6.041/6.431 Fall 2009 Quiz 1
Tuesday, October 13, 12:05 - 12:55 PM.

DO NOT TURN THIS PAGE OVER UNTIL YOU ARE TOLD TO DO SO

Question	Score	Out of
A		2
B.1		10
B.2 (a)		10
B.2 (b i)		12
B.2 (b ii)		12
B.2 (c)		10
B.3 (a)		10
B.3 (b)		12
B.3 (c)		12
B.3 (d i)		5
B.3 (d ii)		5
Your Grade		100

Name: ______________________________

Recitation Instructor: ______________________________

TA: ______________________________

- This quiz has 2 problems, worth a total of 100 points.
- You may tear apart pages 3 and 4, as per your convenience.
- Write your solutions in this quiz booklet, only solutions in this quiz booklet will be graded. Be neat! You will not get credit if we can't read it.
- You are allowed one two-sided, handwritten, 8.5 by 11 formula sheet. Calculators are not allowed.
- Parts B.2 and B.3 can be done independently.
- You may give an answer in the form of an arithmetic expression (sums, products, ratios, factorials) of <u>numbers</u> that could be evaluated using a calculator. Expressions like $\binom{8}{3}$ or $\sum_{k=0}^{5}(1/2)^k$ are also fine.
- You have 50 minutes to complete the quiz.
- Graded quizzes will be returned in recitation on Thursday 10/15.

Summary of Results for Special Random Variables

Discrete Uniform over $[a, b]$:

$$p_X(k) = \begin{cases} \dfrac{1}{b-a+1}, & \text{if } k = a, a+1, \ldots, b, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{E}[X] = \frac{a+b}{2}, \qquad \text{var}(X) = \frac{(b-a)(b-a+2)}{12}.$$

Bernoulli with Parameter p: (Describes the success or failure in a single trial.)

$$p_X(k) = \begin{cases} p, & \text{if } k = 1, \\ 1-p, & \text{if } k = 0, \end{cases}$$

$$\mathbf{E}[X] = p, \qquad \text{var}(X) = p(1-p).$$

Binomial with Parameters p and n: (Describes the number of successes in n independent Bernoulli trials.)

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n,$$

$$\mathbf{E}[X] = np, \qquad \text{var}(X) = np(1-p).$$

Geometric with Parameter p: (Describes the number of trials until the first success, in a sequence of independent Bernoulli trials.)

$$p_X(k) = (1-p)^{k-1} p, \qquad k = 1, 2, \ldots,$$

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \text{var}(X) = \frac{1-p}{p^2}.$$

Problem B: (98 points) As a way to practice his probability skills, Bob goes apple picking. The orchard he goes to grows two varieties of apples: gala and honey crisp.

The **proportion of the gala apples** in the orchard is p $(0 < p < 1)$, the **proportion of the honey crisp apples** is $1 - p$. The number of apples in the orchard is so large that you can assume that picking a few apples does not change the proportion of the two varieties.

Independent of all other apples, the probability that a randomly picked **gala apple is ripe** is g and the probability that a randomly picked **honey crisp apple is ripe** is h.

1. **(10 points)** Suppose that Bob picks an apple at random (uniformly) and eats it. Find the probability that it was a **ripe gala** apple.

   Note: Parts 2 and 3 below can be done independently.

2. Suppose that Bob picks n apples at random (independently and uniformly).
   (a) **(10 points)** Find the probability that exactly k of those are **gala apples.**
   (b) Suppose that there are **exactly** k **gala apples** among the n apples Bob picked. Caleb comes by and gives Bob a **ripe gala** apple to add to his bounty. Bob then picks an apple at random from the $n + 1$ apples and eats it.
      (i) **(12 points)** What is the probability that it was a **ripe apple?**
      (ii) **(12 points)** What is the probability that it was a **gala apple if it was ripe?**
   (c) **(10 points)** Let $n = 20$, and suppose that Bob picked exactly 10 gala apples. What is the probability that the first 10 apples that Bob picked were all gala?

3. Next, Bob tries a different strategy. He starts with a tree of the **gala** variety and picks apples at random from that tree. Once Bob picks an apple off the tree, he carefully examines it to make sure it is ripe. Once he comes across an apple that is not ripe, he moves to **another gala tree.** He does this until he encounters an unripe apple on that second tree. Assume that each tree has a very large, essentially infinite, number of apples.
   (a) **(10 points)** Let X_i be the **number of apples** Bob picks off the ith tree, $(i = 1, 2)$. Write down the PMF, expectation, and variance of X_i.
   (b) **(12 points)** For $i = 1, 2$, let Y_i be the **total** number of **ripe apples** Bob picked from the first i trees. Find the expectation and the variance of Y_2. (Note that $Y_1 = X_1 - 1$ and $Y_2 = (X_1 - 1) + (X_2 - 1)$.)
   (c) **(12 points)** Find the joint PMF of Y_1 and Y_2.
   (d) In the following, answer just "yes" or "no." (Explanations will not be taken into account in grading.)
      (i) **(5 points)** Are X_1 and Y_2 independent?
      (ii) **(5 points)** Are X_2 and Y_1 independent?

Each question is repeated in the following pages. Please write your answer on the appropriate page.

1. **(10 points)** Suppose that Bob picks an apple at random (uniformly) and eats it. Find the probability that it was a **ripe gala** apple.

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(ii) **(5 points)** Are X_2 and Y_1 independent?