tutorial 9 Answers

April 20-21, 2006

1. $\mathbf{P}(\mathrm{D} > \alpha) = \mathbf{P}(|(X-\mu)/\mu| > \alpha) = \mathbf{P}(|X-\mu| > \alpha\mu)$
   Using Chebyshev Inequality,

$$\mathbf{P}(|X-\mu| > \alpha\mu) \leq \frac{\sigma^2}{\alpha^2\mu^2} = \frac{1}{r^2\alpha^2}$$

   Therefore,

$$\mathbf{P}(D > \alpha) \leq \frac{1}{r^2\alpha^2}$$

$$\mathbf{P}(D \leq \alpha) \geq 1 - \frac{1}{r^2\alpha^2}$$

2. (a) Let X_i be random variables indicating the quality of the ith bulb ("1" for good bulbs, "0" for bad ones). Then X_i are independent Bernoulli random variables. Let Z_n be

$$Z_n = \frac{X_1 + X_2 + ... + X_n}{n}.$$

   We apply the Chebyshev inquality and obtain

$$\mathbf{P}\left(|Z_n - p| \geq \epsilon\right) \leq \frac{\sigma^2}{n\epsilon^2},$$

   where σ^2 is the variance of the Bernoulli random variable. Hence, we obtain

$$\lim_{n\to\infty} \mathbf{P}\left(|Z_n - p| \geq \epsilon\right) = 0,$$

   by noticing $\lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} = 0$. This means that Z_n converges to p in probability.

   (b) For any number greater than 500, we know the number of bulbs would be enough for the test by using Chebyshev. Since the variance of a Bernoulli random variable is $p(1-p)$ which is less than or equal to $\frac{1}{4}$, we have $\sigma^2 \leq \frac{1}{4}$. Hence, for $n \geq 500$,

$$\begin{aligned}
\mathbf{P}\left(\left|\frac{X_1 + X_2 + ... + X_n}{n} - p\right| \geq 0.1\right) &\leq \frac{\sigma^2}{n0.1^2} \\
&\leq \frac{\frac{1}{4}}{n \times 0.1^2} \\
&\leq 1 - 0.95 = 0.05.
\end{aligned}$$

   However, for a number less than 500, we can not tell if the number of bulbs is enough for the test because we don't know the variance. If the variance is very small, which is possible when p is quite small, 27 bulbs could be enough actually.

   Thus, the answer is "cannot be decided". In reality, we need to estimate the variance first.

3. (a) $E[X_n] = \frac{1}{n}$
   $var(X_n) = \frac{n-1}{n^2}$
   $E[Y_n] = 1$
   $var(Y_n) = n - 1$

   (b) X_n is convergent in probability

   (c) In this case, Chebyshev tells us nothing.

   (d) Yes, to zero.

   (e) Yes, to zero.