LECTURE 20

- Readings: Section 6.3

Lecture outline

- Markov Processes – II
  - Markov process review.
  - Steady-state behavior.
  - Birth-death processes.

Review

- Discrete state, discrete time, time-homogeneous
  - Transition probabilities p_{ij}
  - Markov property

$$p_{ij} = \mathbf{P}(X_{n+1} = j | X_n = i, X_{n-1}, \cdots, X_0)$$
$$= \mathbf{P}(X_{n+1} = j | X_n = i)$$

- State occupancy probabilities, given initial state i :

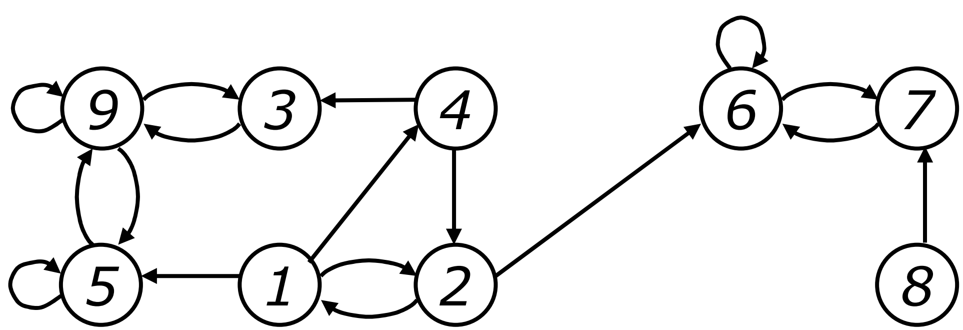
$$r_{ij}(n) = \mathbf{P}(X_n = j | X_0 = i)$$

- Key recursion:

$$r_{ij}(n) = \sum_{k=1}^{m} r_{ik}(n-1) p_{kj}$$

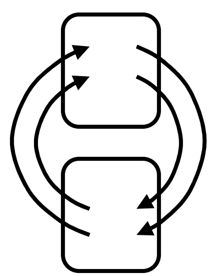
Recurrent and Transient States

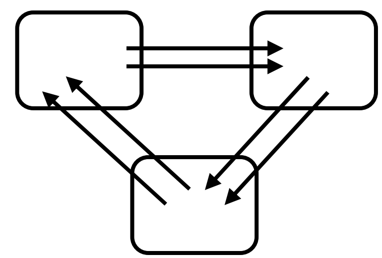
- State i is **recurrent** if:
  - Starting from i, and from wherever you can go, there is a way of returning to i.
- If not recurrent, a state is called **transient**.
  - If i is transient then $\mathrm{P}(X_n = i) \to 0$ as $n \to \infty$.
  - State i is visited a finite number of times.
- **Recurrent Class**:
  - Collection of recurrent states that "communicate" to each other, and to no other state.

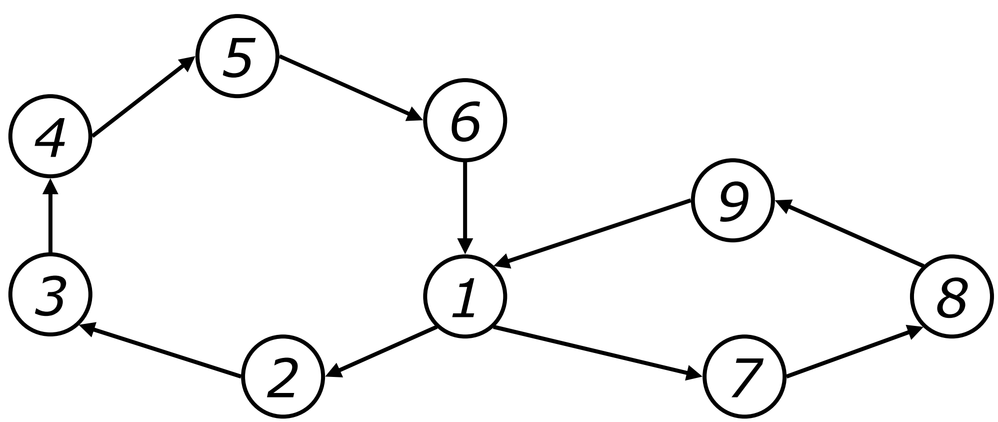


Periodic States

- The states in a recurrent class are **periodic** if:
  - They can be grouped into $d > 1$ groups so that all transitions from one group lead to the next group.







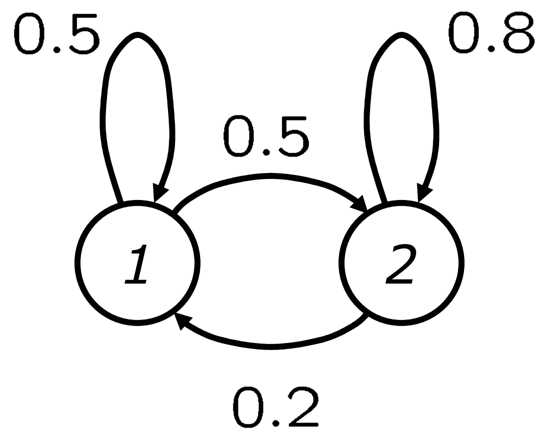
Steady-State Probabilities

- Do the $r_{ij}(n)$ converge to some π_j ? (independent of the initial state i)

- Yes, if:
  - Recurrent states are all in a single class, and
  - No periodicity.

- Start from key recursion: $r_{ij}(n) = \sum_k r_{ik}(n-1)p_{kj}$
  - Take the limit as $n \to \infty$: $\pi_j = \sum_k \pi_k p_{kj}$
  - Additional equation: $\sum_j \pi_j = 1$

Example

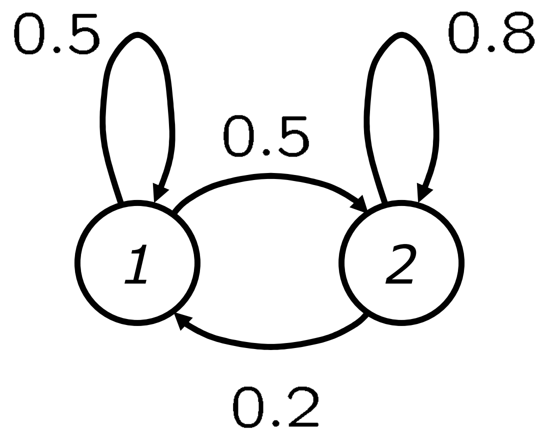


$$\pi_1 = 0.5\pi_1 + 0.2\pi_2$$

$$\pi_2 = 0.5\pi_1 + 0.8\pi_2$$

$$\pi_1 + \pi_2 = 1$$

Example



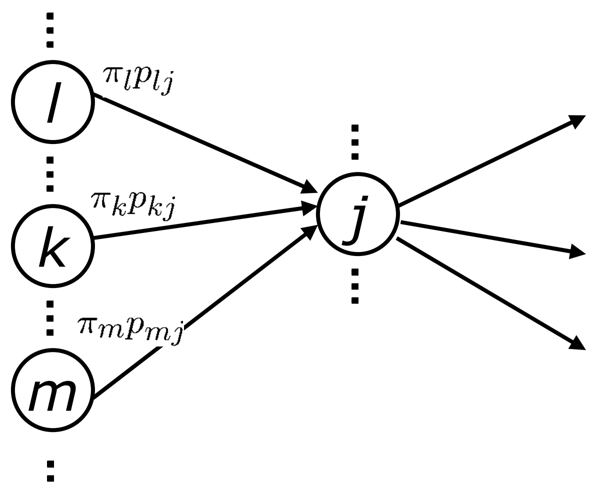
$$\pi_1 = 2/7 \qquad \pi_2 = 5/7$$

- Assume process starts at state 1.
- $\mathbf{P}(X_1 = 1, \text{ and } X_{100} = 1) =$ $2/7$
- $\mathbf{P}(X_{100} = 1, \text{ and } X_{101} = 2) =$ $(\frac{2}{7})(\frac{1}{2}) = 1/7$

Visit Frequency Interpretation

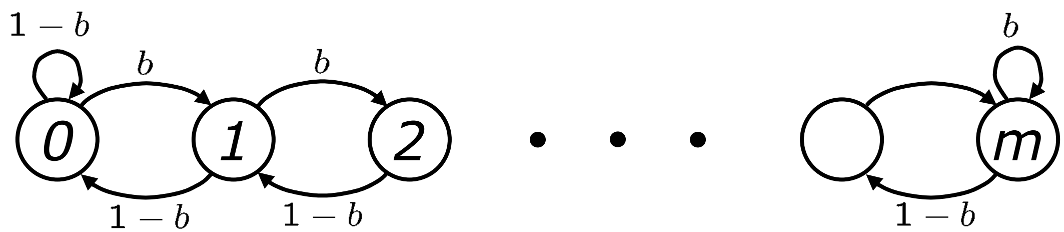
$$\pi_j = \sum_k \pi_k p_{kj}$$

- (Long run) frequency of being in j : π_j
- Frequency of transitions $k \to j$: $\pi_k p_{kj}$
- Frequency of transitions into j : $\sum_k \pi_k p_{kj}$

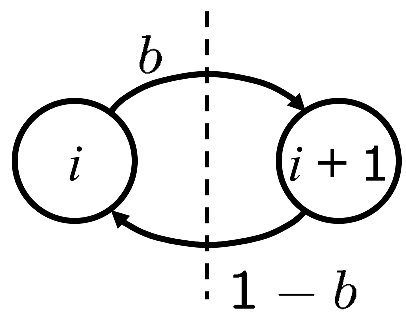


Random Walk (1)

- A person walks between two (m -spaced) walls:
  - To the right with probability b
  - To the left with probability $1-b$
  - Pushes against the walls with the same probabilities.



- Locally, we have:



- Balance equations: $\pi_i b = \pi_{i+1}(1-b)$

Random Walk (2)

- Justification:

$$\pi_0 = \pi_0(1-b) + \pi_1(1-b) \rightarrow$$

$$\pi_0 b = \pi_1(1-b)$$

$$\pi_1 = \pi_0(b) + \pi_1(0) + \pi_2(1-b) \rightarrow$$

$$\pi_1 b = \pi_2(1-b)$$

$$\vdots$$

Random Walk (3)

- Define: $\rho = \dfrac{b}{1-b}$
- Then: $\pi_{i+1} = \pi_i \dfrac{b}{1-b} = \pi_i \rho$

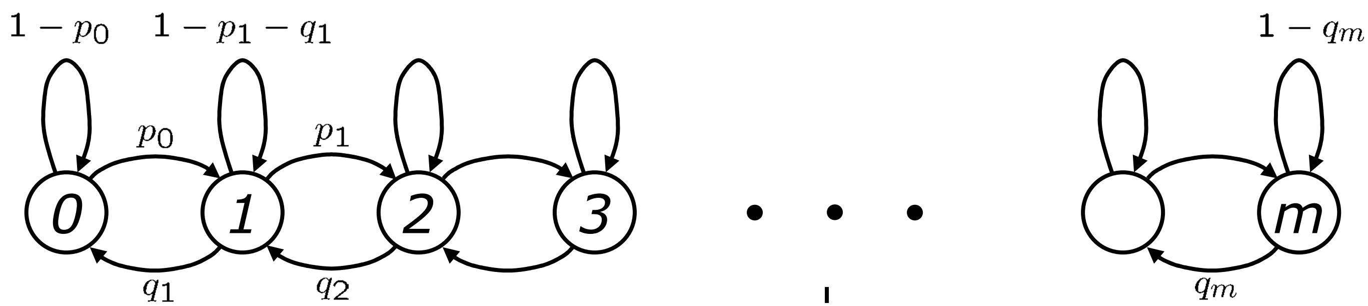
$$\pi_i = \pi_0 \rho^i, \quad i = 0, 1, \cdots, m$$

- To get π_0, use: $\sum_j \pi_j = 1$

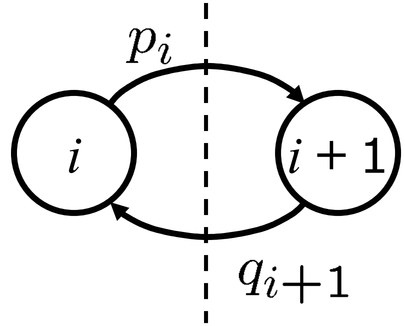
$$\pi_0 = \frac{1}{1 + \rho + \cdots + \rho^m} = \frac{1-\rho}{1-\rho^{m+1}}$$

Birth-Death Process (1)

- General (state-varying) case:



- Locally, we have:



- Balance equations: $\pi_i p_i = \pi_{i+1} q_{i+1}$

- Why? (More powerful, e.g. queues, etc.)

Birth-Death Process (2)

- Special case: $p_i = p$ and $q_i = q$ for all i and, again, define $\rho = p/q$ (called "load factor").
  - Less general (but more so than the random walk).

$$\pi_{i+1} = \pi_i \frac{p}{q} = \pi_i \rho$$

$$\pi_i = \pi_0 \rho^i, \qquad i = 0, 1, \cdots, m$$

- Assume $p < q$ and $m \approx \infty$

$$\pi_0 = 1 - \rho$$

$$\mathbf{E}[X_n] = \frac{\rho}{1-\rho} \qquad \text{(in steady-state)}$$