Tutorial 3: Solutions

1. In general we have that $\mathbf{E}[aX + bY + c] = a\mathbf{E}[X] + b\mathbf{E}[Y] + c$. Therefore,

$$\mathbf{E}[Z] = 2 \cdot \mathbf{E}[X] - 3 \cdot \mathbf{E}[Y].$$

For the case of independent random variables, we have that if $Z = a \cdot X + b \cdot Y$, then

$$\text{var}(Z) = a^2 \cdot \text{var}(X) + b^2 \cdot \text{var}(Y).$$

Therefore, $\text{var}(Z) = 4 \cdot \text{var}(X) + 9 \cdot \text{var}(Y)$.

2. See online solutions.

3. (a) We can find c knowing that the probability of the entire sample space must equal 1.

$$\begin{aligned} 1 &= \sum_{x=1}^{3}\sum_{y=1}^{3} p_{X,Y}(x,y) \\ &= c + c + 2c + 2c + 4c + 3c + c + 6c \\ &= 20c \end{aligned}$$

Therefore, $c = \frac{1}{20}$.

(b) $p_Y(2) = \sum_{x=1}^{3} p_{X,Y}(x,2) = 2c + 0 + 4c = 6c = \frac{3}{10}$.

(c) $Z = YX^2$

$$\begin{aligned} \mathbf{E}[Z \mid Y = 2] &= \mathbf{E}[YX^2 \mid Y = 2] \\ &= \mathbf{E}[2X^2 \mid Y = 2] \\ &= 2\mathbf{E}[X^2 \mid Y = 2] \end{aligned}$$

$p_{X|Y}(x \mid 2) = \frac{p_{X,Y}(x,2)}{p_Y(2)}$.
Therefore,

$$p_{X|Y}(x \mid 2) = \begin{cases} \frac{1/10}{3/10} = \frac{1}{3} & \text{if } x = 1 \\ \frac{1/5}{3/10} = \frac{2}{3} & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathbf{E}[Z \mid Y = 2] &= 2\sum_{x=1}^{3} x^2 p_{X|Y}(x \mid 2) \\ &= 2\left((1^2) \cdot \frac{1}{3} + (3^2) \cdot \frac{2}{3}\right) \\ &= \frac{38}{3} \end{aligned}$$

(d) Yes. Given $X \neq 2$, the distribution of X is the same given $Y = y$.
$\mathbf{P}(X = x \mid Y = y, X \neq 2) = \mathbf{P}(X = x \mid X \neq 2)$.
For example,

$$\mathbf{P}(X = 1 \mid Y = 1, X \neq 2) = \mathbf{P}(X = 1 \mid Y = 3, X \neq 2) = \mathbf{P}(X = 1 \mid X \neq 2) = \frac{1}{3}$$

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(e) $p_{Y|X}(y \mid 2) = \frac{p_{X,Y}(2,y)}{p_X(2)}$.
$p_X(2) = \sum_{y=1}^{3} p_{X,Y}(2, y) = c + 0 + c = 2c = \frac{1}{10}$.
Therefore,

$$p_{Y|X}(y \mid 2) \quad = \quad \begin{cases} \frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 1 \\ \frac{1/20}{1/10} = \frac{1}{2} & \text{if } y = 3 \\ 0 & \text{otherwise} \end{cases}$$

$\mathbf{E}[Y^2 \mid X = 2] = \sum_{y=1}^{3} y^2 p_{Y|X}(y \mid 2) = (1^2) \cdot \frac{1}{2} + (3^2) \cdot \frac{1}{2} = 5.$
$\mathbf{E}[Y \mid X = 2] = \sum_{y=1}^{3} y p_{Y|X}(y \mid 2) = 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 2.$
$\text{var}(Y \mid X = 2) = \mathbf{E}[Y^2 \mid X = 2] - \mathbf{E}[Y \mid X = 2]^2 = 5 - 2^2 = 1.$

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