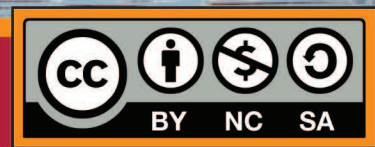


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Introduction to Electrical Engineering
and Computer Science I

by Leslie Kaelbling & figures
by Dennis Freeman



Chapter 1

Course Overview

1.1 Goals for 6.01

We have many goals for this course. Our primary goal is for you to learn to appreciate and use the fundamental design principles of modularity and abstraction in a variety of contexts from electrical engineering and computer science. To achieve this goal, we will study electrical engineering (EE) and computer science (CS) largely from the perspective of how to build systems that interact with, and attempt to control, an external environment. Such systems include everything from low-level controllers like heat regulators or cardiac pacemakers, to medium-level systems like automated navigation or virtual surgery, to high-level systems that provide more natural human-computer interfaces.

Our second goal is to show you that making mathematical models of real systems can help in the design and analysis of those systems; and to give you practice with the difficult step of deciding which aspects of the real world are important to the problem being solved and how to model them in ways that give insight into the problem.

We also hope to engage you more actively in the educational process. Most of the work of this course will not be like typical problems from the end of a chapter. You will work individually and in pairs to solve problems that are deeper and more open-ended. There will not be a unique right answer. Argument, explanation, and justification of approach will be more important than the answer. We hope to expose you to the ubiquity of trade-offs in engineering design: it is rare that an approach will be best in every dimension; some will excel in one way, others in a different way. Deciding how to make such trade-offs is a crucial part of engineering.

Another way in which we hope to engage you in the material is by having many of you return to the course as lab assistants in future semesters. Having a large number of lab assistants in the class means that students can be given more open-ended problems, and have people around to help them when they are stuck. Even more importantly, the lab assistants are meant to question the students as they go; to challenge their understanding and help them see and evaluate a variety of approaches. This process is of great intellectual value to student and lab assistant alike.

Finally, of course, we have the more typical goals of teaching exciting and important basic material from electrical engineering and computer science, including modern software engineering, linear systems analysis, electronic circuits, and decision-making. This material all has an internal elegance and beauty, as well as crucial role in building modern EE and CS systems.

1.2 Modularity, abstraction, and modeling

Whether proving a theorem by building up from lemmas to basic theorems to more specialized results, or designing a circuit by building up from components to modules to complex processors, or designing a software system by building up from generic procedures to classes to class libraries, humans deal with complexity by exploiting the power of abstraction and modularity. Without such tools, a single person would be overwhelmed by the complexity of a system, as there is only so much detail that a single person can consciously manage at a time.

Modularity is the idea of building components that can be re-used; and *abstraction* is the idea that after constructing a module (be it software or circuits or gears), most of the details of the module construction can be ignored and a simpler description used for module interaction (the module computes the square root, or doubles the voltage, or changes the direction of motion).

Given basic modules, one can move up a level of abstraction and construct a new module by putting together several previously-built modules, thinking only of their abstract descriptions, and not their implementations. And, of course, this process can be repeated over many stages. This process gives one the ability to construct systems with complexity far beyond what would be possible if it were necessary to understand each component in detail.

Any module can be described in a large number of ways. We might describe the circuitry in a digital watch in terms of how it behaves as a clock and a stopwatch, or in terms of voltages and currents within the circuit, or in terms of the heat produced at different parts of the circuitry. Each of these is a different *model* of the watch. Different models will be appropriate for different tasks: there is no single correct model. Rather, each model exposes different dimensions of the system, allowing us to explore many aspects of the design space of a system, and to trade off different factors in the performance of a system.

The primary theme of this course will be to learn about different methods for building modules out of primitives, and of building different abstract models of them, so that we can analyze or predict their behavior, and so we can recombine them into even more complex systems. The same fundamental principles will apply to software, to control systems, and to circuits.

1.2.1 Example problem

Imagine that you need to make a robot that will roll up close to a light bulb and stop a fixed distance from it. The first question is, how can we get electrical signals to relate to the physical phenomena of light readings and robot wheel rotations? There is a large part of electrical engineering related to the design of physical devices that connect to the physical world in such a way that some electrical property of the device relates to a physical process in the world. For example, a light-sensitive resistor (photo-resistor) is a sensor whose resistance changes depending on light intensity impinging on it; a motor is an effector whose rotor speed is related to the voltage across its two terminals. In this course, we will not examine the detailed physics of sensors and effectors, but will concentrate on ways of designing systems that use sensors and effectors to perform both simple and more complicated tasks. To get a robot to stop in front of a light bulb, the problem will be to find a way to connect the photo-resistor to the motor, so that the robot will stop at an appropriate distance from the bulb. Thus, we will already use the idea of abstraction to treat sensors

and effectors as primitive modules whose internal details we can ignore, and whose performance characteristics we can use as we design systems built on these elements.

1.2.2 An abstraction hierarchy of mechanisms

Given the light-sensitive resistor and the motor, we could find many ways of connecting them, using bits of metal and ceramic of different kinds, or making some kind of magnetic or mechanical linkages. The space of possible designs of machines is enormous.

One of the most important things that engineers do, when faced with a set of design problems, is to standardize on a *basis set* of components to use to build their systems. There are many reasons for standardizing on a basis set of components, mostly having to do with efficiency of understanding and of manufacturing. It is important, as a designer, to develop a repertoire of standard bits and pieces of designs that you understand well and can put together in various ways to make more complex systems. If you use the same basis set of components as other designers, you can learn valuable techniques from them, rather than having to re-invent the techniques yourself. And other people will be able to readily understand and modify your designs.

We can often make a design job easier by limiting the space of possible designs, and by standardizing on:

- a basis set of *primitive* components;
- ways of *combining* the primitive components to make more complex systems;
- ways of “*packaging*” or *abstracting* pieces of a design so they can be reused (in essence creating new “*primitives*”); and
- ways of capturing common *patterns* of abstraction (essentially, abstracting our abstractions).

Very complicated design problems can become tractable using such a *primitive-combination-abstraction-pattern* (PCAP) approach. In this class, we will examine and learn to use a variety of PCAP strategies common in EE and CS, and will even design some of our own, for special purposes. In the rest of this section, we will hint at some of the PCAP systems we will be developing in much greater depth throughout the class. [Figure 1.1](#) shows one view of this development, as a successive set of restrictions of the design space of mechanisms.

One very important thing about abstract models is that once we have fixed the abstraction, it will usually be possible to implement it using a variety of different underlying substrates. So, as shown in [figure 1.2](#), we can construct general-purpose computers out of a variety of different kinds of systems, including digital circuits and general-purpose computers. And systems satisfying the digital circuit abstraction can be constructed from analog circuits, but also from gears or water or light.

Another demonstration of the value of abstracted models is that we can use them, by analogy, to describe quite different systems. So, for example, the constraint models of circuits that we will study can be applied to describing transmission of neural signals down the spinal cord, or of heat through a network of connected bodies.

Let’s explore the abstraction hierarchy [figure 1.1](#) in some more detail, moving up abstraction levels while observing common patterns.

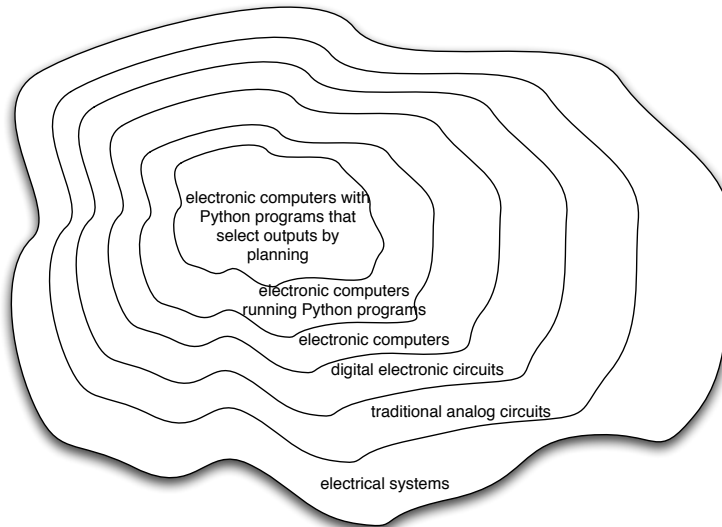


Figure 1.1 Increasingly constrained systems.

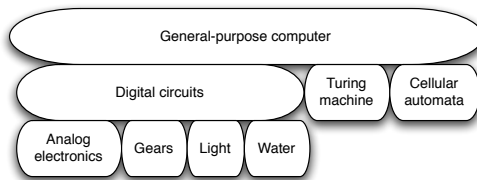


Figure 1.2 A single abstraction may have a variety of different underlying implementations.

Circuits

Typical electronics circuits are built out of a basis set of primitive components such as voltage sources, resistors, capacitors, inductors and transistors. This set of component types was chosen to be closely related to primitive concepts in our physical understanding of electronic systems in terms of voltage, current, resistance, and the rate of change of those quantities over time. So, when we buy a physical resistor, we tend to think only of its resistance; and when we buy a capacitor, we think of its ability to store charge. But, of course, that physical resistor has some capacitance, and the capacitor, some resistance. Still, it helps us in our designs to have devices that are as close to the ideal models as possible; and it helps that people have been designing with these components for years, so that we can adopt the strategies that generations of clever people have developed.

The method of combining circuit elements is to connect their terminals with conductors (wires, crimps, solder, etc.), a process that is generically referred to as *wiring*. And our method of abstraction is to describe the *constraints* that a circuit element exerts on the currents and voltages of terminals to which the element is connected.

So, armed with the standard basis set of analog circuit components, we can try to build a circuit to control our robot. We have a resistance that varies with the light level, but we need a voltage that does so, as well. We can achieve this by using a *voltage divider*, which is shown in [figure 1.3A](#). Using an abstract, constraint-based model of the behavior of circuit components that we will

study in detail later in the course, we can determine the following relationship between V_{out} and V_{in} :

$$V_{out} = \frac{R_B}{R_A + R_B} V_{in} .$$

So, for example, if $R_A = R_B$, then $V_{out} = V_{in}/2$. Or, in our case, if R_A actually varies with the amount of light shining on it, then V_{out} will also vary with the light.¹

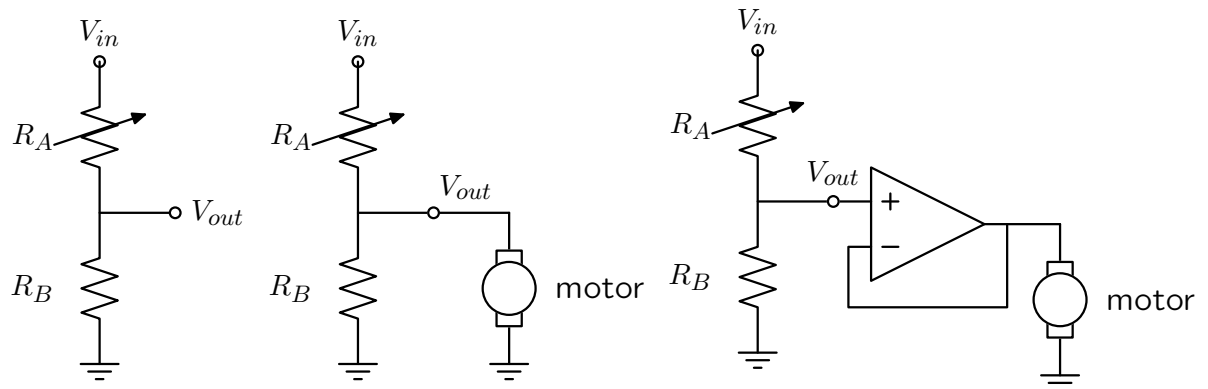


Figure 1.3 Voltage dividers: A. Resistor divider. B. Connected to a motor, in a way that *breaks* the abstraction from part A. C. Connected to a motor, with a buffer.

That is great. So, now, we might imagine that we could use this voltage difference that we have created to drive the motor at different speeds, depending on the light hitting the resistor, using a circuit something like the one shown in [figure 1.3B](#). But, sadly, that will not work. It turns out that once we connect the motor to the circuit, it actually changes the voltages, and we can no longer maintain the voltage difference needed to drive the motor.

So, although we have developed an abstract model of the behavior of circuit components, which lets us analyze the behavior of a particular complete circuit design, it does not give us *modularity*. That is, we cannot design two parts of a circuit, understand each of their behaviors, and then predict the behavior of the composite system based on the behavior of the separate components. Instead, we would have to re-analyze the joint behavior of the whole composite system. Lack of modularity makes it very difficult to design large systems, because two different people, or the same person at two different times, cannot design pieces and put them together without understanding the whole.

To solve this problem, we can augment our analog-circuit toolbox with some additional components that allow us to design components with modular behavior; they “buffer” or “isolate” one part of the circuit from another in ways that allow us to combine the parts more easily. In this class, we will use op-amps to build buffers, which will let us solve our sample problem using a slightly more complex circuit, as shown in [figure 1.3C](#).

Thus, the key point is that good modules preserve abstraction barriers between the use of a module and internal details of how they are constructed. We will see this theme recur as we discuss different PCAP systems.

¹ Do not worry if this example does not make much sense to you; we will explore this all in great detail later.

Digital circuits

In analog circuits, we think about voltages on terminals ranging over real values. This gives us the freedom to create an enormous variety of circuits, but sometimes that freedom actually makes the design process more difficult. To design ever more complex circuits, we can move to a much stronger, *digital* abstraction, in which the voltages on the terminals are thought of as only taking on values that are either “low” or “high” (often called 0 and 1). This PCAP system is made up of a basis set of elements, called gates, that are built out of simpler analog circuit components, such as transistors and resistors. Adopting the digital abstraction is a huge limitation on the kinds of circuits that can be built. However, digital circuits can be easy to design and build and also can be inexpensive because the basic elements (gates) are simple (fewer transistors are needed to construct a gate than to construct an op amp), versatile (only a small number of different kinds of logic elements are necessary to construct an arbitrary digital circuit), and combinations are easy to think about (using Boolean logic). These properties allow designers to build incredibly complex machines by designing small parts and putting them together into increasingly larger pieces. Digital watches, calculators, and computers are all built this way.

Digital design is a very important part of EECS, and it is treated in a number of our courses at basic and advanced levels, but is not one of the topics we will go into in detail in 6.01. Nevertheless, the same central points that we explore in this course apply to this domain as well.

Computers

One of the most important developments in the design of digital circuits is that of the general-purpose “stored program” computer. Computers are a particular class of digital circuits that are general purpose: the same actual circuit can perform (almost) any transformation between its inputs and its outputs. Which particular transformation it performs is governed by a program, which is some settings of physical switches, or information written on an external physical memory, such as cards or a tape, or information stored in some sort of internal memory.

The “almost” in the previous section refers to the actual memory capacity of the computer. Exactly what computations a computer can perform depends on the amount of memory it has; and also on the time you are willing to wait. So, although a general-purpose computer can do anything a special-purpose digital circuit can do, in the information-processing sense, the computer might be slower or use more power. However, using general-purpose computers can save an enormous amount of engineering time and effort. It is *much* cheaper and easier to debug and modify and manufacture software than hardware. The modularities and abstractions that software PCAP systems give us are even more powerful than those derived from the digital circuit abstraction.

Again, we can see how abstraction separates use from details; we don’t need to know how the circuits inside a computer are designed, we just need to know the rules by which we can use them and the constraints under which they perform.

Python programs

Every general-purpose computer has a different detailed design, which means that the way its program needs to be specified is different. Furthermore, the “machine languages” for describing computer programs, at the most primitive level, are awkward for human programmers. So, we

have developed a number of computer *programming languages* for specifying a desired computation. These languages are converted into instructions in the computer's native machine language by other computer programs, called *compilers* or *interpreters*. One of the coolest and most powerful things about general-purpose computers is this ability to write computer programs that process or create or modify other computer programs.

Computer programming languages are PCAP systems. They provide primitive operations, ways of combining them, ways of abstracting over them, and ways of capturing common abstraction patterns. We will spend a considerable amount of time in this course studying the particular primitives, combination mechanisms, and abstraction mechanisms made available in the Python programming language. But the choice of Python is relatively unimportant. Virtually all modern programming languages supply a similar set of mechanisms, and the choice of programming language is primarily a matter of taste and convenience.

In a computer program we have both data primitives and computation primitives. At the most basic level, computers generally store binary digits (bits) in groups of 32 or 64, called *words*. These words are data primitives, and they can be interpreted by our programs as representing integers, floating-point numbers, strings, or addresses of other data in the computer's memory. The computational primitives supported by most computers include numerical operations such as addition and multiplication, and locating and extracting the contents of a memory at a given address. In Python, we will work at a level of abstraction that does not require us to think very much about addresses of data, but it is useful to understand that level of description, as well.

Primitive data and computation elements can be combined and abstracted in a variety of different ways, depending on choices of programming style. We will explore these in more detail in [section 1.3.2](#).

1.2.3 Models

So far, we have discussed a number of ways of framing the problem of designing and constructing mechanisms. Each PCAP system is accompanied by a modeling system that lets us make mathematical models of the systems we construct.

What is a model? It is a new system that is considerably simpler than the system being modeled, but which captures the important aspects of the original system. We might make a physical model of an airplane to test in a wind tunnel. It does not have to have passenger seats or landing lights; but it has to have surfaces of the correct shape, in order to capture the important properties for this purpose. Mathematical models can capture properties of systems that are important to us, and allow us to discover things about the system much more conveniently than by manipulating the original system itself.

One of the hardest things about building models is deciding which aspects of the original system to model and which ones to ignore. Many classic, dramatic engineering failures can be ascribed to failing to model important aspects of the system. One example (which turned out to be an ethical success story, rather than a failure) is LeMessurier's Citicorp Building², in which an engineering design change was made, but tested in a model in which the wind came only from a limited set of directions.

² See "The Fifty-Nine Story Crisis", *The New Yorker*, May 29, 1995, pp 45-53.

Another important dimension in modeling is whether the model is deterministic or not. We might, for example, model the effect of the robot executing a command to set its velocity as making an instantaneous change to the commanded velocity. But, of course, there is likely to be some delay and some error. We have to decide whether to ignore that delay and error in our model and treat it as if it were ideal; or, for example, to make a probabilistic model of the possible results of that command. Generally speaking, the more different the possible outcomes of a particular action or command, and the more diffuse the probability distribution over those outcomes, the more important it is to explicitly include uncertainty in the model.

Once we have a model, we can use it in a variety of different ways, which we explore below.

Analytical models

By far the most prevalent use of models is in analysis: Given a circuit design, what will the voltage on this terminal be? Given a computer program, will it compute the desired answer? How long will it take? Given a control system, implemented as a circuit or as a program, will the system stop at the right distance from the light bulb?

Analytical tools are important. It can be hard to verify the correctness of a system by trying it in all possible initial conditions with all possible inputs; sometimes it is easier to develop a mathematical model and then prove a theorem about the model.

For some systems, such as pure software computations, or the addition circuitry in a calculator, it is possible to analyze correctness or speed with just a model of the system in question. For other systems, such as fuel injectors, it is impossible to analyze the correctness of the controller without also modeling the environment (or “plant”) to which it is connected, and then analyzing the behavior of the coupled system of the controller and environment.

To demonstrate some of these tradeoffs, we can do a very simple analysis of a robot moving toward a lamp. Imagine that we arrange it so that the robot’s velocity at time t , $V[t]$, is proportional to the difference between the actual light level, $X[t]$, and a desired light level (that is, the light level we expect to encounter when the robot is the desired distance from the lamp), X_{desired} ; that is, we can model our control system with the difference equation

$$V[t] = k(X_{\text{desired}} - X[t])$$

where k is the constant of proportionality, or *gain*, of the controller.

Now, we need to model the world. For simplicity, we equate the light level to the robot’s position (assuming that the units match); in addition we assume the robot’s position at time t is its position at time $t - 1$ plus its velocity at time $t - 1$ (actually, this should be the product of the velocity and the length of time between samples, but we can just assume a unit time step and thus use velocity). When we couple these models, we get the difference equation

$$X[t] = X[t - 1] + k(X_{\text{desired}} - X[t - 1]) .$$

Now, for a given value of k , we can determine how the system will behave over time, by solving the difference equation in X .

Later in the course, we will see how easily-determined mathematical properties of a difference-equation model can tell us whether a control system will have the desired behavior and whether

or not the controller will cause the system to oscillate. These same kinds of analyses can be applied to robot control systems as well as to the temporal properties of voltages in a circuit, and even to problems as unrelated as the consequences of a monetary policy decision in economics. It is important to note that treating the system as moving in discrete time steps is an approximation to underlying continuous dynamics; it can make models that are easier to analyze, but it requires sophisticated understanding of sampling to understand the effects of this approximation on the correctness of the results.

Synthetic models

One goal of many people in a variety of sub-disciplines of computer science and electrical engineering is automatic synthesis of systems from formal descriptions of their desired behavior. For example, you might describe some properties you would want the input/output behavior of a circuit or computer program to have, and then have a computer system discover a circuit or program with the desired property.

This is a well-specified problem, but generally the search space of possible circuits or programs is much too large for an automated process; the intuition and previous experience of an expert human designer cannot yet be matched by computational search methods.

However, humans use informal models of systems for synthesis. The documentation of a software library, which describes the function of the various procedures, serves as an informal model that a programmer can use to assemble those components to build new, complex systems.

Internal models

As we wish to build more and more complex systems with software programs as controllers, we find that it is often useful to build additional layers of abstraction on top of the one provided by a generic programming language. This is particularly true when the nature of the exact computation required depends considerably on information that is only received during the execution of the system.

Consider, for example, an automated taxi robot (or, more prosaically, the navigation system in your new car). It takes, as input, a current location in the city and a goal location, and gives, as output, a path from the current location to the goal. It has a map built into it.

It is theoretically possible to build an enormous digital circuit or computer program that contains a look-up table, in which we precompute the path for all pairs of locations and store them away. Then when we want to find a path, we could simply look in the table at the location corresponding to the start and goal location and retrieve the precomputed path. Unfortunately, the size of that table is too huge to contemplate. Instead, what we do is construct an abstraction of the problem as one of finding a path from any start state to any goal state, through a graph (an abstract mathematical construct of “nodes” with “arcs” connecting them). We can develop and implement a general-purpose algorithm that will solve *any* shortest-path problem in a graph. That algorithm and implementation will be useful for a wide variety of possible problems. And for the navigation system, all we have to do is represent the map as a graph, specify the start and goal states, and call the general graph-search algorithm.

We can think of this process as actually putting a model of the system's interactions with the world *inside* the controller; it can consider different courses of action in the world and pick the one that is the best according to some criterion.

Another example of using internal models is when the system has some significant lack of information about the state of the environment. In such situations, it may be appropriate to explicitly model the set of possible states of the external world and their associated probabilities, and to update this model over time as new evidence is gathered. We will pursue an example of this approach near the end of the course.

1.3 Programming embedded systems

There are many different ways of thinking about modularity and abstraction for software. Different models will make some things easier to say and do and others more difficult, making different models appropriate for different tasks. In the following sections, we explore different strategies for building software systems that interact with an external environment, and then different strategies for specifying the purely computational part of a system.

1.3.1 Interacting with the environment

Increasingly, computer systems need to interact with the world around them, receiving information about the external world, and taking actions to affect the world. Furthermore, the world is dynamic, so that as the system is computing, the world is changing, requiring future computation to adapt to the new state of the world.

There are a variety of different ways to organize computations that interact with an external world. Generally speaking, such a computation needs to:

1. get information from sensors (light sensors, sonars, mouse, keyboard, etc.),
2. perform computation, remembering some of the results, and
3. take actions to change the outside world (move the robot, print, draw, etc.).

These operations can be put together in different styles.

1.3.1.1 Sequential

The most immediately straightforward style for constructing a program that interacts with the world is the basic imperative style, in which the program gives a sequence of 'commands' to the system it is controlling. A library of special procedures is defined, some of which read information from the sensors and others of which cause actions to be performed. Example procedures might move a robot forward a fixed distance, or send a file out over the network, or move video-game characters on the screen.

In this model, we could naturally write a program that moves an idealized robot in a square, if there is space in front of it.

```
if noObstacleInFront:
    moveDistance(1)
    turnAngle(90)
    moveDistance(1)
```

```
turnAngle(90)
moveDistance(1)
turnAngle(90)
moveDistance(1)
turnAngle(90)
```

The major problem with this style of programming is that the programmer has to remember to check the sensors sufficiently frequently. For instance, if the robot checks for free space in front, and then starts moving, it might turn out that a subsequent sensor reading will show that there is something in front of the robot, either because someone walked in front of it or because the previous reading was erroneous. It is hard to have the discipline, as a programmer, to remember to keep checking the sensor conditions frequently enough, and the resulting programs can be quite difficult to read and understand.

For the robot moving toward the light, we might write a program like this:

```
while lightValue < desiredValue:
    moveDistance(0.1)
```

This would have the robot creep up, step by step, toward the light. We might want to modify it so that the robot moved a distance that was related to the difference between the current and desired light values. However, if it takes larger steps, then during the time that it is moving it will not be sensitive to possible changes in the light value and cannot react immediately to them.

1.3.1.2 Event-Driven

User-interface programs are often best organized differently, as *event-driven* (also called *interrupt driven*) programs. In this case, the program is specified as a collection of procedures (called 'handlers' or 'callbacks') that are attached to particular events that can take place. So, for example, there might be procedures that are called when the mouse is clicked on each button in the interface, when the user types into a text field, or when the temperature of a reactor gets too high. An "event loop" runs continuously, checking to see whether any of the triggering events have happened, and, if they have, calling the associated procedure.

In our simple example, we might imagine writing a program that told the robot to drive forward by default; and then install an event handler that says that if the light level reaches the desired value, it should stop. This program would be reactive to sudden changes in the environment.

As the number and frequency of the conditions that need responses increases, it can be difficult to both keep a program like this running well and guarantee a minimum response time to any event.

1.3.1.3 Transducer

An alternative view is that programming a system that interacts with an external world is like building a *transducer* with internal state. Think of a transducer as a processing box that runs continuously. At regular intervals (perhaps many times per second), the transducer reads all of the sensors, does a small amount of computation, stores some values it will need for the next computation, and then generates output values for the actions.

This computation happens over and over and over again. Complex behavior can arise out of the temporal pattern of inputs and outputs. So, for example, a robot might try to move forward without hitting something by defining a procedure:

```
def step():
    distToFront = min(frontSonarReadings)
    motorOutput(gain * (distToFront - desiredDistance), 0.0)
```

Executed repeatedly, this program will automatically modulate the robot's speed to be proportional to the free space in front of it.

The main problem with the transducer approach is that it can be difficult to do tasks that are fundamentally sequential (like the example of driving in a square, shown above). We will start with the transducer model, and then, as described in [section 4.3](#), we will build a new abstraction layer on top of it that will make it easy to do sequential commands, as well.

1.3.2 Programming models

Just as there are different strategies for organizing entire software systems, there are different strategies for formally expressing computational processes within those structures.

1.3.2.1 Imperative computation

Most of you have probably been exposed to an imperative model of computer programming, in which we think of programs as giving a sequential set of instructions to the computer to *do* something. And, in fact, that is how the internals of the processors of computers are typically structured. So, in Java or C or C++, you write typically procedures that consist of lists of instructions to the computer:

1. Put this value in this variable
2. Square the variable
3. Divide it by pi
4. If the result is greater than 1, return the result

In this model of computation, the primitive computational elements are basic arithmetic operations and assignment statements. We can combine the elements using sequences of statements, and control structures such as `if`, `for`, and `while`. We can abstract away from the details of a computation by defining a procedure that does it. Now, the engineer only needs to know the specifications of the procedure, but not the implementation details, in order to use it.

1.3.2.2 Functional computation

Another style of programming is the functional style. In this model, we gain power through function calls. Rather than telling the computer to do things, we ask it questions: What is $4 + 5$? What is the square root of 6? What is the largest element of the list?

These questions can all be expressed as asking for the value of a function applied to some arguments. But where do the functions come from? The answer is, from other functions. We start with some set of basic functions (like “plus”), and use them to construct more complex functions.

This method would not be powerful without the mechanisms of conditional evaluation and recursion. Conditional functions ask one question under some conditions and another question under other conditions. Recursion is a mechanism that lets the definition of a function refer to the function being defined. Recursion is as powerful as iteration.

In this model of computation, the primitive computational elements are typically basic arithmetic and list operations. We combine elements using function composition (using the output of one function as the input to another), `if`, and recursion. We use function definition as a method of abstraction, and the idea of higher-order functions (passing functions as arguments to other functions) as a way of capturing common high-level patterns.

1.3.2.3 Data structures

In either style of asking the computer to do work for us, we have another kind of modularity and abstraction, which is centered around the organization of data.

At the most primitive level, computers operate on collections of (usually 32 or 64) bits. We can interpret such a collection of bits as representing different things: a positive integer, a signed integer, a floating-point number, a Boolean value (true or false), one or more characters, or an address of some other data in the memory of the computer. Python gives us the ability to work directly with all of these primitives, except for addresses.

There is only so much you can do with a single number, though. We would like to build computer programs that operate on representations of documents or maps or circuits or social networks. To do so, we need to aggregate primitive data elements into more complex *data structures*. These can include lists, arrays, dictionaries, and other structures of our own devising.

Here, again, we gain the power of abstraction. We can write programs that do operations on a data structure representing a social network, for example, without having to worry about the details of how the social network is represented in terms of bits in the machine.

1.3.2.4 Object-oriented programming: computation + data structures

Object-oriented programming is a style that applies the ideas of modularity and abstraction to execution and data at the same time.

An *object* is a data structure, together with a set of procedures that operate on the data. Basic procedures can be written in an imperative or a functional style, but ultimately there is imperative assignment to state variables in the object.

One major new type of abstraction in OO programming is “generic” programming. It might be that all objects have a procedure called `print` associated with them. So, we can ask any object to print itself, without having to know how it is implemented. Or, in a graphics system, we might have a variety of different objects that know their `x`, `y` positions on the screen. So each of them can be asked, in the same way, to say what their position is, even though they might be represented very differently inside the objects.

In addition, most object-oriented systems support inheritance, which is a way to make new kinds of objects by saying that they are mostly like another kind of object, but with some exceptions. This is another way to take advantage of abstraction.

Programming languages

Python as well as other modern programming languages, such as Java, Ruby and C++, support all of these programming models. The programmer needs to choose which programming model best suits the task. This is an issue that we will return to throughout the course.

1.4 Summary

We hope that this course will give you a rich set of conceptual tools and practical techniques, as well as an appreciation of how math, modeling, and implementation can work together to enable the design and analysis of complex computational systems that interact with the world.

Acknowledgments

Thanks to Jacob White, Hal Abelson, Tomas Lozano-Perez, Sarah Finney, Sari Canelake, Eric Grimson, Ike Chuang, and Berthold Horn for useful comments and criticisms, and to Denny Freeman for significant parts and most of the figures in the linear systems chapter. - LPK

Chapter 2

Learning to Program in Python

Depending on your previous programming background, we recommend different paths through the available readings:

- **If you have never programmed before:** you should start with a general introduction to programming and Python. We recommend *Python for Software Design: How to Think Like a Computer Scientist*, by Allen Downey. This is a good introductory text that uses Python to present basic ideas of computer science and programming. It is available for purchase in hardcopy, or as a free download from:

<http://www.greenteapress.com/thinkpython>

After that, you can go straight to the next chapter.

- **If you have programmed before, but not in Python:** you should read the rest of this chapter for a quick overview of Python, and how it may differ from other programming languages with which you are familiar.
- **If you have programmed in Python:** you should skip to the next chapter.

Everyone should have a bookmark in their browser for *Python Tutorial*, by Guido Van Rossum. This is the standard tutorial reference by the inventor of Python. It is accessible at:

<http://docs.python.org/tut/tut.html>

In the rest of this chapter, we will assume you know how to program in some language, but are new to Python. We will use Java as an informal running comparative example. In this section we will cover what we think are the most important differences between Python and what you may already know about programming; but these notes are by no means complete.

2.1 Using Python

Python is designed for easy interaction between a user and the computer. It comes with an interactive mode called a *listener* or *shell*. The shell gives a prompt (usually something like `>>>`) and waits for you to type in a Python expression or program. Then it will evaluate the expression you entered, and print out the value of the result. So, for example, an interaction with the Python shell might look like this:

```
>>> 5 + 5
10
>>> x = 6
>>> x
6
>>> x + x
12
>>> y = 'hi'
>>> y + y
'hihi'
>>>
```

So, you can use Python as a fancy calculator. And as you define your own procedures in Python, you can use the shell to test them or use them to compute useful results.

2.1.1 Indentation and line breaks

Every programming language has to have some method for indicating grouping of instructions. Here is how you write an if-then-else structure in Java:

```
if (s == 1){
    s = s + 1;
    a = a - 10;
} else {
    s = s + 10;
    a = a + 10;
}
```

The braces specify what statements are executed in the if case. It is considered good style to indent your code to agree with the brace structure, but it is not required. In addition, the semicolons are used to indicate the end of a statement, independent of the locations of the line breaks in the file. So, the following code fragment has the same meaning as the previous one, although it is much harder to read and understand.

```
if (s == 1){
s = s
+ 1;    a = a - 10;
} else {
        s = s + 10;
a = a + 10;
}
```

In Python, on the other hand, there are no braces for grouping or semicolons for termination. Indentation indicates grouping and line breaks indicate statement termination. So, in Python, we would write the previous example as

```
if s == 1:
    s = s + 1
    a = a - 10
else:
    s = s + 10
    a = a + 10
```

There is no way to put more than one statement on a single line.³ If you have a statement that is too long for a line, you can signal it with a backslash:

```
aReallyLongVariableNameThatMakesMyLinesLong = \  
    aReallyLongVariableNameThatMakesMyLinesLong + 1
```

It is easy for Java programmers to get confused about colons and semi-colons in Python. Here is the deal: (1) Python does not use semi-colons; (2) Colons are used to start an indented block, so they appear on the first line of a procedure definition, when starting a `while` or `for` loop, and after the condition in an `if`, `elif`, or `else`.

Is one method better than the other? No. It is entirely a matter of taste. The Python method is pretty unusual. But if you are going to use Python, you need to remember that indentation and line breaks are significant.

2.1.2 Types and declarations

Java programs are what is known as *statically and strongly typed*. Thus, the types of all the variables must be known at the time that the program is written. This means that variables have to be declared to have a particular type before they are used. It also means that the variables cannot be used in a way that is inconsistent with their type. So, for instance, you would declare `x` to be an integer by saying

```
int x;  
x = 6 * 7;
```

But you would get into trouble if you left out the declaration, or did

```
int x;  
x = "thing";
```

because a *type checker* is run on your program to make sure that you don't try to use a variable in a way that is inconsistent with its declaration.

In Python, however, things are a lot more flexible. There are no variable declarations, and the same variable can be used at different points in your program to hold data objects of different types. So, the following is fine, in Python:

```
if x == 1:  
    x = 89.3  
else:  
    x = "thing"
```

The advantage of having type declarations and compile-time type checking, as in Java, is that a compiler can generate an executable version of your program that runs very quickly, because it can be certain about what kind of data is stored in each variable, and it does not have to check it at runtime. An additional advantage is that many programming mistakes can be caught at compile

³ Actually, you can write something like `if a > b: a = a + 1` all on one line, if the work you need to do inside an `if` or `for` is only one line long.

time, rather than waiting until the program is being run. Java would complain even before your program started to run that it could not evaluate

```
3 + "hi"
```

Python would not complain until it was running the program and got to that point.

The advantage of the Python approach is that programs are shorter and cleaner looking, and possibly easier to write. The flexibility is often useful: In Python, it is easy to make a list or array with objects of different types stored in it. In Java, it can be done, but it is trickier. The disadvantage of the Python approach is that programs tend to be slower. Also, the rigor of compile-time type checking may reduce bugs, especially in large programs.

2.1.3 Modules

As you start to write bigger programs, you will want to keep the procedure definitions in multiple files, grouped together according to what they do. So, for example, we might package a set of utility functions together into a single file, called `utility.py`. This file is called a module in Python.

Now, if we want to use those procedures in another file, or from the the Python shell, we will need to say

```
import utility
```

so that all those procedures become available to us and to the Python interpreter. Now, if we have a procedure in `utility.py` called `foo`, we can use it with the name `utility.foo`. You can read more about modules, and how to reference procedures defined in modules, in the Python documentation.

2.1.4 Interaction and Debugging

We encourage you to adopt an interactive style of programming and debugging. Use the Python shell a lot. Write small pieces of code and test them. It is much easier to test the individual pieces as you go, rather than to spend hours writing a big program, and then find it does not work, and have to sift through all your code, trying to find the bugs.

But, if you find yourself in the (inevitable) position of having a big program with a bug in it, do not despair. Debugging a program does not require brilliance or creativity or much in the way of insight. What it requires is persistence and a systematic approach.

First of all, have a test case (a set of inputs to the procedure you are trying to debug) and know what the answer is supposed to be. To test a program, you might start with some special cases: what if the argument is 0 or the empty list? Those cases might be easier to sort through first (and are also cases that can be easy to get wrong). Then try more general cases.

Now, if your program gets your test case wrong, what should you do? Resist the temptation to start changing your program around, just to see if that will fix the problem. Do not change any code until you know what is wrong with what you are doing now, and therefore believe that the change you make is going to correct the problem.

Ultimately, for debugging big programs, it is most useful to use a software development environment with a serious debugger. But these tools can sometimes have a steep learning curve, so in this class we will learn to debug systematically using “print” statements.

One good way to use “print” statements to help in debugging is to use a variation on binary search. Find a spot roughly halfway through your code at which you can predict the values of variables, or intermediate results your computation. Put a print statement there that lists expected as well as actual values of the variables. Run your test case, and check. If the predicted values match the actual ones, it is likely that the bug occurs after this point in the code; if they do not, then you have a bug prior to this point (of course, you might have a second bug after this point, but you can find that later). Now repeat the process by finding a location halfway between the beginning of the procedure and this point, placing a print statement with expected and actual values, and continuing. In this way you can narrow down the location of the bug. Study that part of the code and see if you can see what is wrong. If not, add some more print statements near the problematic part, and run it again. **Don’t try to be smart....be systematic and indefatigable!**

You should learn enough of Python to be comfortable writing basic programs, and to be able to efficiently look up details of the language that you don’t know or have forgotten.

2.2 Procedures

In Python, the fundamental abstraction of a computation is as a procedure (other books call them “functions” instead; we will end up using both terms). A procedure that takes a number as an argument and returns the argument value plus 1 is defined as:

```
def f(x):
    return x + 1
```

The indentation is important here, too. All of the statements of the procedure have to be indented one level below the `def`. It is crucial to remember the `return` statement at the end, if you want your procedure to return a value. So, if you defined `f` as above, then played with it in the shell,⁴ you might get something like this:

```
>>> f
<function f at 0x82570>
>>> f(4)
5
>>> f(f(f(4)))
7
```

If we just evaluate `f`, Python tells us it is a function. Then we can apply it to 4 and get 5, or apply it multiple times, as shown.

What if we define

⁴ Although you can type procedure definitions directly into the shell, you will not want to work that way, because if there is a mistake in your definition, you will have to type the whole thing in again. Instead, you should type your procedure definitions into a file, and then get Python to evaluate them. Look at the documentation for Idle or the 6.01 FAQ for an explanation of how to do that.


```
def g(x):  
    x + 1
```

Now, when we play with it, we might get something like this:

```
>>> g(4)  
>>> g(g(4))  
Traceback (most recent call last):  
  File "<stdin>", line 1, in ?  
  File "<stdin>", line 2, in g  
TypeError: unsupported operand type(s) for +: 'NoneType' and 'int'
```

What happened!! First, when we evaluated `g(4)`, we got nothing at all, because our definition of `g` did not return anything. Well...strictly speaking, it returned a special value called `None`, which the shell does not bother printing out. The value `None` has a special type, called `NoneType`. So, then, when we tried to apply `g` to the result of `g(4)`, it ended up trying to evaluate `g(None)`, which made it try to evaluate `None + 1`, which made it complain that it did not know how to add something of type `NoneType` and something of type `int`.

Whenever you ask Python to do something it cannot do, it will complain. You should learn to read the error messages, because they will give you valuable information about what is wrong with what you were asking.

Print vs Return

Here are two different function definitions:

```
def f1(x):  
    print x + 1  
def f2(x):  
    return x + 1
```

What happens when we call them?

```
>>> f1(3)  
4  
>>> f2(3)  
4
```

It looks like they behave in exactly the same way. But they don't, really. Look at this example:

```
>>> print(f1(3))  
4  
None  
>>> print(f2(3))  
4
```

In the case of `f1`, the function, when evaluated, prints 4; then it returns the value `None`, which is printed by the Python shell. In the case of `f2`, it does not print anything, but it returns 4, which is printed by the Python shell. Finally, we can see the difference here:

```
>>> f1(3) + 1
4
Traceback (most recent call last):
  File "<stdin>", line 1, in ?
TypeError: unsupported operand type(s) for +: 'NoneType' and 'int'
>>> f2(3) + 1
5
```

In the first case, the function does not return a value, so there is nothing to add to 1, and an error is generated. In the second case, the function returns the value 4, which is added to 1, and the result, 5, is printed by the Python read-eval-print loop.

The book *Think Python*, which we recommend reading, was translated from a version for Java, and it has a lot of `print` statements in it, to illustrate programming concepts. But for just about everything we do, it will be returned values that matter, and printing will be used only for debugging, or to give information to the user.

Print is very useful for debugging. It is important to know that you can print out as many items as you want in one line:

```
>>> x = 100
>>> print 'x', x, 'x squared', x*x, 'xiv', 14
x 100 x squared 10000 xiv 14
```

We have also snuck in another data type on you: strings. A string is a sequence of characters. You can create a string using single or double quotes; and access individual elements of strings using indexing.

```
>>> s1 = 'hello world'
>>> s2 = "hello world"
>>> s1 == s2
True
>>> s1[3]
'l'
```

As you can see, indexing refers to the extraction of a particular element of a string, by using square brackets `[i]` where `i` is a number that identifies the location of the character that you wish to extract (*note that the indexing starts with 0*).

Look in the Python documentation for more about strings.

2.3 Control structures

Python has control structures that are slightly different from those in other languages.

2.3.1 Conditionals

Booleans

Before we talk about conditionals, we need to clarify the Boolean data type. It has values `True` and `False`. Typical expressions that have Boolean values are numerical comparisons:

```
>>> 7 > 8
False
>>> -6 <= 9
True
```

We can also test whether data items are equal to one another. Generally we use `==` to test for equality. It returns `True` if the two objects have equal values. Sometimes, however, we will be interested in knowing whether the two items are the exact same object (in the sense discussed in [section 3.3](#)). In that case we use `is`:

```
>>> [1, 2] == [1, 2]
True
>>> [1, 2] is [1, 2]
False
>>> a = [1, 2]
>>> b = [1, 2]
>>> c = a
>>> a == b
True
>>> a is b
False
>>> a == c
True
>>> a is c
True
```

Thus, in the examples above, we see that `==` testing can be applied to nested structures, and basically returns true if each of the individual elements is the same. However, `is` testing, especially when applied to nested structures, is more refined, and only returns `True` if the two objects point to exactly the same instance in memory.

In addition, we can combine Boolean values conveniently using `and`, `or`, and `not`:

```
>>> 7 > 8 or 8 > 7
True
>>> not 7 > 8
True
>>> 7 == 7 and 8 > 7
True
```

If

Basic conditional statements have the form:⁵

```
if <booleanExpr>:
    <statementT1>
    ...
    <statementTk>
else:
    <statementF1>
```

⁵ See the Python documentation for more variations.

```
...
<statementFn>
```

When the interpreter encounters a conditional statement, it starts by evaluating `<boolean-Expr>`, getting either `True` or `False` as a result.⁶ If the result is `True`, then it will evaluate `<statementT1>, ..., <statementTk>`; if it is `False`, then it will evaluate `<statementF1>, ..., <statementFn>`. Crucially, it always evaluates only one set of the statements.

Now, for example, we can implement a procedure that returns the absolute value of its argument.

```
def abs(x):
    if x >= 0:
        return x
    else:
        return -x
```

We could also have written

```
def abs(x):
    if x >= 0:
        result = x
    else:
        result = -x
    return result
```

Python uses the level of indentation of the statements to decide which ones go in the groups of statements governed by the conditionals; so, in the example above, the `return result` statement is evaluated once the conditional is done, no matter which branch of the conditional is evaluated.

For and While

If we want to do some operation or set of operations several times, we can manage the process in several different ways. The most straightforward are `for` and `while` statements (often called `for` and `while` loops).

A `for` loop has the following form:

```
for <var> in <listExpr>:
    <statement1>
    ...
    <statementn>
```

The interpreter starts by evaluating `listExpr`. If it does not yield a list, tuple, or string⁷, an error occurs. If it does yield a list or list-like structure, then the block of statements will, under normal circumstances, be executed one time for every value in that list. At the end, the variable `<var>` will remain bound to the last element of the list (and if it had a useful value before the `for` was evaluated, that value will have been permanently overwritten).

Here is a basic `for` loop:

⁶ In fact, Python will let you put any expression in the place of `<booleanExpr>`, and it will treat the values `0`, `0.0`, `[]`, `''`, and `None` as if they were `False` and everything else as `True`.

⁷ or, more esoterically, another object that can be iterated over.

```
result = 0
for x in [1, 3, 4]:
    result = result + x * x
```

At the end of this execution, `result` will have the value 26, and `x` will have the value 4.

One situation in which the body is not executed once for each value in the list is when a `return` statement is encountered. No matter whether `return` is nested in a loop or not, if it is evaluated it immediately causes a value to be returned from a procedure call. So, for example, we might write a procedure that tests to see if an item is a member of a list, and returns `True` if it is and `False` if it is not, as follows:

```
def member(x, items):
    for i in items:
        if x == i:
            return True
    return False
```

The procedure loops through all of the elements in `items`, and compares them to `x`. As soon as it finds an item `i` that is equal to `x`, it can quit and return the value `True` from the procedure. If it gets all the way through the loop without returning, then we know that `x` is not in the list, and we can return `False`.

Exercise 2.1. Write a procedure that takes a list of numbers, `nums`, and a limit, `limit`, and returns a list which is the shortest prefix of `nums` the sum of whose values is greater than `limit`. Use `for`. Try to avoid using explicit indexing into the list. (Hint: consider the strategy we used in `member`.)

Range

Very frequently, we will want to iterate through a list of integers, often as indices. Python provides a useful procedure, `range`, which returns lists of integers. It can be used in complex ways, but the basic usage is `range(n)`, which returns a list of integers going from 0 up to, but not including, its argument. So `range(3)` returns `[0, 1, 2]`.

Exercise 2.2. Write a procedure that takes `n` as an argument and returns the sum of the squares of the integers from 1 to `n-1`. It should use `for` and `range`.

Exercise 2.3. What is wrong with this procedure, which is supposed to return True if the element `x` occurs in the list `items`, and False otherwise?

```
def member (x, items):
    for i in items:
        if x == i:
            return True
        else:
            return False
```

While

You should use `for` whenever you can, because it makes the structure of your loops clear. Sometimes, however, you need to do an operation several times, but you do not know in advance how many times it needs to be done. In such situations, you can use a `while` statement, of the form:

```
while <booleanExpr>:
    <statement1>
    ...
    <statementn>
```

In order to evaluate a `while` statement, the interpreter evaluates `<booleanExpr>`, getting a Boolean value. If the value is `False`, it skips all the statements and evaluation moves on to the next statement in the program. If the value is `True`, then the statements are executed, and the `<booleanExpr>` is evaluated again. If it is `False`, execution of the loop is terminated, and if it is `True`, it goes around again.

It will generally be the case that you initialize a variable before the `while` statement, change that variable in the course of executing the loop, and test some property of that variable in the Boolean expression. Imagine that you wanted to write a procedure that takes an argument `n` and returns the largest power of 2 that is smaller than `n`. You might do it like this:

```
def pow2Smaller(n):
    p = 1
    while p*2 < n:
        p = p*2
    return p
```

Lists

Python has a built-in list data structure that is easy to use and incredibly convenient. So, for instance, you can say

```
>>> y = [1, 2, 3]
>>> y[0]
1
>>> y[2]
3
>>> y[-1]
```



```
3
>>> y[-2]
2
>>> len(y)
3
>>> y + [4]
[1, 2, 3, 4]
>>> [4] + y
[4, 1, 2, 3]
>>> [4,5,6] + y
[4, 5, 6, 1, 2, 3]
>>> y
[1, 2, 3]
```

A list is written using square brackets, with entries separated by commas. You can get elements out by specifying the index of the element you want in square brackets, but note that, like for strings, the indexing starts with 0. Note that you can index elements of a list starting from the initial (or zeroth) one (by using integers), or starting from the last one (by using negative integers).

You can add elements to a list using '+', taking advantage of Python operator overloading. Note that this operation does not change the original list, but makes a new one.

Another useful thing to know about lists is that you can make *slices* of them. A slice of a list is sublist; you can get the basic idea from examples.

```
>>> b = range(10)
>>> b
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9]
>>> b[1:]
[1, 2, 3, 4, 5, 6, 7, 8, 9]
>>> b[3:]
[3, 4, 5, 6, 7, 8, 9]
>>> b[:7]
[0, 1, 2, 3, 4, 5, 6]
>>> b[:-1]
[0, 1, 2, 3, 4, 5, 6, 7, 8]
>>> b[:-2]
[0, 1, 2, 3, 4, 5, 6, 7]
```

Iteration over lists

What if you had a list of integers, and you wanted to add them up and return the sum? Here are a number of different ways of doing it.⁸

⁸ For any program you will ever need to write, there will be a huge number of different ways of doing it. How should you choose among them? The most important thing is that the program you write be correct, and so you should choose the approach that will get you to a correct program in the shortest amount of time. That argues for writing it in the way that is cleanest, clearest, shortest. Another benefit of writing code that is clean, clear and short is that you will be better able to understand it when you come back to it in a week or a month or a year, and that other people will also be better able to understand it. Sometimes, you will have to worry about writing a version of a program that runs very quickly, and it might be that in order to make that happen, you will have to write it less cleanly or clearly or briefly. But it is important to have a version that is correct before you worry about getting one that is fast.

First, here is a version in a style you might have learned to write in a Java class (actually, you would have used `for`, but Python does not have a `for` that works like the one in C and Java).

```
def addList1(l):
    sum = 0
    listLength = len(l)
    i = 0
    while (i < listLength):
        sum = sum + l[i]
        i = i + 1
    return sum
```

It increments the index `i` from 0 through the length of the list - 1, and adds the appropriate element of the list into the sum. This is perfectly correct, but pretty verbose and easy to get wrong.

Here is a version using Python's `for` loop.

```
def addList2(l):
    sum = 0
    for i in range(len(l)):
        sum = sum + l[i]
    return sum
```

A loop of the form

```
for x in l:
    something
```

will be executed once for each element in the list `l`, with the variable `x` containing each successive element in `l` on each iteration. So,

```
for x in range(3):
    print x
```

will print `0 1 2`. Back to `addList2`, we see that `i` will take on values from 0 to the length of the list minus 1, and on each iteration, it will add the appropriate element from `l` into the sum. This is more compact and easier to get right than the first version, but still not the best we can do!

This one is even more direct.

```
def addList3(l):
    sum = 0
    for v in l:
        sum = sum + v
    return sum
```

We do not ever really need to work with the indices. Here, the variable `v` takes on each successive value in `l`, and those values are accumulated into `sum`.

For the truly lazy, it turns out that the function we need is already built into Python. It is called `sum`:

```
def addList4(l):
    return sum(l)
```

In section ??, we will see another way to do `addList`, which many people find more beautiful than the methods shown here.

List Comprehensions

Python has a very nice built-in facility for doing many iterative operations, called *list comprehensions*. The basic template is

```
[<resultExpr> for <var> in <listExpr> if <conditionExpr>]
```

where `<var>` is a single variable (or a tuple of variables), `<listExpr>` is an expression that evaluates to a list, tuple, or string, and `<resultExpr>` is an expression that may use the variable `<var>`. The `if <conditionExpr>` is optional; if it is present, then only those values of `<var>` for which that expression is `True` are included in the resulting computation.

You can view a list comprehension as a special notation for a particular, very common, class of `for` loops. It is equivalent to the following:

```
*resultVar* = []
for <var> in <listExpr>:
    if <conditionExpr>:
        *resultVar*.append(<resultExpr>)
*resultVar*
```

We used a kind of funny notation `*resultVar*` to indicate that there is some anonymous list that is getting built up during the evaluation of the list comprehension, but we have no real way of accessing it. The result is a list, which is obtained by successively binding `<var>` to elements of the result of evaluating `<listExpr>`, testing to see whether they meet a condition, and if they meet the condition, evaluating `<resultExpr>` and collecting the results into a list.

Whew. It is probably easier to understand it by example.

```
>>> [x/2.0 for x in [4, 5, 6]]
[2.0, 2.5, 3.0]
>>> [y**2 + 3 for y in [1, 10, 1000]]
[4, 103, 1000003]
>>> [a[0] for a in [['Hal', 'Abelson'], ['Jacob', 'White'],
                  ['Leslie', 'Kaelbling']]]
['Hal', 'Jacob', 'Leslie']
>>> [a[0]+'!' for a in [['Hal', 'Abelson'], ['Jacob', 'White'],
                      ['Leslie', 'Kaelbling']]]
['Hal!', 'Jacob!', 'Leslie!']
```

Imagine that you have a list of numbers and you want to construct a list containing just the ones that are odd. You might write

```
>>> nums = [1, 2, 5, 6, 88, 99, 101, 10000, 100, 37, 101]
>>> [x for x in nums if x%2==1]
[1, 5, 99, 101, 37, 101]
```

Note the use of the `if` conditional here to include only particular values of `x`.

And, of course, you can combine this with the other abilities of list comprehensions, to, for example, return the squares of the odd numbers:

```
>>> [x*x for x in nums if x%2==1]
[1, 25, 9801, 10201, 1369, 10201]
```

You can also use structured assignments in list comprehensions

```
>>> [first for (first, last) in [['Hal', 'Abelson'], ['Jacob', 'White'],
                              ['Leslie', 'Kaelbling']]]
['Hal', 'Jacob', 'Leslie']
>>> [first+last for (first, last) in [['Hal', 'Abelson'], ['Jacob', 'White'],
                              ['Leslie', 'Kaelbling']]]
['HalAbelson', 'JacobWhite', 'LeslieKaelbling']
```

Another built-in function that is useful with list comprehensions is `zip`. Here are some examples of how it works:

```
> zip([1, 2, 3], [4, 5, 6])
[(1, 4), (2, 5), (3, 6)]
> zip([1,2], [3, 4], [5, 6])
[(1, 3, 5), (2, 4, 6)]
```

Here is an example of using `zip` with a list comprehension:

```
>>> [first+last for (first, last) in zip(['Hal', 'Jacob', 'Leslie'],
                                       ['Abelson', 'White', 'Kaelbling'])]
['HalAbelson', 'JacobWhite', 'LeslieKaelbling']
```

Note that this last example is very different from this one:

```
>>> [first+last for first in ['Hal', 'Jacob', 'Leslie'] \
      for last in ['Abelson', 'White', 'Kaelbling']]
['HalAbelson', 'HalWhite', 'HalKaelbling', 'JacobAbelson', 'JacobWhite',
'JacobKaelbling', 'LeslieAbelson', 'LeslieWhite', 'LeslieKaelbling']
```

Nested list comprehensions behave like nested for loops, the expression in the list comprehension is evaluated for every combination of the values of the variables.

2.4 Common Errors and Messages

Here are some common Python errors and error messages to watch out for. Please let us know if you have any favorite additions for this list.

- A complaint about `NoneType` often means you forgot a return.

```
def plus1(x):
    x + 1
>>> y = plus1(x)
>>> plus1(x) + 2
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
TypeError: unsupported operand type(s) for +: 'NoneType' and 'int'
```

- Weird results from math can result from integer division

```
>>> 3/ 9
0
```

- “Unsubscriptable object” means you are trying to get an element out of something that isn’t a dictionary, list, or tuple.

```
>>> x = 1
>>> x[3]
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
TypeError: 'int' object is unsubscriptable
```

- “Object is not callable” means you are trying to use something that isn’t a procedure or method as if it were.

```
>>> x = 1
>>> x(3)
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
TypeError: 'int' object is not callable
```

- “List index out of range” means you are trying to read or write an element of a list that is not present.

```
>>> a = range(5)
>>> a
[0, 1, 2, 3, 4]
>>> a[5]
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
IndexError: list index out of range
```

- “Maximum recursion depth exceeded” means you have a recursive procedure that is nested very deeply or your base case is not being reached due to a bug.

```
def fizz(x):
    return fizz(x - 1)
>>> fizz(10)
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
  File "<stdin>", line 2, in fizz
  File "<stdin>", line 2, in fizz
  ...
  File "<stdin>", line 2, in fizz
RuntimeError: maximum recursion depth exceeded
```

- “Key Error” means that you are trying to look up an element in a dictionary that is not present.

```
>>> d = {'a':7, 'b':8}
>>> d['c']
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
KeyError: 'c'
```

- Another common error is forgetting the `self` before calling a method. This generates the same error that you would get if you tried to call a function that wasn't defined at all.

```
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
  File "V2.py", line 22, in __add__
    return add(v)
NameError: global name 'add' is not defined
```

2.5 Python Style

Software engineering courses often provide very rigid guidelines on the style of programming, specifying the appropriate amount of indentation, or what to capitalize, or whether to use underscores in variable names. Those things can be useful for uniformity and readability of code, especially when a lot of people are working on a project. But they are mostly arbitrary: a style is chosen for consistency and according to some person's aesthetic preferences.

There are other matters of style that seem, to us, to be more fundamental, because they directly affect the readability or efficiency of the code.

- **Avoid recalculation of the same value.**

You should compute it once and assign it to a variable instead; otherwise, if you have a bug in the calculation (or you want to change the program), you will have to change it multiple times. It is also inefficient.

- **Avoid repetition of a pattern of computation.**

You should use a function instead, again to avoid having to change or debug the same basic code multiple times.

- **Avoid numeric indexing.**

You should use destructuring if possible, since it is much easier to read the code and therefore easier to get right and to modify later.

- **Avoid excessive numeric constants.**

You should name the constants, since it is much easier to read the code and therefore easier to get right and to modify later.

Here are some examples of simple procedures that exhibit various flaws. We'll talk about what makes them problematic.

2.5.1 Normalize a vector

Let's imagine we want to normalize a vector of three values; that is to compute a new vector of three values, such that its length is 1. Here is our first attempt; it is a procedure that takes as input a list of three numbers, and returns a list of three numbers:

```
def normalize3(v):
    return [v[0]/math.sqrt(v[0]**2+v[1]**2+v[2]**2),
            v[1]/math.sqrt(v[0]**2+v[1]**2+v[2]**2),
            v[2]/math.sqrt(v[0]**2+v[1]**2+v[2]**2)]
```


This is correct, but it looks pretty complicated. Let's start by noticing that we're recalculating the denominator three times, and instead save the value in a variable.

```
def normalize3(v):
    magv = math.sqrt(v[0]**2+v[1]**2+v[2]**2)
    return [v[0]/magv,v[1]/magv,v[2]/magv]
```

Now, we can see a repeated pattern, of going through and dividing each element by magv. Also, we observe that the computation of the magnitude of a vector is a useful and understandable operation in its own right, and should probably be put in its own procedure. That leads to this procedure:

```
def mag(v):
    return math.sqrt(sum([vi**2 for vi in v]))

def normalize3(v):
    return [vi/mag(v) for vi in v]
```

This is especially nice, because now, in fact, it applies not just to vectors of length three. So, it's both shorter and more general than what we started with. But, one of our original problems has snuck back in: we're recomputing mag(v) once for each element of the vector. So, finally, here's a version that we're very happy with:⁹

```
def mag(v):
    return math.sqrt(sum([vi**2 for vi in v]))

def normalize3(v):
    magv = mag(v)
    return [vi/magv for vi in v]
```

2.5.2 Perimeter of a polygon

Now, let's consider the problem of computing the length of the perimeter of a polygon. The input is a list of vertices, encoded as a list of lists of two numbers (such as `[[1, 2], [3.4, 7.6], [-4.4, 3]]`). Here is our first attempt:

```
def perim(vertices):
    result = 0
    for i in range(len(vertices)-1):
        result = result + math.sqrt((vertices[i][0]-vertices[i+1][0])**2 + \
                                   (vertices[i][1]-vertices[i+1][1])**2)
    return result + math.sqrt((vertices[-1][0]-vertices[0][0])**2 + \
                              (vertices[-1][1]-vertices[0][1])**2)
```

⁹ Note that there is still something for someone to be unhappy with here: the use of a list comprehension means that we're creating a new list, which we are just going to sum up; that's somewhat less efficient than adding the values up in a loop. However, as we said at the outset, for almost every program, clarity matters more than efficiency. And once you have something that's clear and correct, you can selectively make the parts that are executed frequently more efficient.

Again, this works, but it ain't pretty. The main problem is that someone reading the code doesn't immediately see what all that subtraction and squaring is about. We can fix this by defining another procedure:

```
def perim(vertices):
    result = 0
    for i in range(len(vertices)-1):
        result = result + pointDist(vertices[i],vertices[i+1])
    return result + pointDist(vertices[-1],vertices[0])

def pointDist(p1,p2):
    return math.sqrt(sum([(p1[i] - p2[i])**2 for i in range(len(p1))]))
```

Now, we've defined a new procedure `pointDist`, which computes the Euclidean distance between two points. And, in fact, we've written it generally enough to work on points of any dimension (not just two). Just for fun, here's another way to compute the distance, which some people would prefer and others would not.

```
def pointDist(p1,p2):
    return math.sqrt(sum([(c1 - c2)**2 for (c1, c2) in zip(p1, p2)]))
```

For this to make sense, you have to understand `zip`. Here's an example of how it works:

```
> zip([1, 2, 3],[4, 5, 6])
[(1, 4), (2, 5), (3, 6)]
```

2.5.3 Bank transfer

What if we have two values, representing bank accounts, and want to transfer an amount of money `amt` between them? Assume that a bank account is represented as a list of values, such as `['Alyssa', 8300343.03, 0.05]`, meaning that 'Alyssa' has a bank balance of \$8,300,343.03, and has to pay a 5-cent fee for every bank transaction. We might write this procedure as follows. It moves the amount from one balance to the other, and subtracts the transaction fee from each account.

```
def transfer(a1, a2, amt):
    a1[1] = a1[1] - amt - a1[2]
    a2[1] = a2[1] + amt - a2[2]
```

To understand what it's doing, you really have to read the code at a detailed level. Furthermore, it's easy to get the variable names and subscripts wrong.

Here's another version that abstracts away the common idea of a deposit (which can be positive or negative) into a procedure, and uses it twice:

```
def transfer(a1, a2, amt):
    deposit(a1, -amt)
    deposit(a2, amt)

def deposit(a, amt):
    a[1] = a[1] + amt - a[2]
```

Now, `transfer` looks pretty clear, but `deposit` could still use some work. In particular, the use of numeric indices to get the components out of the bank account definition is a bit cryptic (and easy to get wrong).¹⁰

```
def deposit(a, amt):
    (name, balance, fee) = a
    a[1] = balance + amt - fee
```

Here, we've used a destructuring assignment statement to give names to the components of the account. Unfortunately, when we want to change an element of the list representing the account, we still have to index it explicitly. Given that we have to use explicit indices, this approach in which we name them might be better.

```
acctName = 0
acctBalance = 1
acctFee = 2
def deposit(a, amt):
    a[acctBalance] = a[acctBalance] + amt - a[acctFee]
```

Strive, in your programming, to make your code as simple, clear, and direct as possible. Occasionally, the simple and clear approach will be too inefficient, and you'll have to do something more complicated. In such cases, you should still start with something clear and simple, and in the end, you can use it as documentation.

2.5.4 Coding examples

Following are some attempts at defining a procedure `isSubset`, which takes two arguments, `a` and `b`, and returns `True` if `a` is a subset of `b`, assuming that `a` and `b` are represented as lists of elements.

Here is one solution to the problem which is in the Pythonic functional style.

```
def isSubset(a, b):
    return reduce(operator.and_, [x in b for x in a])
```

This is short and direct. The expression `x in b` tests to see if the item `x` is in the list `b`. So, `[x in b for x in a]` is a list of Booleans, one for each element in `a`, indicating whether that element is in `b`. Finally, we reduce that list using the `and` operator¹¹, which will have the value `True` if *all* of the elements of the list are `True`, and `False` otherwise.

An alternative is to do it recursively:

```
def isSubset(a, b):
    if a == []:
        return True
    else:
        return a[0] in b and isSubset(a[1:], b)
```

¹⁰ We'll see other approaches to this when we start to look at object-oriented programming. But it's important to apply basic principles of naming and clarity no matter whether you're using assembly language or Java.

¹¹ To get versions of basic Python operations in the form of procedures, you need to do `import operator`. Now, you can do addition with `operator.add(3, 4)`. Because `and` already has special syntactic significance in Python, they had to name the operator version of it something different, and so it is `operator.and_`.

The base case of the recursion is that `a` is the empty list; in that case, it's clear that `a` is a subset of `b`. Otherwise, we can define our answer in terms of `isSubset`, but asking a simpler question (in this case, on a list for `a` which is one element smaller). So, we say that `a` is a subset of `b` if the first element of `a` is a member of `b` and the set of rest of the elements of `a` is a subset of `b`.

We could go even farther in compressing the recursive solution:

```
def isSubset(a, b):
    return a == None or a[0] in b and isSubset(a[1:], b)
```

Here, we are taking advantage of the fact that in Python (and most other languages), the `or` operator has the “early out” property. That is, if we are evaluating `e1 or e2`, we start by evaluating `e1`, and if it is `True`, then we know the result of the expression has to be `True`, and therefore we return without evaluating `e2`. So, `or` can act as a kind of conditional operator. Some people would find this example too abstruse (shorter isn't *always* better), and some would find it very beautiful.

Here is another good solution, this time in the imperative style:

```
def isSubset(a, b):
    for x in a:
        if not x in b:
            return False
    return True
```

It works by going through the elements in `a`, in order. If any of those elements is not in `b`, then we can immediately quit and return `False`. Otherwise, if we have gotten through all of the elements in `a`, and each of them has been in `b`, then `a` really is a subset of `b` and we can return `True`.

Here is another good imperative example:

```
def isSubset(a, b):
    result = True
    for x in a:
        result = result and x in b
    return result
```

This procedure starts by initializing a result to `True`, which is the identity element for `and` (it plays the same role as `0` for `add`). Then it goes all the way through the list `a`, and ands the old result with `x in b`. This computation is very similar to the computation done in the functional version, but rather than creating the whole list of Booleans and then reducing, we interleave the individual membership tests with combining them into a result.

All of these examples are short and clear, and we'd be happy to see you write any of them. However, there are lots of versions of this procedure that students have written that we are not happy with. Here are some examples of procedures that are correct, in the sense that they return the right answer, but incorrect, in the sense that they are long or confused or hard to read.

Bad Example 1.

```
def isSubset(a,b):
    list1=[]
    for i in a:
        for j in b:
            if i==j:
                list1.append(i)
                break
    if len(list1)==len(a):
        return True
    else:
        return False
```

This procedure works by going through both lists and making a list of the items that are in common (basically computing the intersection). Then, it checks to see whether the intersection is of the same length as *a*. There are several problems here:

- Using the idea of computing the intersection and then seeing whether it is equal to *a* is a nice idea. But it's hard to see that's what this code is doing. Much better would be to make a procedure to compute the intersection, and then call it explicitly and compare the result to *a*.
- Comparing the length of the intersection and the length of *a* is dangerous. Because we allow repetitions of elements in the lists representing sets, two sets that are equal, in the set sense, may not have the same length.
- The `break` statement exits one level of the loop you are currently executing. It is often hard to understand and best avoided.
- If you ever find yourself writing:

```
if condition:
    return True
else:
    return False
```

You could always replace that with

```
return condition
```

which is shorter and clearer.

Bad Example 2.

```
def isSubset(a,b):
    itera = 0
    okay = True
    while itera < len(a):
        iterb = 0
        tempokay = False
        while iterb < len(b):
            if a[itera] == b[iterb]:
                tempokay = True
            iterb = iterb + 1
        if tempokay == False:
            okay = False
        itera = itera + 1
    return okay
```

This procedure is basically iterating over both lists, which is fine, but it is using `while` rather than `for` to do so, which makes it hard to follow. We can rewrite it with `for`:

```
def isSubset(a,b):
    okay = True
    for itera in range(len(a)):
        tempokay = False
        for iterb in range(len(b)):
            if a[itera] == b[iterb]:
                tempokay = True
        if tempokay == False:
            okay = False
    return okay
```

continued

Bad Example 3. Previous bad example, being made over. Now, the remaining lack of clarity is in the handling of the Booleans. We can replace

```
if a[itera] == b[iterb]:
    tempokay = True
```

with

```
tempokay = tempokay or (a[itera] == b[iterb])
```

which will set `tempokay` to `True` if `a[itera] == b[iterb]`, and otherwise leave it the same. And we can replace

```
if tempokay == False:
    okay = False
```

with

```
okay = okay and tempokay
```

It still has the effect that if `tempokay` is false, then `okay` will be false, and otherwise it will be as it was before. So, now we have:

```
def isSubset(a,b):
    okay = True
    for itera in range(len(a)):
        tempokay = False
        for iterb in range(len(b)):
            tempokay = tempokay or a[itera] == b[iterb]
        okay = okay and tempokay
    return okay
```

The logical structure is becoming clearer. It might be even better if we were to write:

```
def isSubset(a,b):
    foundAllAsSoFar = True
    for itera in range(len(a)):
        foundThisA = False
        for iterb in range(len(b)):
            foundThisA = foundThisA or a[itera] == b[iterb]
        foundAllAsSoFar = foundAllAsSoFar and foundThisA
    return okay
```

Exercise 2.4. Now, see if you can, first, explain, and then improve this example!

```
def issubset(a,b):
    i = 0
    j = 0
    while i < len(a):
        c = False
        while j < len(b):
            if a[i] == b[j]:
                c = True
                j = j+1
            if c:
                c = False
            else:
                return False
        j = 0
        i = i+1
    return True
```


Chapter 3

Programs and Data

Object-oriented programming is a popular way of organizing programs, which groups together data with the procedures that operate on them, thus facilitating some kinds of modularity and abstraction. In the context of our PCAP framework, object-oriented programming will give us methods for capturing common patterns in data and the procedures that operate on that data, via classes, generic functions, and inheritance.

In this chapter, we will try to develop a deep understanding of object-oriented programming by working through the mechanism by which an interpreter evaluates a computer program. The first part of the chapter will focus on interpretation of typical expressions, starting from the simplest single-statement programs and working up through list structures and procedures. Many of the observations made through this process apply to styles of program organization as well as object-oriented programming. Once we understand how an interpreter evaluates standard expressions, we will move to objects and classes. Although we use Python as an example, the discussion in this chapter is intended to be illustrative of principles of computer languages, more generally.

In many computer languages, including Python, programs are understood and executed by a computer program called an *interpreter*. Interpreters are surprisingly simple: the rules defining the meaning or *semantics* of a programming language are typically short and compact; and the interpreter basically encodes these rules and applies them to any legal expression in the language. The enormous richness and complexity of computer programs comes from the composition of primitive elements with simple rules. The interpreter, in essence, defines the semantics of the language by capturing the rules governing the value or behavior of program primitives, and of what it means to combine the primitives in various ways. We will study the meaning of computer programs by understanding how the interpreter operates on them.

An interpreter is made up of four pieces:

- The *reader* or *tokenizer* takes as input a string of characters and divides them into *tokens*, which are numbers (like `-3.42`), words (like `while` or `a`), and special characters (like `:`).
- The *parser* takes as input the string of tokens and understands them as constructs in the programming language, such as while loops, procedure definitions, or return statements.
- The *evaluator* (which is also sometimes called the *interpreter*, as well) has the really interesting job of determining the value and effects of the program that you ask it to interpret.
- The *printer* takes the value returned by the evaluator and prints it out for the user to see.

Programs should never be a mystery to you: you can learn the simple semantic rules of the language and, if necessary, simulate what the interpreter would do, in order to understand *any* computer program which you are facing. Of course, in general, one does not want to work through

the tedious process of simulating the interpreter, but this foundation of understanding the interpreter's process enables you to reason about the evaluation of any program.

3.1 Primitives, Composition, Abstraction, and Patterns

We will start by thinking about how the PCAP framework applies to computer programs, in general. We can do this by filling in [table 3.1](#), exploring the PCAP ideas in data, procedures, and objects.

Data

The primitive data items in most programming languages are things like integers, floating point numbers, and strings. We can combine these into data structures (we discuss some basic Python data structures in [section 3.3](#)) such as lists, arrays, dictionaries and records. Making a data structure allows us, at the most basic level, to think of a collection of primitive data elements as if it were one thing, freeing us from details. Sometimes, we just want to think of a collection of data, not in terms of its underlying representation, but in terms of what it represents. So, we might want to think of a set of objects, or a family tree, without worrying whether it is an array or a list in its basic representation. *Abstract data types* provide a way of abstracting away from representational details and allowing us to focus on what the data really means.

Procedures

The primitive procedures of a language are things like built-in numeric operations and basic list operations. We can combine these using the facilities of the language, such as `if` and `while`, or by using function composition ($f(g(x))$). If we want to abstract away from the details of how a particular computation is done, we can define a new function; defining a function allows us to use it for computational jobs without thinking about the details of how those computational jobs get done. You can think of this process as essentially creating a new primitive, which we can then use while ignoring the details of how it is constructed. One way to capture common patterns of abstraction in procedures is to abstract over procedures themselves, with higher-order procedures, which we discuss in detail in [section 3.4.6](#).

Objects

Object-oriented programming provides a number of methods of abstraction and pattern capture in both data and procedures. At the most basic level, objects can be used as records, combining together primitive data elements. More generally, they provide strategies for jointly abstracting a data representation and the procedures that work on it. The features of *inheritance* and *polymorphism* are particularly important, and we will discuss them in detail later in this chapter.

	Procedures	Data
Primitives	<code>+, *, ==</code>	numbers, strings
Means of combination	<code>if, while, f(g(x))</code>	lists, dictionaries, objects
Means of abstraction	<code>def</code>	ADTS, classes
Means of capturing patterns	higher-order procedures	generic functions, inheritance

Table 3.1 Primitives, combination, abstraction, patterns framework for computer programs

3.2 Expressions and assignment

We can think of most computer programs as performing some sort of transformation on data. Our program might take as input the exam scores of everyone in the class and generate the average score as output. Or, in a transducer model, we can think about writing the program that takes the current memory state of the transducer and an input, and computes a new memory state and output.

To represent data in a computer, we have to encode it, ultimately as sequences of binary digits (0s and 1s). The memory of a computer is divided into ‘words’, which typically hold 32 or 64 bits; a word can be used to store a number, one or several characters, or a pointer to (the address of) another memory location.

A computer program, at the lowest level, is a set of primitive instructions, also encoded into bits and stored in the words of the computer’s memory. These instructions specify operations to be performed on the data (and sometimes the program itself) that are stored in the computer’s memory. In this class, we will not work at the level of these low-level instructions: a high-level programming language such as Python lets us abstract away from these details. But it is important to have an abstract mental model of what is going on within the computer.

3.2.1 Simple expressions

A cornerstone of a programming language is the ability to evaluate expressions. We will start here with arithmetic expressions, just to get the idea. An expression consists of a sequence of ‘tokens’ (words, special symbols, or numerals) that represent the application of operators to data elements. Each expression has a value, which can be computed recursively by evaluating primitive expressions, and then using standard rules to combine their values to get new values.

Numerals, such as 6 or -3.7 are primitive expressions, whose values are numeric constants. Their values can be *integers*, within some fixed range dictated by the programming language, or *floating point numbers*. Floating point numbers are used to represent non-integer values, but they are different, in many important ways, from the real numbers. There are infinitely many real numbers within a finite interval, but only finitely many floating-point numbers exist at all (because they all must be representable in a fixed number of bits). In fact, the usual laws of real arithmetic (transitivity, associativity, etc.) are violated in floating-point arithmetic, because the results of any given sub-computation may not be representable in the given number of bits.

We will illustrate the evaluation of expressions in Python by showing short transcripts of interactive sessions with the *Python shell*: the shell is a computer program that

- Prompts the user for an expression, by typing `>>>`,
- Reads what the user types in, and converts it into a set of tokens,
- Parses the tokens into a data structure representing the syntax of the expression,
- Evaluates the parsed expression using an interpreter, and
- Prints out the resulting value

So, for example, we might have this interaction with Python:

```
>>> 2 + 3
5
>>> (3 * 8) - 2
22
>>> ((3 * 8) - 2) / 11
2
>>> 2.0
2.0
>>> 0.1
0.10000000000000001
>>> 1.0 / 3.0
0.33333333333333331
>>> 1 / 3
0
```

There are a couple of things to observe here. First, we can see how floating point numbers only approximately represent real numbers: when we type in `0.1`, the closest Python can come to it in floating point is `0.10000000000000001`. The last interaction is particularly troubling: it seems like the value of the expression `1 / 3` should be something like `0.33333`. However, in Python, if both operands to the `/` operator are integers, then it will perform an integer division, truncating any remainder.¹²

These expressions can be arbitrarily deeply nested combinations of primitives. The rules used for evaluation are essentially the same as the ones you learned in school; the interpreter proceeds by applying the operations in precedence order¹³, evaluating sub-expressions to get new values, and then evaluating the expressions those values participate in, until a single value results.

3.2.2 Variables

We cannot go very far without *variables*. A variable is a name that we can *bind* to have a particular value and then later use in an expression. When a variable is encountered in an expression, it is evaluated by looking to see to what value it is bound.

An interpreter keeps track of which variables are bound to what values in *binding environments*. An environment specifies a mapping between variable names and values. The values can be integers, floating-point numbers, characters, or pointers to more complex entities such as procedures or larger collections of data.

Here is an example binding environment:

¹² This behavior will no longer be the default in Python 3.0.

¹³ Please Excuse My Dear Aunt Sally (Parentheses, Exponentiation, Multiplication, Division, Addition, Subtraction)

b	3
x	2.2
foo	-1012

Each row represents a binding: the entry in the first column is the variable name and the entry in the second column is the value it to which it is bound.

When you start up the Python shell, you immediately start interacting with a local binding environment. You can add a binding or change an existing binding by evaluating an assignment statement of the form:

```
<var> = <expr>
```

where `<var>` is a variable name (a string of letters or digits or the character `_`, not starting with a digit) and `<expr>` is a Python expression.¹⁴

Expressions are always evaluated in some environment.

We might have the following interaction in a fresh Python shell:

```
>>> a = 3
>>> a
3
>>> b
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
NameError: name 'b' is not defined
>>>
```

We started by assigning the variable `a` to have the value 3. That added a binding for `a` to the local environment.

Next, we evaluated the expression `a`. The value of an expression with one or more variable names in it cannot be determined unless we know with respect to what environment it is being evaluated. Thus, we will always speak of evaluating expressions *in* an environment. During the process of evaluating an expression in some environment `E`, if the interpreter comes to a variable, it looks up that variable in `E`: if `E` contains a binding for the variable, then the associated value is returned; if it does not, then an error is generated. In the Python shell interaction above, we can see that the interpreter was able to find a binding for `a` and return a value, but it was not able to find a binding for `b`.

Why do we bother defining values for variables? They allow us to re-use an intermediate value in a computation. We might want to compute a formula in two steps, as in:

```
>>> c = 952**4
>>> c**2 + c / 2.0
6.7467650588636822e+23
```

¹⁴ When we want to talk about the abstract form or syntax of a programming language construct, we will often use *meta-variables*, written with angle brackets, like `<var>`. This is meant to signify that `<var>` could be any Python variable name, for example.

They will also play a crucial role in abstraction and the definition of procedures. By giving a name to a value, we can isolate the use of that value in other computations, so that if we decide to change the value, we only have to change the definition (and not change a value several places in the code).

It is fine to reassign the value of a variable; although we use the equality symbol = to stand for assignment, we are not making a mathematical statement of equality. So, for example, we can write:

```
>>> a = 3
>>> a = a + 1
>>> a
4
```

Exercise 3.1. What is the result of evaluating this sequence of assignment statements and the last expression? Determine this by hand-simulating the Python interpreter. Draw an environment and update the stored values as you work through this example.

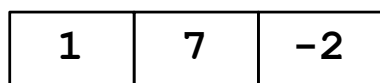
```
>>> a = 3
>>> b = a
>>> a = 4
>>> b
```

3.3 Structured data

We will often want to work with large collections of data. Rather than giving each number its own name, we want to organize the data into natural structures: grocery lists, matrices, sets of employee medical records. In this section, we will explore a simple but enormously useful and flexible data structure, which is conveniently built into Python: the list. The precise details of how lists are represented inside a computer vary from language to language. We will adopt an abstract model in which we think of a list as an ordered sequence of memory locations that contain values. So, for example, in Python, we can express a list of three integers as:

```
>>> [1, 7, -2]
[1, 7, -2]
```

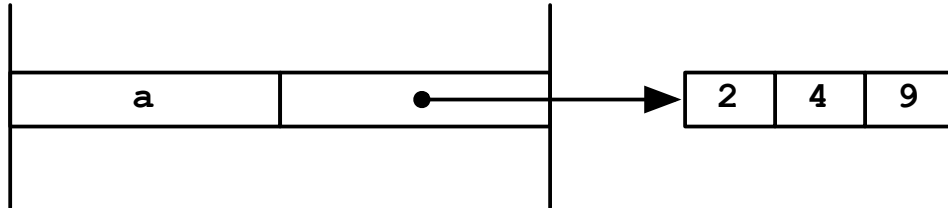
which we will draw in an abstract memory diagram as:



We can assign a list to a variable:

```
>>> a = [2, 4, 9]
```

A binding environment associates a name with a single fixed-size data item. So, if we want to associate a name with a complex structure, we associate the name directly with a 'pointer' to (actually, the memory address of) the structure. So we can think of `a` as being bound to a 'pointer' to the list:



Now that we have lists, we have some new kinds of expressions, which let us extract components of a list by specifying their indices. An index of 0 corresponds to the first element of a list. An index of -1 corresponds to the last element (no matter how many elements there are).¹⁵ So, if `a` is bound as above, then we would have:

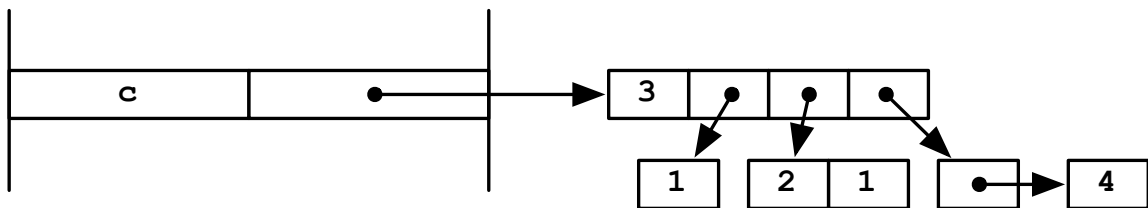
```
>>> a[0]
2
>>> a[2]
9
>>> a[-1]
9
>>> a[3]
Traceback (most recent call last):
  File "<stdin>", line 1, in <module>
IndexError: list index out of range
```

Note that if we attempt to access an element of the list that is not present (in this case, the fourth element of a three-element list), then an error is generated.

Lists can be nested inside one another. The Python expression:

```
>>> c = [3, [1], [2, 1], [[4]]]
```

creates a list that looks, in memory, like this:

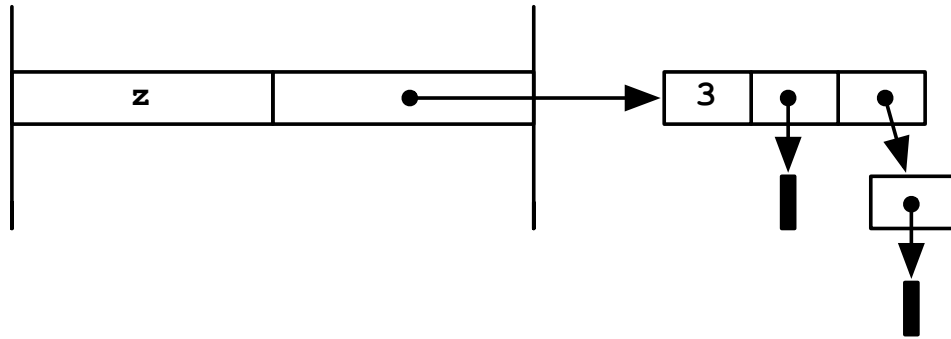


It is also possible to have an empty list, which is written in Python as `[]`. We will draw it in our memory diagrams as a small black box. So, for example, this list

```
>>> z = [3, [], [[]]]
```

looks like this in memory:

¹⁵ See the Python tutorial for much more on list indexing.



Python has a useful function, `len`, which takes a list as an argument and returns its length. It does not look inside the elements of the list—it just returns the number of elements at the top level of structure. So, we have

```
>>> len([1, 2, 3])
3
>>> len([[1, 2, 3]])
1
```

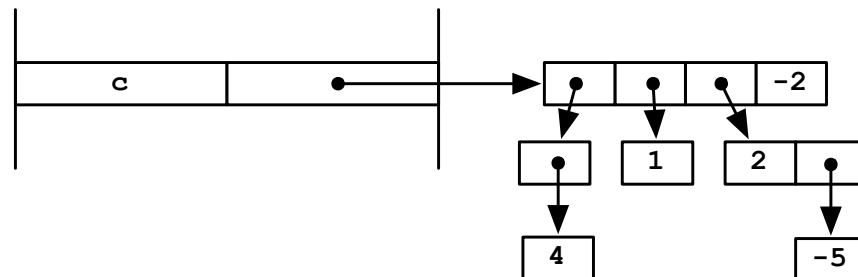
Exercise 3.2. Draw a diagram of the binding environment and memory structure after the following statement has been evaluated:

```
a = [[1], 2, [3, 4]]
```

Exercise 3.3. Draw a diagram of the binding environment and memory structure after the following statement has been evaluated:

```
a = [[]]
```

Exercise 3.4. Give a Python statement which, when evaluated, would give rise to this memory structure:



What is the value, in this environment, of the following expressions:

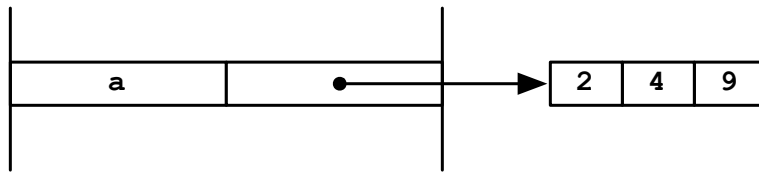
- `c[1]`
- `c[-1]`
- `c[2][1]`

3.3.1 List mutation and shared structure

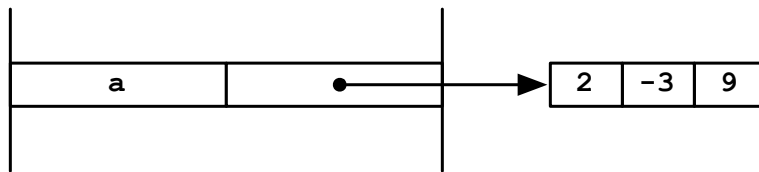
Lists are *mutable* data structures, which means that we can actually change the values stored in their elements. We do this by using element-selection expressions, like `a[1]` on the left-hand side of an assignment statement. So, the assignment

```
a[1] = -3
```

assigns the second element of `a` to be `-3`. In more detail, the left-hand side of this expression evaluates to a pointer to a specific location in memory (just as `a`'s value is a pointer to a location in memory); then the assignment statement changes the value stored there by inserting the value of the right-hand side of the expression. If that statement were evaluated in this environment,



then the resulting environment would be:



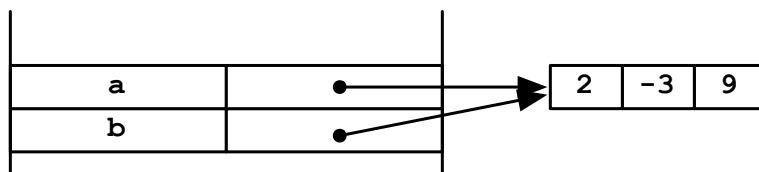
We have permanently changed the list named `a`.

In this section, we will explore the consequences of the mutability of lists; programs that change list structure can become *very* confusing, but you can always work your way through what is happening by drawing out the memory diagrams.

Continuing the previous example, let us remember that `a` is bound directly to a pointer to a list (or a sequence of memory cells), and think about what happens if we do:

```
>>> b = a
```

Now, `a` and `b` are both names for *the same list structure*, resulting in a memory diagram like this:



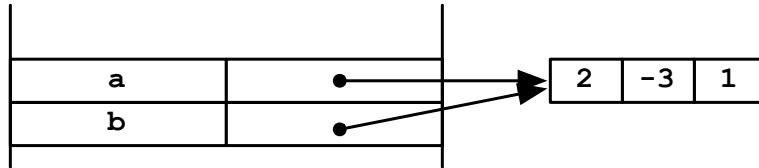
Now, we can reference parts of the list through `b`, and even change the list structure that way:

```
>>> b[0]
2
>>> b[2] = 1
```

Notice that, because `a` and `b` point to the same list, changing `b` changes `a`!

```
>>> a
[2, -3, 1]
```

Here is the memory picture now:

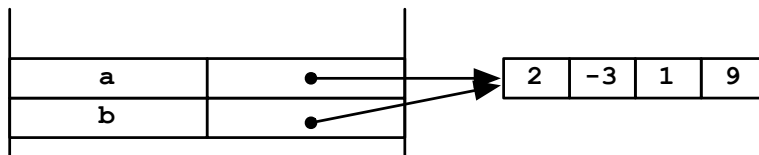


This situation is called *aliasing*: the name `b` has become an alias for `a`. Aliasing can be useful, but it can also cause problems, because you might inadvertently change `b` (by passing it into a procedure that changes one of its structured arguments, for example) when it is important to you to keep `a` unmodified.

Another important way to change a list is to add or delete elements. We will demonstrate adding elements to the end of a list, but see the Python documentation for more operations on lists. This statement

```
>>> a.append(9)
```

causes a new element to be added to the end of the list name `a`. The resulting memory state is:



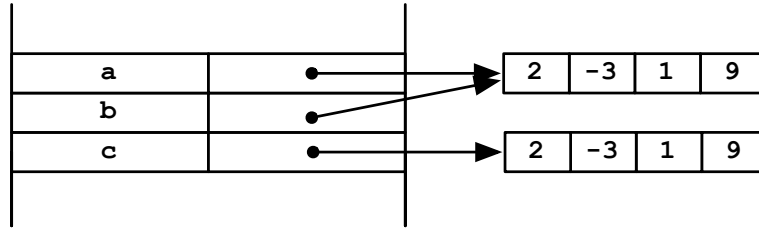
As before, because `a` and `b` are names for the same list (i.e., they point to the same memory sequence), `b` is changed too. This is a side effect of the aliasing between `a` and `b`:

```
>>> b
[2, -3, 1, 9]
```

Often, it will be important to make a fresh copy of a list so that you can change it without affecting the original one. Here are two equivalent ways to make a copy (use whichever one you can remember):

```
>>> c = list(a)
>>> c = a[:]
```

Here is a picture of the memory at this point:



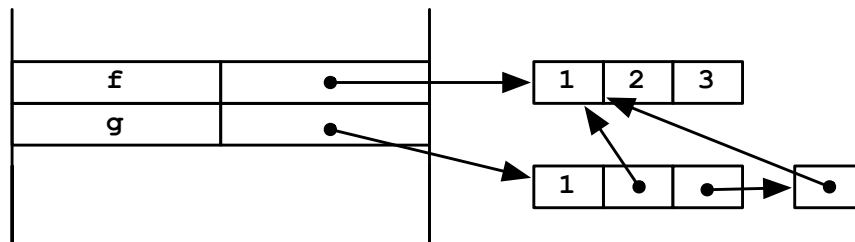
Now, if we change an element of *c*, it does not affect *a* (or *b*):

```
>>> c[0] = 100
>>> c
[100, -3, 1, 9]
>>> a
[2, -3, 1, 9]
```

We can make crazy lists that share structure within a single list:

```
>>> f = [1, 2, 3]
>>> g = [1, f, [f]]
>>> g
[1, [1, 2, 3], [[1, 2, 3]]]
```

which results in this memory structure:



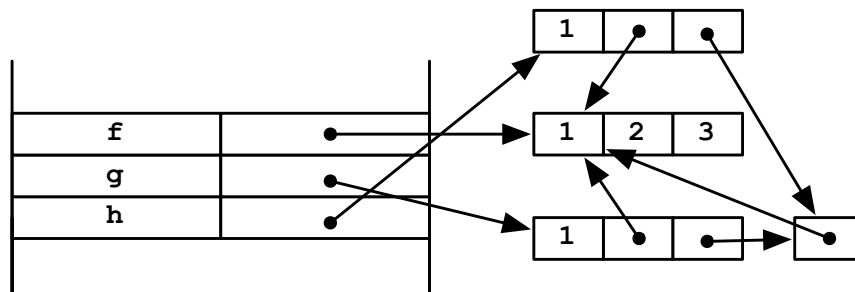
If you want to add an element to a list and get a new copy at the same time, you can do

```
>>> a + [1]
```

The `+` operator makes a new list that contains the elements of both of its arguments, but does not share any top-level structure. All of our methods of copying only work reliably if your lists do not contain other lists, because it only copies one level of list structure. So, for example, if we did:

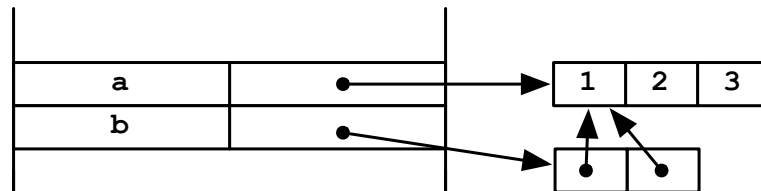
```
>>> h = list(g)
```

we would end up with this picture:

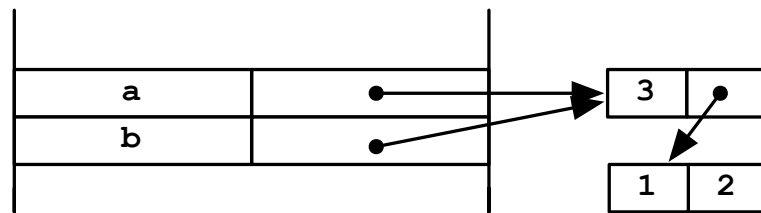


It is clear that if we were to change `f`, it would change `h`, so this is not a completely new copy. If you need to copy deep structures, that is, to make a copy not only of the top level list structure, but of the structures of any lists that list contains, and the lists those lists contain, etc., you will need to use the Python `copy . deepcopy` procedure.

Exercise 3.5. Give a sequence of Python statements which, when evaluated, would give rise to this memory structure:



Exercise 3.6. Give a sequence of Python statements which, when evaluated, would give rise to this memory structure:



Exercise 3.7. Show the memory structure after this sequence of expressions.

```
>>> a = [1, 2, 3]
>>> b = [a, a]
>>> a.append(100)
```

What will be the value of `b` at this point?

Exercise 3.8. Show the memory structure after this sequence of expressions.

```
>>> a = [5, 6]
>>> b = [1, 2]
>>> c = b + a
```

We will use this “curvy road” symbol to indicate sections of the notes or exercises that are somewhat more difficult and not crucial to understanding the rest of the notes. Feel free to skip them on first reading; but, of course, we think they are cool.¹⁶

¹⁶ Thanks to Don Knuth’s *Art of Computer Programming* for the idea of the curvy road sign.

Exercise 3.9.

Show the memory structure after this sequence of expressions.



```
>>> a = [1, 2, 3]
>>> a[1] = a
```

What will be the value of a at this point?

3.3.2 Tuples and strings

Python has two more list-like data types that are important to understand.

A *tuple* is a structure that is like a list, but is *not* mutable. You can make new tuples, but you cannot change the contents of a tuple or add elements to it. A tuple is typically written like a list, but with round parentheses instead of square ones:

```
>>> a = (1, 2, 3)
```

In fact, it is the commas and not the parentheses that matter here. So, you can write

```
>>> a = 1, 2, 3
>>> a
(1, 2, 3)
```

and still get a tuple. The only tricky thing about tuples is making a tuple with a single element. We could try

```
>>> a = (1)
>>> a
1
```

but it does not work, because in the expression `(1)` the parentheses are playing the standard grouping role (and, in fact, because parentheses do not make tuples). So, to make a tuple with a single element, we have to use a comma:

```
>>> a = 1,
>>> a
(1,)
```

This is a little inelegant, but so it goes.

Tuples will be important in contexts where we are using structured objects as 'keys', that is, to index into another data structure, and where inconsistencies would occur if those keys could be changed.

An important special kind of tuple is a *string*. A string can almost be thought of as a tuple of characters. The details of what constitutes a character and how they are encoded is complicated, because modern character sets include characters from nearly all the world's languages. We will stick to the characters we can type easily on our keyboards. In Python, you can write a string with either single or double quotes: `'abc'` or `"abc"`. You can select parts of it as you would a list:

```
>>> s = 'abc'
>>> s[0]
'a'
>>> s[-1]
'c'
```

The strange thing about this is that `s` is a string, and because Python has no special data type to represent a single character, `s[0]` is also a string.

We will frequently use `+` to concatenate two existing strings to make a new one:

```
>>> to = 'Jody'
>>> fromP = 'Robin'
>>> letter = 'Dear ' + to + ",\n It's over.\n" + fromP
>>> print letter
Dear Jody,
  It's over.
Robin
```

As well as using `+` to concatenate strings, this example illustrates several other small but important points:

- You can put a single quote inside a string that is delimited by double-quote characters (and vice versa).
- If you want a new line in your string, you can write `\n`. Or, if you delimit your string with a *triple* quote, it can go over multiple lines.
- The `print` statement can be used to print out results in your program.
- Python, like most other programming languages, has some *reserved words* that have special meaning and cannot be used as variables. In this case, we wanted to use `from`, but that has a special meaning to Python, so we used `fromP` instead.

Structured assignment

Once we have lists and tuples, we can use a nice trick in assignment statements, based on the packing and unpacking of tuples.

```
>>> a, b, c = 1, 2, 3
>>> a
1
>>> b
2
>>> c
3
```

Or, with lists,

```
>>> [a, b, c] = [1, 2, 3]
>>> a
1
>>> b
2
>>> c
3
```

When you have a list (or a tuple) on the left-hand side of an assignment statement, you have to have a list (or tuple) of matching structure on the right-hand side. Then Python will “unpack” them both, and assign to the individual components of the structure on the left hand side. You can get fancier with this method:

```
>>> thing = [8, 9, [1, 2], 'John', [33.3, 44.4]]
>>> [a, b, c, d, [e1, e2]] = thing
>>> c
[1, 2]
>>> e1
33.299999999999997
```

3.4 Procedures

Procedures are computer-program constructs that let us capture common patterns of computation by:

- Gathering together sequences of statements
- Abstracting away from particular data items on which they operate.

Here is a procedure definition,¹⁷ and then its use:

```
def square(x):
    return x * x

>>> square(6)
36
>>> square(2 - square(2))
4
```

We will work through, in detail, what happens when the interpreter evaluates a procedure definition, and then the application of that procedure.

3.4.1 Definition

A procedure definition has the abstract form:

```
def <name>(<fp1>, ..., <fpn>):
    <statement1>
    ...
    <statementk>
```

There are essentially three parts:

¹⁷ In the code displayed in these notes, we will show procedures being defined and then used, as if the definitions were happening in the Python shell (but without the prompts). In fact, you should not type procedure definitions into the shell, because if you make a mistake, you will have to re-type the whole thing, and because multi-line objects are not handled very well. Instead, type your procedure definitions into a file in Idle, and then test them by ‘running’ the file in Idle (which will actually evaluate all of the expressions in your file) and then evaluating test expressions in Idle’s shell.

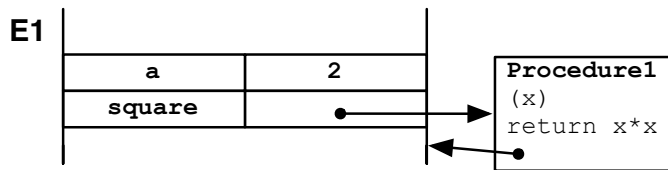
- `<name>` is a name for the procedure, with the same restrictions as a variable name;
- `<fp1>`, ..., `<fnp>` is a list of *formal parameters*, which will stand for the data items on which this procedure will operate; and
- `<statement1>`, ..., `<statementk>`, known as the *body of the procedure*, is a list of Python statements (right now, we know about assignment statements, print statements, and basic expressions, but there will be more.)

When we evaluate a procedure definition in an environment,¹⁸ E, Python does two things:

1. Makes a procedure object¹⁹ that contains the formal parameters, the body of the procedure, and a pointer to E; and then
2. Binds the name to have this procedure as its value.

Here is an example of an environment after we have evaluated

```
def square(x):
    return x * x
```



Note how the construct to which `square` points is a procedure object, with a list of formal parameters, a body, and a pointer to its environment.

3.4.2 Procedure calls

When you evaluate an expression of the form

```
<expr0>(<expr1>, ..., <exprn>)
```

the Python interpreter treats this as a procedure call. It will be easier to talk about a specific case of calling a procedure, so we will illustrate with the example

```
>>> square(a + 3)
```

evaluated in environment E1. Here are the steps:

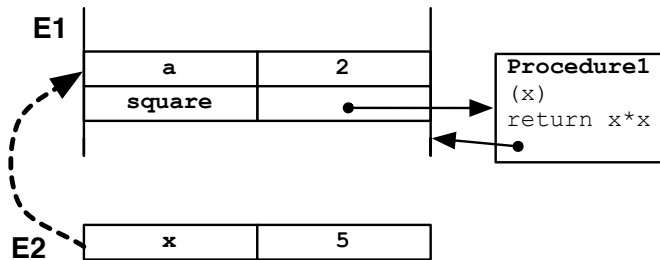
1. The expression that determines the procedure (`<expr0>`) is evaluated. In this case, we evaluate `square` in E1 and get Procedure1.
2. The expressions that determine the arguments (`<expr1>`, ..., `<exprn>`) are evaluated. In this case, we evaluate `a + 3` in E1 and get 5.

¹⁸ Remember that every expression is always evaluated with respect to some environment.

¹⁹ In our memory diagrams, we will show the procedure object as a box with the word **Procedure**`<N>`, where `N` is some integer, at the top; we give the procedure objects these numbers so we can refer to them easily in the text.

3. A new environment (in this case E2) is created, which:
 - binds the formal parameters of the procedure (in this case `x`) to the values of its arguments (in this case, 5); and
 - has as its parent the environment in which the procedure was defined (we find a pointer to this environment stored in the procedure; in this case E1 is its *parent*).

At this point, our memory looks like this:



The dotted line between E2 and E1 is intended to indicate that E1 is the *parent environment* of E2.

4. The statements of the procedure body are evaluated in the new environment until either a return statement or the end of the list of statements is reached. If a return statement is evaluated, the expression after the return is evaluated and its value is returned as the value of the procedure-call expression. Otherwise, the procedure has no return value, and the expression has the special Python value `None`.

In our example, the only statement in the body is a return statement. So, we evaluate the expression `x * x` in E2, obtaining a value of 25. That value is returned as the value of the entire procedure-call expression `square(a + 3)`.

This basic mechanism can generate behavior of arbitrary complexity, when coupled with recursion or other control structures, discussed in [section 2.3](#).

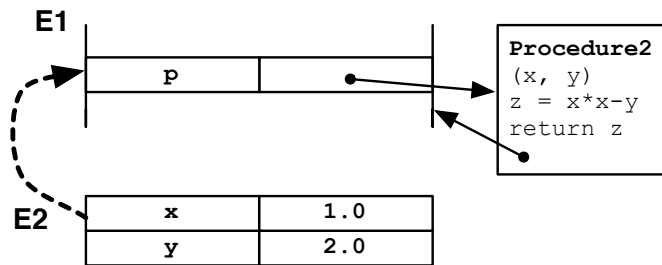
Worked example 1

Let's examine what happens when we evaluate the following Python code:

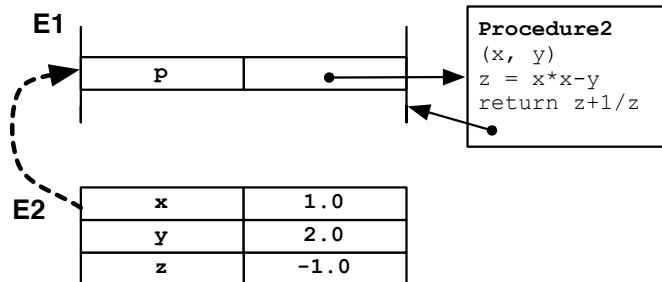
```
def p(x, y):
    z = x*x - y
    return z + 1/z
```

```
>>> p(1.0, 2.0)
-2.0
```

Here is a picture of the calling environment (E1) and the procedure-call environment (E2) just before the body of the procedure is evaluated:



After evaluating $z = x*x - y$ in E2, we have:



Finally, we evaluate $z + 1/z$ in E2 and get -2.0, which we return.

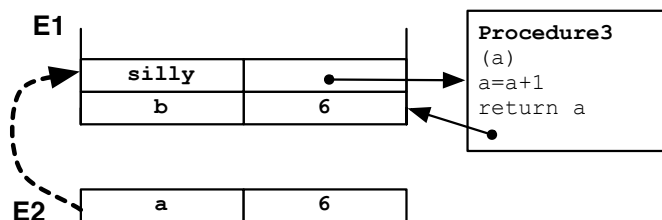
Worked example 2

Here is another example:

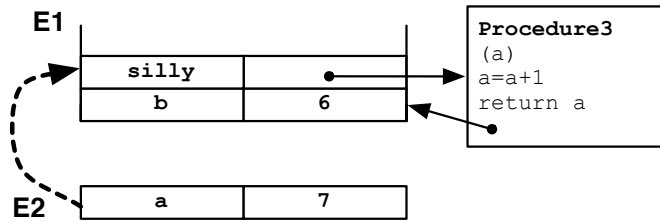
```
def silly(a):
    a = a + 1
    return a
```

```
>>> b = 6
>>> silly(b)
7
```

Here is a picture of the calling environment (E1) and the procedure-call environment (E2) just before the body of the procedure is evaluated:



After evaluating $a = a + 1$ in E2, we have:



Finally, we evaluate `a` in E2 and get 7, which we return.

Convince yourself that the execution below works fine, and notice that having a binding for `a` in E2 means that we leave the binding for `a` in E1 unmolested:

```
>>> a = 2
>>> silly(a)
3
>>> a
2
```

Exercise 3.10. Draw the environment diagram at the time the statement with the arrow is being executed:

```
def fizz(x, y):
    p = x + y
    q = p*p
    return q + x    # <-----

>>> fizz(2, 3)
```

Exercise 3.11. Draw the environment diagram at the time the statement with the arrow is being executed:

```
def fuzz(a, b):
    return a + b    # <-----

def buzz(a):
    return fuzz(a, square(a))

>>> buzz(2)
```

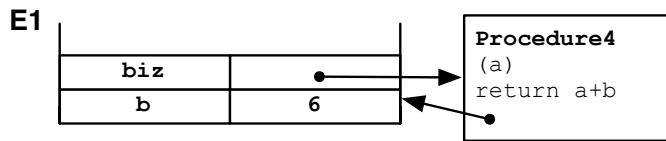
3.4.3 Non-local references

So far, whenever we needed to evaluate a variable, there was a binding for that variable in the 'local' environment (the environment in which we were evaluating the expression). But consider this example:

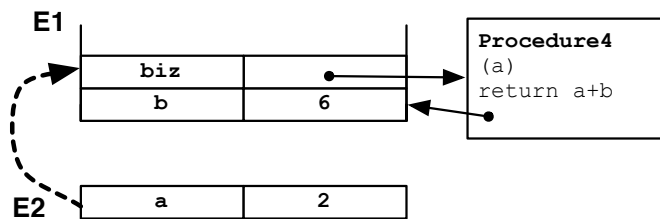
```
def biz(a):
    return a + b
```

```
>>> b = 6
>>> biz(2)
```

This actually works fine, and returns 8. Let's see why. Here is the environment, E1, in which the Python shell is working, after we execute the `b = 6` statement:



Now, when we evaluate `biz(2)` in E1, we make a new environment, E2, which binds `a` to 2 and points to E1 as its parent environment.



We need to elaborate, slightly, how it is that a variable v is evaluated in an environment E :

- We look to see if there is a binding for v in E ; if so, we stop and return it.
- If not, we evaluate v in the parent environment of E . If E has no parent, we generate an error.

It is important to appreciate that this process will continue up an arbitrarily long chain of environments and their parents until either a binding for v is found or there are no more environments to examine.

So, in our case, we will evaluate the expression `a + b` in environment E2. We start by evaluating `a` and finding value 2 in E2. Then, we evaluate `b` and cannot find it in E2...but we don't panic! We follow the parent pointer to E1 and try again. We find a binding for `b` in E1 and get the value 6. So, the value of `a + b` in E2 is 8.

Exercise 3.12. Draw a picture of the relevant binding environments when the statement with the arrow is being executed. What value does the procedure return?

```
def f(a):
    return a + g(a)

def g(b):
    return a + b    # <-----

>>> f(3)
```

Exercise 3.13. Draw a picture of the relevant binding environments when the statement with the arrow is being executed. What value does the procedure return?

```
def f(a):
    def g(b):
        return a + b    # <-----
    return a + g(a)

>>> f(3)
```

Exercise 3.14. Draw a picture of the relevant binding environments when the statement with the arrow is being executed. What value does the procedure return? Does it cause an error?

```
def a(a):
    return a * a

>>> a(2)
```

3.4.4 Environments in Python

Generally, Python establishes the following binding environments:

1. `__builtin__`: the mother of all environments: it contains the definitions of all sorts of basic symbols, like `list` and `sum`. It is the parent of all module environments.
2. Module: each separate file that contains Python code is called a module and establishes its own environment, whose parent is `__builtin__`.
3. Procedure calls: as described in this section. A procedure that is defined at the 'top level' of a module (that is, not nested in the definition of another procedure) has the module's environment as its parent, and has its name defined in the module's environment. Procedures that are defined inside other procedures have the procedure-call environment of the containing procedure as their parent.

We have seen two operations that cause bindings to be created: assignments and procedure calls. Bindings are also created when you evaluate an `import` statement. If you evaluate

```
import math
```

then a file associated with the `math` module is evaluated and the name `math` is bound, in the current environment, to the `math` module, which is an environment. No other names are added to the current environment, and if you want to refer to names in that module, you have to qualify them, as in `math.sqrt`. (As we will see in a subsequent section, the evaluation of this expression first evaluates `math`, which returns an environment. We can then evaluate `sqrt` with respect to that environment, thus returning a pointer to the procedure stored there.) If you execute

```
from math import sqrt
```

then the `math` file is evaluated, and the name `sqrt` is bound, in the current environment, to whatever the name `sqrt` is bound in the `math` module. But note that if you do this, the name `math` is not bound to anything, and you cannot access any other procedures in the `math` module unless you import them explicitly, as well.

3.4.5 Non-local references in procedures

There is an important subtlety in the way names are handled in the environment created by a procedure call. When a name that is not bound in the local environment is referenced, then it is looked up in the chain of parent environments. So, as we have seen, it is fine to have

```
a = 2
def b():
    return a
```

When a name is assigned inside a procedure, a new binding is created for it in the environment associated with the current call of that procedure. So, it is fine to have

```
a = 2
def b():
    a = 3
    c = 4
    return a + c
```

Both assignments cause new bindings to be made in the local environment, and it is those bindings that are used to supply values in the return expression. It will not change `a` in the global environment.

But here is a code fragment that causes trouble:

```
a = 3
def b():
    a = a + 1
    print a
```

It seems completely reasonable, and you might expect `b()` to return 4. But, instead, it generates an error. What is going on?? It all has to do with when Python decides to add a binding to the local environment. When it sees this procedure definition, it sees that the name `a` occurs on the left-hand-side of an assignment statement, and so, at the very beginning, it puts a new entry for `a` in the local environment, but without any value bound to it. Now, when it is time to evaluate the statement

```
a = a + 1
```

Python starts by evaluating the expression on the right hand side: `a + 1`. When it tries to look up the name `a` in the procedure-call environment, it finds that `a` has been added to the environment, but has not yet had a value specified. So it generates an error.

In Python, we can write code to increment a number named in the global environment, by using the `global` declaration:

```
a = 3
def b():
    global a
    a = a + 1
    print a
>>> b()
4
>>> b()
5
>>> a
5
```

The statement `global a` asks that a new binding for `a` *not* be made in the procedure-call environment. Now, all references to `a` are to the binding in the module’s environment, and so this procedure actually changes `a`.

In Python, we can only make assignments to names in the procedure-call environment or to the module environment, but not to names in intermediate environments. So, for example,

```
def outer():
    def inner():
        a = a + 1
    a = 0
    inner()
```

In this example, we get an error, because Python has made a new binding for `a` in the environment for the call to `inner`. We would really like `inner` to be able to see and modify the `a` that belongs to the environment for `outer`, but there is no way to arrange this. Some other programming languages, such as Scheme, offer more fine-grained control over how the scoping of variables is handled.

3.4.6 Procedures as first-class objects

In Python, unlike many other languages, procedures are treated in much the same way as numbers: they can be stored as values of variables, can be passed as arguments to procedures, and can be returned as results of procedure calls. Because of this, we say that procedures are treated as “first-class” objects. We will explore this treatment of “higher-order” procedures (procedures that manipulated procedures) throughout this section.

First of all, it is useful to see that we can construct (some) procedures without naming them using the `lambda` constructor:

```
lambda <var1>, ..., <varn> : <expr>
```

The formal parameters are `<var1>, ..., <varn>` and the body is `<expr>`. There is no need for an explicit `return` statement; the value of the expression is always returned. A single expression can only be one line of Python, such as you could put on the right hand side of an assignment statement. Here are some examples:

```
>>> f = lambda x: x*x
>>> f
<function <lambda> at 0x4ecf0>
```

```
>>> f(4)
16
```

Here is a procedure of two arguments defined with `lambda`:

```
>>> g = lambda x,y : x * y
>>> g(3, 4)
12
```

Using the expression-evaluation rules we have already established, we can do some fancy things with procedures, which we will illustrate throughout the rest of this section.

```
>>> procs = [lambda x: x, lambda x: x + 1, lambda x: x + 2]
>>> procs[0]
<function <lambda> at 0x83d70>
>>> procs[1](6)
7
```

Here, we have defined a list of three procedures. We can see that an individual element of the list (e.g., `procs[0]`) is a procedure.²⁰ So, then `procs[1](6)` applies the second procedure in the list to the argument 6. Since the second procedure returns its argument plus 1, the result is 7.

Here is a demonstration that procedures can be assigned to names in just the way other values can.

```
>>> fuzz = procs[2]
>>> fuzz(3)
5
```

```
def thing(a):
    return a * a
```

```
>>> thing(3)
9
>>> thang = thing
>>> thang(3)
9
```

Passing procedures as arguments

Just as we can pass numbers or lists into procedures as arguments, we can pass in procedures as arguments, as well.

What if we find that we are often wanting to perform the same procedure twice on an argument? That is, we seem to keep writing `square(square(x))`. If it were always the same procedure we were applying twice, we could just write a new procedure

```
def squaretwice(x):
    return square(square(x))
```

²⁰ In the programming world, people often use the words “function” and “procedure” either interchangeable or with minor subtle distinctions. The Python interpreter refers to the objects we are calling procedures as “functions.”

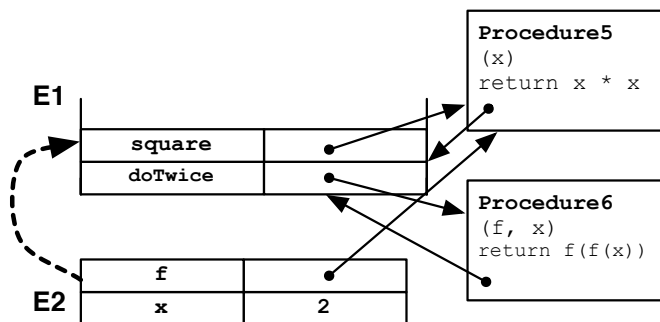
But what if it is different procedures? We can write a new procedure that takes a procedure `f` as an argument and applies it twice:

```
def doTwice(f, x):
    return f(f(x))
```

So, if we wanted to square twice, we could do:

```
>>> doTwice(square, 2)
16
```

Here is a picture of the environment structure in place when we are evaluating the `return f(f(x))` statement in `doTwice`:



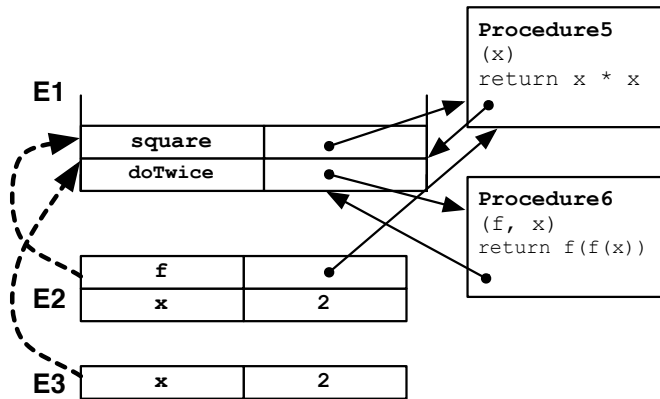
Environment E1 is the module environment, where procedures `square` and `doTwice` were defined and where the expression `doTwice(square, 2)` is being evaluated. The interpreter:

- Evaluates `doTwice` in E1 and gets Procedure6.
- Evaluates `square` in E1 and gets Procedure5.
- Evaluates `2` in E1 and gets 2.
- Makes the new binding environment E2, binding the formal parameters, `f` and `x`, of Procedure 6, to actual arguments Procedure5 and 2.
- Evaluates the body of Procedure6 in E2.

Now, let's peek one level deeper into the process. To evaluate `f(f(x))` in E2, the interpreter starts by evaluating the inner `f(x)` expression. It

- Evaluates `f` in E2 and gets Procedure5.
- Evaluates `x` in E2 and gets 2.
- Makes the new binding environment E3, binding the formal parameter `x`, of Procedure5, to 2. (Remember that the parent of E3 is E1 because Procedure5 has E1 as a parent.)
- Evaluates the body of Procedure5 in E3, getting 4.

The environments at the time of this last evaluation step are:



A similar thing happens when we evaluate the outer application of `f`, but now with argument 4, and a return value of 16.

Exercise 3.15. Here is the definition of a procedure `sumOfProcs` that takes two procedures, `f` and `g`, as well as another value `x`, as arguments, and returns `f(x) + g(x)`. The `sumOfProcs` procedure is then applied to two little test procedures:

```
def sumOfProcs(f, g, x):
    return f(x) + g(x)

def thing1(a):
    return a*a*a
def thing2(b):
    return b+1    # <-----

>>> sumOfProcs(thing1, thing2, 2)
```

Draw a picture of all of the relevant environments at the moment the statement with the arrow is being evaluated. What is the return value of the call to `sumOfProcs`?

Returning procedures as values

Another way to apply a procedure multiple times is this:

```
def doTwiceMaker(f):
    return lambda x: f(f(x))
```

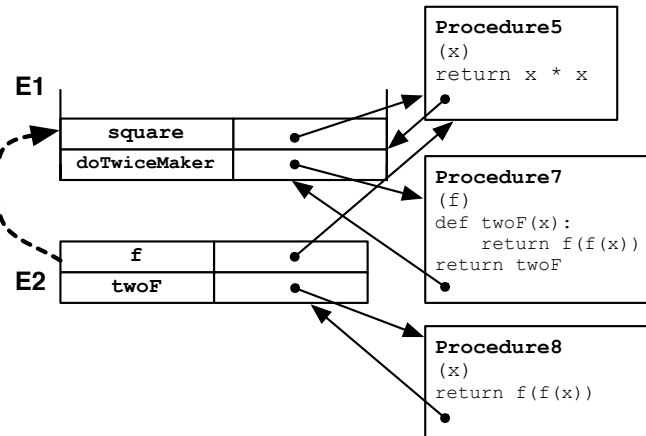
This is a procedure that *returns a procedure!* If you would rather not use `lambda`, you could write it this way:

```
def doTwiceMaker(f):
    def twoF(x):
        return f(f(x))
    return twoF
```

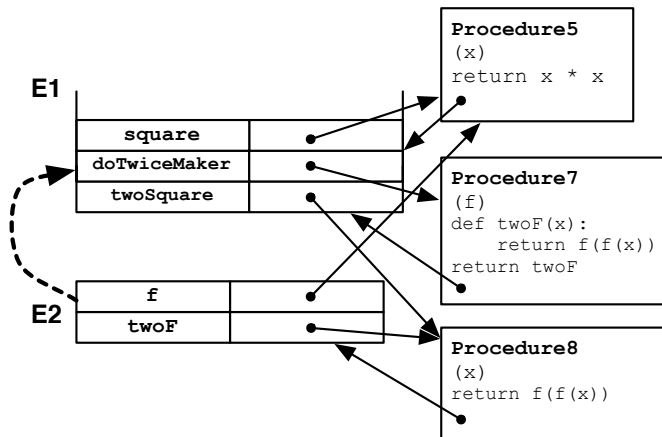
Now, to use `doTwiceMaker`, we might start by calling it with a procedure, such as `square`, as an argument and naming the resulting procedure.

```
>>> twoSquare = doTwiceMaker(square)
```

Here is a picture of the environments just before the `return twoF` statement in `doTwiceMaker` is evaluated.



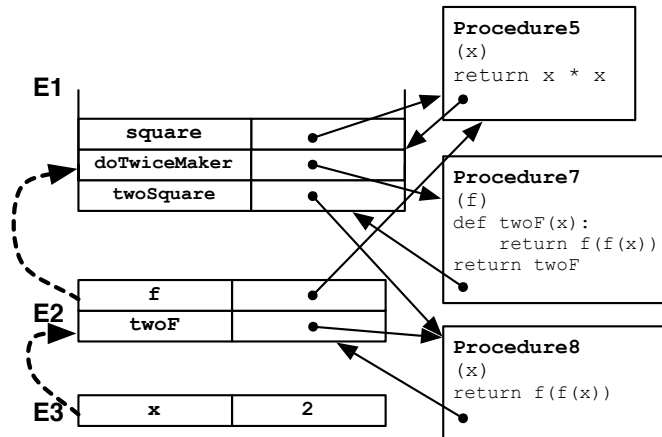
Here is a picture of the environments after the `doTwiceMaker` returns its value and it is assigned to `twoSquare` in E1. It is important to see that Procedure8 is the return value of the call to `doTwiceMaker` and that, because Procedure8 retains a pointer to the environment in which it was defined, we need to keep E2 around. And it is E2 that remembers which procedure (via its binding for `f`) is going to be applied twice.



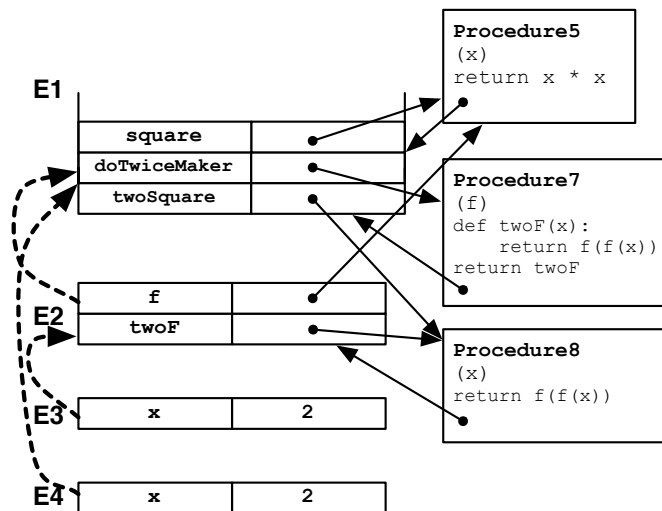
Now, when we evaluate this expression in E1

```
>>> twoSquare(2)
```

we start by making a new binding environment, E3, for the procedure call. Note that, because the procedure we are calling, Procedure8, has E2 stored in it, we set the parent of E3 to be E2.



Next, we evaluate the body of Procedure8, which is `return f(f(x))` in E3. Let's just consider evaluating the inner expression `f(x)` in E3. We evaluate `f` in E3 and get Procedure5, and evaluate `x` and get 2. Now, we make a new binding environment, E4, to bind the formal parameter of Procedure5 to 2. Because Procedure5 has a stored pointer to E1, E4's parent is E1, as shown here:



Evaluating the body of Procedure5 in E4 yields 4. We will repeat this process to evaluate the outer application of `f`, in `f(f(x))`, now with argument 4, and end with result 16.

Essentially the same process would happen when we evaluate

```
>>> doTwiceMaker(square)(2)
16
```

except the procedure that is created by the expression `doTwiceMaker(square)` is not assigned a name; it is simply used as an intermediate result in the expression evaluation.

3.5 Object-oriented programming

We have seen structured data and interesting procedures that can operate on that data. It will often be useful to make a close association between collections of data and the operations that apply

to them. The style of programming that takes this point of view is *object-oriented programming* (OOP). It requires adding some simple mechanisms to our interpreter, but is not a big conceptual departure from the things we have already seen. It is, however, a different style of organizing large programs.

3.5.1 Classes and instances

In OOP, we introduce the idea of *classes* and *instances*. An *instance* is a collection of data that describe a single entity in our domain of interest, such as a person or a car or a point in 3D space. If we have many instances that share some data values, or upon which we would want to perform similar operations, then we can represent them as being members of a *class* and store the shared information once with the class, rather than replicating it in the instances.

Consider the following staff database for a large undergraduate course:

name	role	age	building	room	course
Pat	Prof	60	34	501	6.01
Kelly	TA	31	34	501	6.01
Lynn	TA	29	34	501	6.01
Dana	LA	19	34	501	6.01
Chris	LA	20	34	501	6.01

There are lots of shared values here, so we might want to define a class. A class definition has the form:

```
class <name>:
  <statement1>
  ...
  <statementn>
```

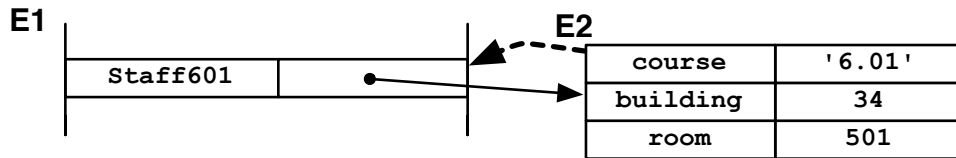
Here is the definition of simple class in our example domain:

```
class Staff601:
  course = '6.01'
  building = 34
  room = 501
```

From the implementation perspective, the most important thing to know is that **classes and instances are environments**.

When we define a new class, we make a new environment. In this case, the act of defining class Staff601 in an environment E1 results in a binding from Staff601 to E2, an empty environment whose parent is E1, the environment in which the class definition was evaluated. Now, the statements inside the class definition are evaluated in the new environment, resulting in a memory state like this:²¹

²¹ In fact, the string '6.01' should be shown as an external memory structure, with a pointer stored in the binding environment; for compactness in the following diagrams we will sometimes show the strings themselves as if they were stored directly in the environment.



Note how the common values and names have been captured in a separate environment.

Caveat: when we discuss methods in [section 3.5.2](#), we will see that the rules for evaluating procedure definitions inside a class definition are slightly different from those for evaluating procedure definitions that are not embedded in a class definition.

We will often call names that are bound in a class's environment *attributes* of the class. We can access these attributes of the class after we have defined them, using the *dot* notation:

```
<envExpr>.<var>
```

When the interpreter evaluates such an expression, it first evaluates `<envExpr>`; if the result is not an environment, then an error is signaled. If it is an environment, `E`, then the name `<var>` is looked up in `E` (using the general process of looking in the parent environment if it is not found directly in `E`) and the associated value returned.

So, for example, we could do:

```
>>> Staff601.room
501
```

We can also use the dot notation on the left-hand side of an assignment statement, if we wish to modify an environment. An assignment statement of the form

```
<envExpr>.<var> = <valExpr>
```

causes the name `<var>` in the environment that is the result of evaluating `<envExpr>` to be bound to the result of evaluating `<valExpr>`.

So, we might change the room in which 6.01 meets with:

```
Staff601.room = Staff601.room - 100
```

or add a new attribute with

```
Staff601.coolness = 11 # out of 10, of course...
```

Now, we can make an *instance* of a class with an expression of the form:

```
<classExpr>()
```

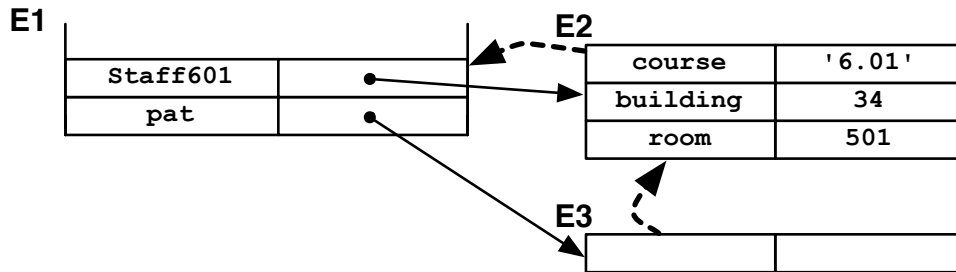
This expression has as its value a new empty environment whose parent pointer is the environment obtained as a result of evaluating `<classExpr>`.²²

So, if we do:

```
>>> pat = Staff601()
```

we will end up with an environment state like this:

²² Another way of thinking of this is that whenever you define a class, Python defines a procedure with the same name, which is used to create an instance of that class.



At this point, given our standard rules of evaluation and the dot notation, we can say:

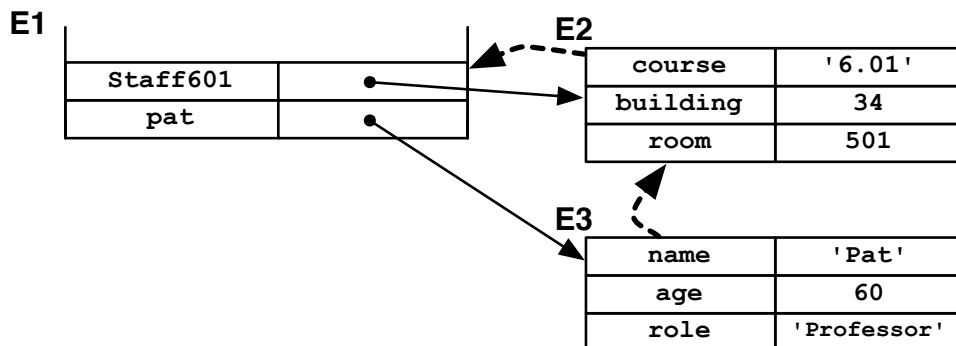
```
>>> pat.course
'6.01'
```

The interpreter evaluates `pat` in `E1` to get the environment `E3` and then looks up the name `course`. It does not find it in `E3`, so it follows the parent pointer to `E2`, and finds there that it is bound to `'6.01'`.

Similarly, we can set attribute values in the instance. So, if we were to do:

```
pat.name = 'Pat'
pat.age = 60
pat.role = 'Professor'
```

we would get this environment structure.

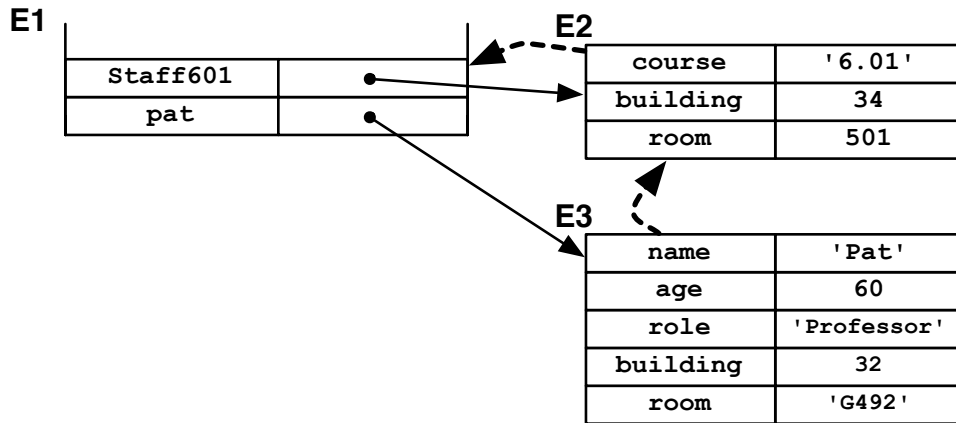


Note that these names are bound in the instance environment, not the class.

These structures are quite flexible. If we wanted to say that Professor Pat is an exception, holding office hours in a different place from the rest of the 6.01 staff, we could say:

```
pat.building = 32
pat.office = 'G492'
```

Here is the new environment state. Nothing is changed in the `Staff601` class: these assignments just make new bindings in `pat`'s environment, which 'shadow' the bindings in the class, so that when we ask for `pat`'s `building`, we get 32.



3.5.2 Methods

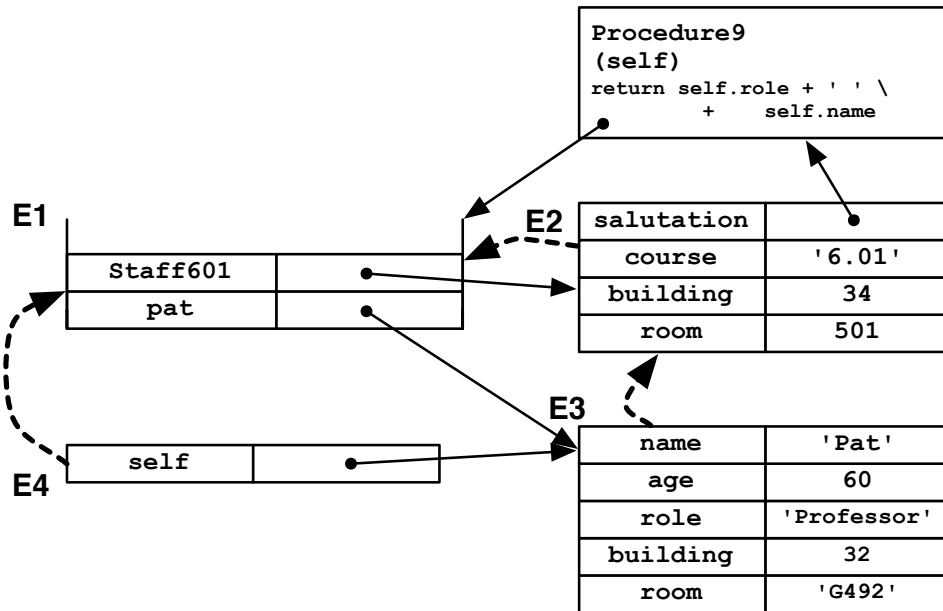
Objects and classes are a good way to organize procedures, as well as data. When we define a procedure that is associated with a particular class, we call it a *method* of that class. Method definition requires only a small variation on our existing evaluation rules.

So, imagine we want to be able to greet 6.01 staff members appropriately on the staff web site. We might add the definition of a `salutation` method:

```
class Staff601:
    course = '6.01'
    building = 34
    room = 501

    def salutation(self):
        return self.role + ' ' + self.name
```

This procedure definition, made inside the class definition, is evaluated in *almost* the standard way, resulting in a binding of the name `salutation` in the class environment to the new procedure. The way in which this process deviates from the standard procedure evaluation process is that the environment stored in the procedure is the module (file) environment, no matter how deeply nested the class and method definitions are:



Note (especially for Java programmers!) that the way the body of `salutation` has access to the attributes of the instance on which it is operating and to attributes of the class is via the instance we passed to it as the parameter `self`. The parent environment is E1, which means that methods cannot simply use the names of class attributes without accessing them through the instance.

The notation

```
Staff601.salutation(pat)
```

is a little clumsy; it requires that we remember to what class `pat` belongs, in order to get the appropriate `salutation` method. We ought, instead, to be able to write

```
pat.salutation(pat) ##### Danger: not legal Python
```

This should have exactly the same result. (Verify this for yourself if you do not see it, by tracing through the environment diagrams. Even though the `pat` object has no binding for `salutation`, the environment-lookup process will proceed to its parent environment and find a binding in the class environment.)

But this is a bit redundant, having to mention `pat` twice. So, Python added a special rule that says: *If you access a class method by looking in an **instance**, then that instance is automatically passed in as the first argument of the method.*

So, we can write

```
pat.salutation()
```

This is **exactly** equivalent to

```
Staff601.salutation(pat)
```

A consequence of this decision is that every method must have an initial argument that is an instance of the class to which the method belongs. That initial argument is traditionally called `self`, but it is not necessary to do so.

Here is a new method. Imagine that `Staff601` instances have a numeric `salary` attribute. So, we might say

```
pat.salary = 100000
```

Now, we want to write a method that will give a 6.01 staff member a k-percent raise:

```
class Staff601:
    course = '6.01'
    building = 34
    room = 501

    def salutation(self):
        return self.role + ' ' + self.name

    def giveRaise(self, percentage):
        self.salary = self.salary + self.salary * percentage
```

As before, we could call this method as

```
Staff601.giveRaise(pat, 0.5)
```

or we could use the short-cut notation and write:

```
pat.giveRaise(0.5)
```

This will change the `salary` attribute of the `pat` instance to 150000.²³

3.5.3 Initialization

When we made the `pat` instance, we first made an empty instance, and then added attribute values to it. Repeating this process for every new instance can get tedious, and we might wish to guarantee that every new instance we create has some set of attributes defined. Python has a mechanism to streamline the process of initializing new instances. If we define a class method with the special name `__init__`, Python promises to call that method whenever a new instance of that class is created.

We might add an initialization method to our `Staff601` class:

```
class Staff601:
    course = '6.01'
    building = 34
    room = 501
```

²³ Something to watch out for!!! A common debugging error happens when you: make an instance of a class (such as our `pat` above); then change the class definition and re-evaluate the file; then try to test your changes using your old instance, `pat`. Instances remember the definitions of all the methods in their class *when they were created*. So if you change the class definition, you need to make a new instance (it could still be called `pat`) in order to get the new definitions.

```

def __init__(self, name, role, years, salary):
    self.name = name
    self.role = role
    self.age = years
    self.salary = salary

def salutation(self):
    return self.role + ' ' + self.name

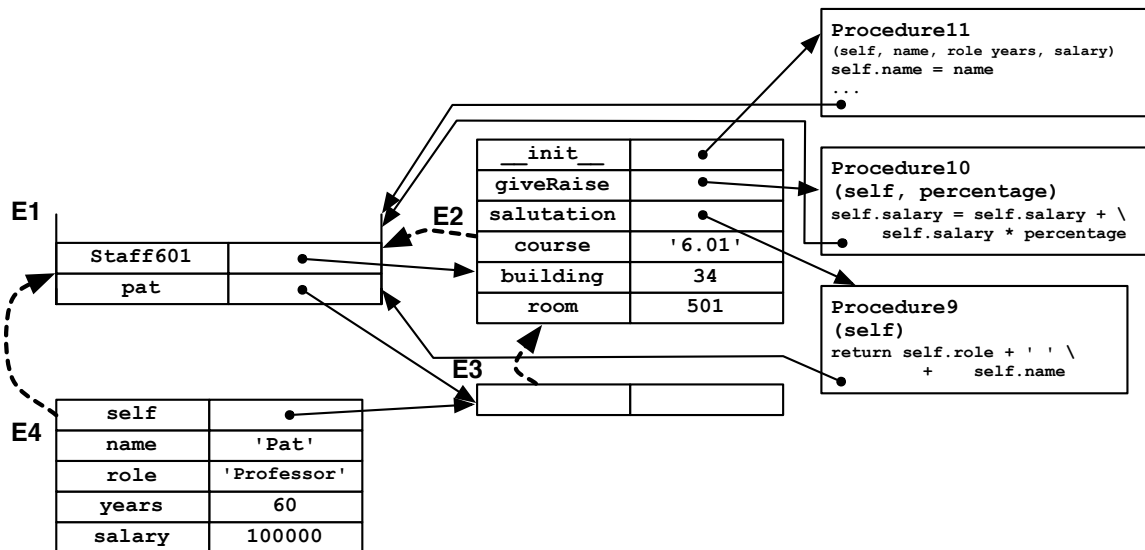
def giveRaise(self, percentage):
    self.salary = self.salary + self.salary * percentage

```

Now, to create an instance, we would do:²⁴

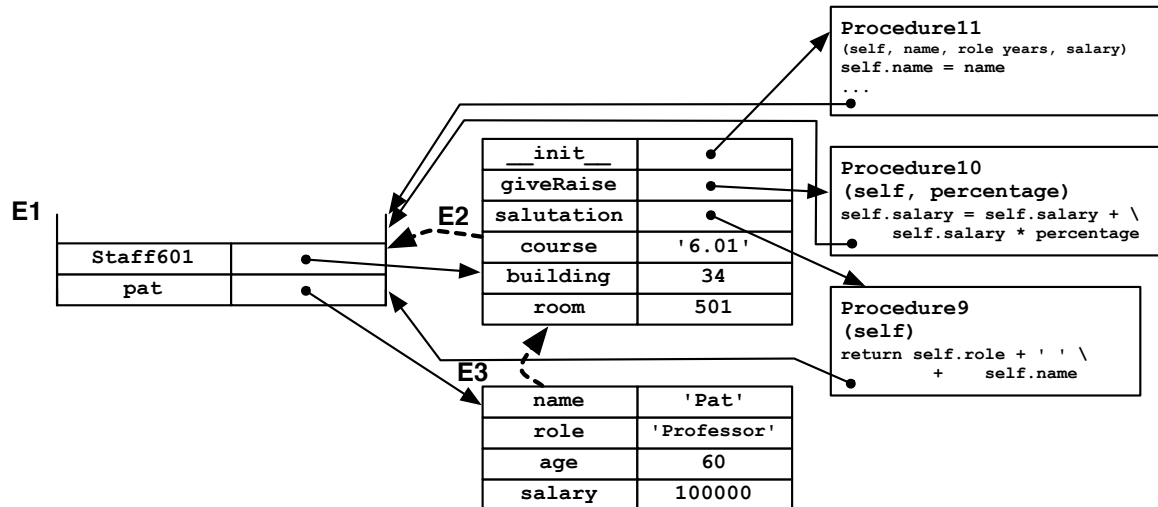
```
pat = Staff601('Pat', 'Professor', 60, 100000)
```

Here is a diagram of the environments when the body of the `__init__` procedure is about to be executed:



Note that the formal parameter `self` has been bound to the newly-created instance. Here is the situation after the initialization method has finished executing:

²⁴ We called the fourth formal parameter `years`, when `age` would have been clearer, just to illustrate that the names of formal parameters do not have to match the attributes to which they are bound inside the object.



This method seems very formulaic, but it is frequently all we need to do. To see how initialization methods may vary, we might instead do something like this, which sets the salary attribute based on the role and age arguments passed into the initializer.

```
class Staff601:
    def __init__(self, name, role, age):
        self.name = name
        self.role = role
        if self.role == 'Professor':
            self.salary = 100000
        elif self.role = 'TA':
            self.salary = 30000
        else:
            self.salary = 10000
        self.salary = self.giveRaise((age - 18) * 0.03)
```

3.5.4 Inheritance

We see that we are differentiating among different groups of 6.01 staff members. We can gain clarity in our code by building that differentiation into our object-oriented representation using *subclasses* and *inheritance*.

At the mechanism level, the notion of a subclass is very simple: if we define a class in the following way:

```
def class <className>(<superclassName>):
    <body>
```

then, when the interpreter makes the environment for this new class, it sets the parent pointer of the class environment to be the environment named by `<superclassName>`.

This mechanism allows us to factor class definitions into related, interdependent aspects. For example, in the 6.01 staff case, we might have a base class where aspects that are common to all kinds of staff are stored, and then subclasses for different roles, such as professors:

```

class Staff601:
    course = '6.01'
    building = 34
    room = 501

    def giveRaise(self, percentage):
        self.salary = self.salary + self.salary * percentage

class Prof601(Staff601):
    salary = 100000

    def __init__(self, name, age):
        self.name = name
        self.giveRaise((age - 18) * 0.03)

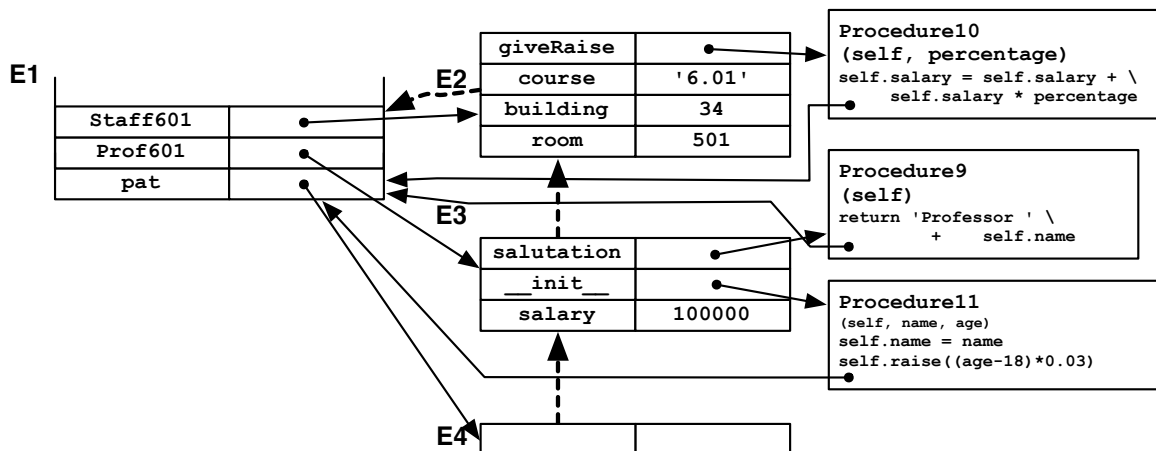
    def salutation(self):
        return 'Professor' + self.name

```

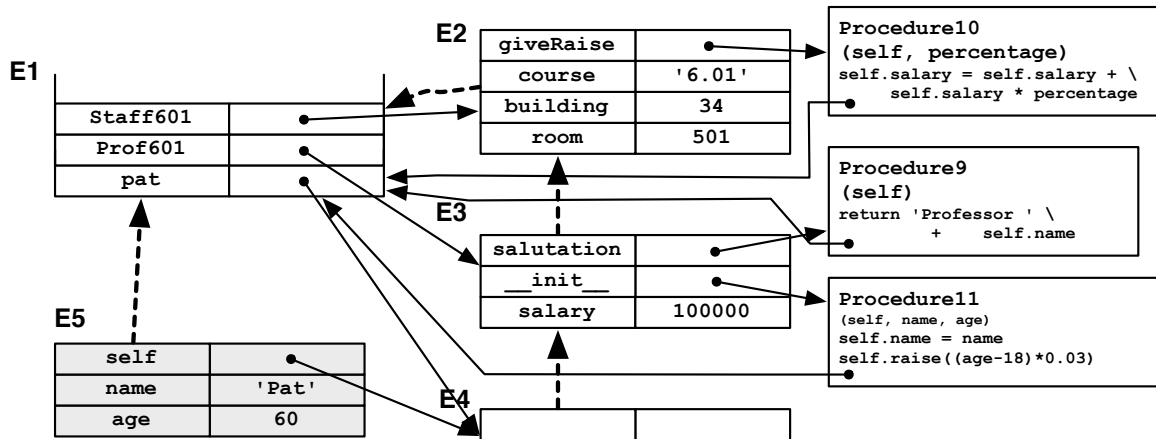
Let's trace what happens when we make a new instance of Prof601 with the expression

```
Prof601('Pat', 60)
```

First a new environment, E4, is constructed, with E3 (the environment associated with the class Prof601 as its parent). Here is the memory picture now:



As soon as it is created, the `__init__` method is called, with the new environment E4 as its first parameter and the rest of its parameters obtained by evaluating the expressions that were passed into to Prof601, in this case, 'Pat' and 60. Now, we need to make the procedure-call environment, binding the formal parameters of Procedure11; it is E5 in this figure:



We evaluate the body of Procedure11 in E5. It starts straightforwardly by evaluating

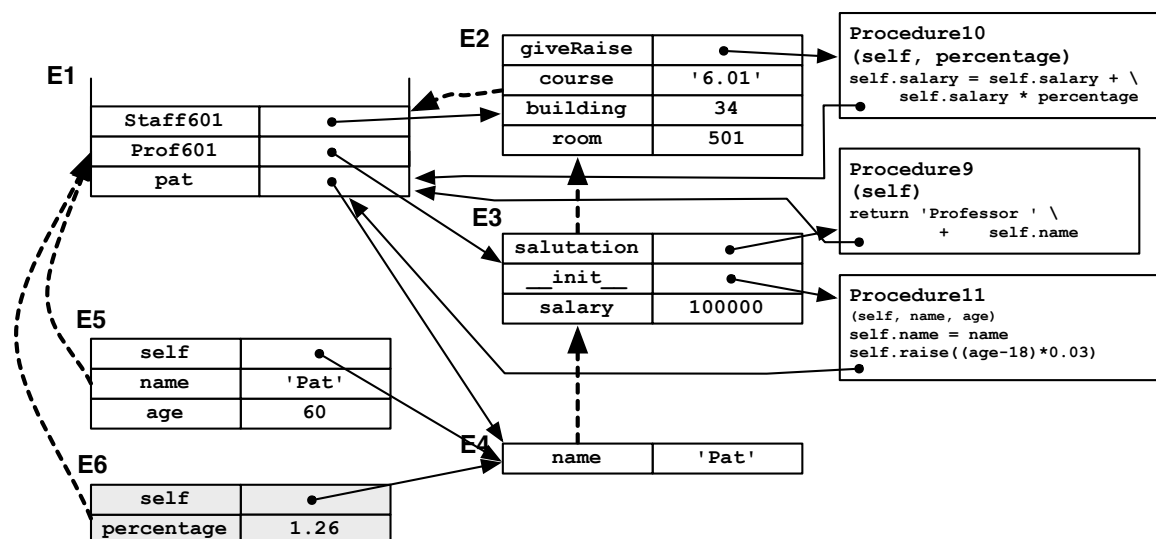
```
self.name = name
```

which creates a binding from name to 'Pat' in the object named self, which is E4. Now, it evaluates

```
self.giveRaise((age - 18) * 0.03)
```

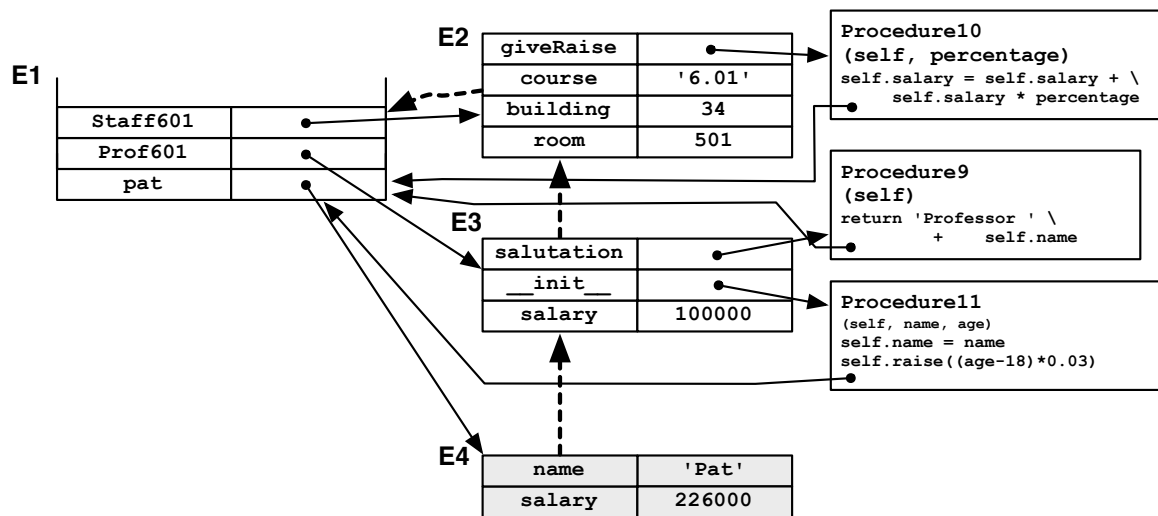
in E5. It starts by evaluating self.giveRaise. self is E4, so we look for a binding of giveRaise. It is not bound in E4, so we look in the parent E3; it is not bound in E3 so we look in E2 and find that it is bound to Procedure10. We are taking advantage of the fact that raises are not handled specially for this individual or for the subclass of 6.01 professors, and use the definition in the general class of 6.01 staff. The interpreter evaluates the argument to Procedure10, $(60 - 18) * 0.03$, getting 1.26.

It is time, now, to call Procedure10. We have to remember that the first argument will be the object through which the method was accessed: E4. So, we make a binding environment for the parameters of Procedure10, called E6:



Now the fun really starts! We evaluate `self.salary = self.salary + self.salary * percentage` in E6. We start by evaluating the right hand side: `self` is E4, `self.salary` is 100000, and `percentage` is 1.26, so the right-hand side is 226000. Now, we assign `self.salary`, which means we make a binding in E4 for `salary`, to 226000. It is important to see that, within a method call, all access to attributes of the object, class, or superclass goes through `self`. It is not possible to 'see' the definition of the `building` attribute directly from the `giveRaise` method: if `giveRaise` needed to depend on the `building` attribute, it would need to access it via the object, with `self.building`. This guarantees that we always get the definition of any attribute or method that is appropriate for that particular object, which may have overridden its definition in the class.

When the procedure calls are all done, the environments are finally like this:



It is useful to back up and see the structure here:

- The instance `pat` is an environment, E4.
- The parent of the instance is an environment, E3, which is the class `Prof601`.
- The parent of the class is an environment, E2, which is the superclass `Staff601`.

3.5.5 Using Inheritance

Frequently when we're using inheritance, we override some methods of a parent class, and use some of its methods unchanged. Occasionally, we will want to modify or augment an existing method from the parent class. In such a situation, we can 'wrap' that method by writing our own version of that method in the child class, but calling the method of the parent to do some of the work. To call a method `bar` from a parent class named `Foo`, you can do something like the following.

```
def SubclassOfFoo(Foo):
    def bar(self, arg):
        Foo.bar(self, arg)
```


When you call a method using the class name, rather than an object name, you need to pass the object in explicitly to the method call.

Here is a concrete example. Imagine that you have a class for managing calendars. It has these methods, plus possibly some others.

```
class Calendar:
    def __init__(self):
        self.appointments = []
    def makeAppointment(self, name, dateTime):
        self.appointments.append((name, dateTime))
    def printCalendar(self, start, end):
        # complicated stuff
```

Now, you'd like to make a calendar that can handle appointments that are made in different time zones. It should store all of the appointments in a single time zone, so that it can sort and display appropriately.

```
class CalendarTimeZone(Calendar):
    def __init__(self, zone):
        Calendar.__init__(self)
        self.zone = zone

    def makeAppointment(self, name, dateTime):
        Calendar.makeAppointment(self, name, dateTime + zone)
```

We make a subclass of `Calendar` which takes a time zone `zone` as an argument to its initialization method. It starts by calling the `Calendar` initialization method, to set up whatever internal data structures are necessary, and then sets the `zone` attribute of the new object. Now, when it's time to make an appointment, we add the time zone offset to the appointed time, and then call the `makeAppointment` method of `Calendar`.

What is nice about this example is that we were able to take advantage of the `Calendar` class, and even add to some of its methods, without knowing anything about its internal representation of calendars.

3.6 Recursion

There are many control structures in Python, and other modern languages, which allow you write short programs that do a lot of work. In this section, we discuss recursion, which is also a way to write programs of arbitrary complexity. It is of particular importance here, because the structure of a language interpreter is recursive, and in the next section we will explore, in detail, how to construct an interpreter.

We have seen how we can define a procedure, and then can use it without remembering or caring about the details of how it is implemented. We sometimes say that we can treat it as a *black box*, meaning that it is unnecessary to look inside it to use it. This is crucial for maintaining sanity when building complex pieces of software. An even more interesting case is when we can think of the procedure that we are in the middle of defining as a black box. That is what we do when we write a recursive procedure.

Recursive procedures are ways of doing a lot of work. The amount of work to be done is controlled by one or more arguments to the procedure. The way we are going to do a lot of work is by calling the procedure, over and over again, from inside itself! The way we make sure this process actually terminates is by being sure that the argument that controls how much work we do gets smaller every time we call the procedure again. The argument might be a number that counts down to zero, or a string or list that gets shorter.

There are two parts to writing a recursive procedure: the base case(s) and the recursive case. The *base case* happens when the thing that is controlling how much work you do has gotten to its smallest value; usually this is 0 or the empty string or list, but it can be anything, as long as you know it is sure to happen. In the base case, you just compute the answer directly (no more calls to the recursive procedure!) and return it. Otherwise, you are in the *recursive case*. In the recursive case, you try to be as lazy as possible, and foist most of the work off on another call to this procedure, but with one of its arguments getting smaller. Then, when you get the answer back from the recursive call, you do some additional work and return the result.

Another way to think about it is this: when you are given a problem of size n , assume someone already gave you a way to solve problems of size $n - 1$. So, now, your job is only to figure out

- To which problem of size $n - 1$ you would like to know the answer, and
- How to do some simple operations to make that answer into the answer to your problem of size n .

What if you wanted to add two positive numbers, but your computer only had the ability to increment and decrement (add and subtract 1)? You could do this recursively, by thinking about it like this. I need to add m and n :

- Presupposing the ability to solve simpler problems, I could get the answer to the problem m plus $n-1$.
- Now, I just need to add 1 to that answer, to get the answer to my problem.

We further need to reason that when n is 0, then the answer is just m . This leads to the following Python definition:

```
def slowAdd(m, n):
    if n == 0:
        return m
    else:
        return 1 + slowAdd(m, n-1)
```

Note how in the final `return` expression, we have reduced the answer to a problem of size n to a simpler version of the same problem (of size $n-1$ plus some simple operations).

Here is an example recursive procedure that returns a string of n 1's:

```
def bunchaOnes(n):
    if n == 0:
        return ''
    else:
        return bunchaOnes(n-1) + '1'
```

The thing that is getting smaller is n . In the base case, we just return the empty string. In the recursive case, we get someone else to figure out the answer to the question of $n-1$ ones, and then we just do a little additional work (adding one more '1' to the end of the string) and return it.

Exercise 3.16. What is the result of evaluating

```
bunchaOnes(-5)
```

Here is another example. It is kind of a crazy way to do multiplication, but logicians love it.

```
def mult(a,b):
    if a==0:
        return 0
    else:
        return b + mult(a-1,b)
```

Trace through an example of what happens when you call `mult(3, 4)`, by adding a print statement that prints arguments `a` and `b` as the first line of the procedure, and seeing what happens.

Here is a more interesting example of recursion. Imagine we wanted to compute the binary representation of an integer. For example, the binary representation of 145 is '10010001'. Our procedure will take an integer as input, and return a string of 1's and 0's.

```
def bin(n):
    if n == 0:
        return '0'
    elif n == 1:
        return '1'
    else:
        return bin(n/2) + bin(n%2)
```

The easy cases (base cases) are when we are down to a 1 or a 0, in which case the answer is obvious. If we do not have an easy case, we divide up our problem into two that are easier. So, if we convert $n/2$ (the integer result of dividing n by 2, which by Python's definition will be a smaller number since it truncates the result, throwing away any remainder), into a string of digits, we will have all but the last digit. And $n\%2$ (n modulo 2) is 1 or 0 depending on whether the number is even or odd, so one more call of `bin` will return a string of '0' or '1'. The other thing that is important to remember is that the `+` operation here is being used for string concatenation, not addition of numbers.

How do we know that this procedure is going to terminate? We know that the number on which it is operating is a positive integer that is getting smaller and smaller, and will eventually be either a 1 or a 0, which can be handled by the base case.

You can also do recursion on lists. Here a way to add up the values of a list of numbers:

```
def addList(elts):
    if elts == []:
        return 0
    else:
        return elts[0] + addList(elts[1:])
```

The `addList` procedure consumed a list and produced a number. The `incrementElements` procedure below shows how to use recursion to do something to every element of a list and make a new list containing the results.

```
def incrementElements(elts):
    if elts == []:
        return []
    else:
        return [elts[0]+1] + incrementElements(elts[1:])
```

If the list of elements is empty, then there is no work to be done, and the result is just the empty list. Otherwise, the result is a new list: the first element of the new list is the first element of the old list, plus 1; the rest of the new list is the result of calling `incrementElement` recursively on the rest of the input list. Because the list we are operating on is getting shorter on every recursive call, we know we will reach the base case, and all will be well.

3.7 Implementing an interpreter

This section can be skipped unless you are interested. It's very cool but a bit tricky.



From the preceding sections, you should have an informal understanding of what happens when a Python program is evaluated. Now, we want to understand it formally enough to actually implement an interpreter in Python.

3.7.1 Spy

We will study a simplified language, which we call *Spy*, because it is a mixture of Scheme and Python. From Scheme, an elegant interpreted programming language with very simple syntax and semantics, we take the syntax of the language, and from Python, we take the basic object-oriented programming system. *Spy* has a small subset of the features of these languages, but it is powerful enough to implement any computer program that can be implemented in any other language (though it could be *mighty* tedious).

To learn more: Interestingly, very minimal versions of several different models of computation (imperative, functional/recursive, rewrite systems) have been shown to be formally equivalent. Some have theorized them to be complete in the sense that there are no computations that cannot be expressed this way. For more information, see:
<http://plato.stanford.edu/entries/church-turing/>

The syntax of *Spy*, like Scheme, is *fully-parenthesized prefix* syntax. *Prefix* means that the name of the function or operation comes before its arguments, and *fully-parenthesized* means that there are parentheses around every sub-expression. So, to write $1 + \text{num} * 4$, we would write

```
(+ 1 (* num 4)) .
```

The reason for adopting this syntax is that it is very easy to manipulate with a computer program (although some humans find it hard to read). In the following, we will assume that someone has already written a *tokenizer*, which is a program that can consume a stream of characters and break it into “words” for us. In particular, we will assume that the tokenizer can break an input Spy program down into a list of lists (of lists...of elements that are strings or integers). So the expression `(+ 1 (* num 4))` would be converted by the tokenizer into a representation such as: `('+', 1, ('*', 'num', 4))`.

Spy has the following features:

- **Integer constants:**

0, 1, -1, 2, -2, ...

- **Basic built-in functions:**

`+`, `-`, `*`, `/`, `=`, with meanings you would expect; note that in this language `=` is a test for equality on two integers, which returns a Boolean.

- **Assignment:**

`(set a 7)` will set (or assign) the value of variable `a` to be 7

- **Function application:**

A list of expressions, the first of which is not a special word in Spy, is treated as function application, or function call. So, `(+ 3 a)` would return 3 plus the value of variable `a`, and `(f)` is a call of a function `f` with no arguments. Note that the first element can, itself, be an expression, so

```
((myFunctionMaker 3) (+ 4 5))
```

is also a valid expression, as long as the value of `(myFunctionMaker 3)` is a function that will consume a single argument (which, in this case, will be the value of the function application `(+ 4 5)`).

- **Function definition:**

New functions can be defined, much as in Python or Scheme. So,

```
(def myFun (x y) (* (+ x y) y))
```

defines a new function named `myFun` of two arguments, which can be applied with an expression like `(myFun 4 9)`.

- **If:**

`(if a b c)` will evaluate and return the value of expression `b` if the value of expression `a` is equal to `True`, and otherwise will evaluate and return the value of expression `c`.

- **Compound expression:**

In Spy, the body of a function definition, and the branches of an `if` expression, are expected to be a single expression; in order to evaluate multiple expressions in one of those locations, we need to group them together with `begin`, like this:

```
(begin (set a 7) (set b 8) (+ a b)) .
```

(The reason we need `begin` to be a special word is so that we can distinguish this list of expressions from a function application, which is also, syntactically, a list of expressions.) The value of a compound expression is the value of its last component. So, in our example, the value of the whole expression would be 15.

Note that the names `begin`, `set`, `def`, and `if` have special meaning in Spy and cannot be used as the names of user-defined functions.

Spy also has some object-oriented features, but we will introduce those in [section 3.7.3](#).

Exercise 3.17. Write a Spy procedure that takes two positive integers as input and returns True if the first is greater than the second. Use only the primitive functions = and + and recursion.

Exercise 3.18. Write a Spy procedure that takes `n` as input and returns the `n`th number in the Fibonacci series.

3.7.2 Evaluating Spy expressions

In this section, we will describe, in complete detail, how it is that a program in Spy can be executed. The basic operations are evaluating expressions, and assigning values to names. In the following sections, we will develop a recursive definition that allows us to compute the value resulting from any Spy program. We will start with simple expressions, and work up to complex ones.

The `spyEval` function will consume an expression and some additional information, and return the value of the expression. It will have the structure of a long set of if-elif-else clauses, each of which tests to see whether the expression has a particular form and performs the appropriate evaluation.

To learn more: An *interpreter* is a program that takes in the text of a computer program and executes it directly. We are going to build an interpreter for Spy in this section, and when we use Python, we use an interpreter. A *compiler* is a program that takes in the text of a computer program and turns it into low-level instructions for a particular computer, which can later be executed. A gross characterization of the difference is that it is easier to debug when working with an interpreter, but that your program runs faster when compiled. For more information start with Wikipedia articles on **Compiler** and **Interpreter_(computing)**.

3.7.2.1 Numbers

What is the value of the program '7'? It's 7, of course. So, we can begin defining a function that will consume a Spy expression (described as a list of lists of strings) and return a value.

```
Spy Eval      def spyEval(form, env):
                if isinstance(form, int):
                    return form
```

Here, we need to be able to examine an expression to see if it is an integer (the `isinstance` procedure in Python takes any Python object and a type or class, and returns `True` if the object is of that type or class, and `False` otherwise). If it is an integer, then we just return that integer value. (You might have imagined that `form` would be a string, and that we'd have to convert that string to an integer; in fact, the tokenizer, when it finds a token that is a number, converts it for us in advance.)

For now, ignore the `env` argument; we'll explain it soon.

3.7.2.2 Compound expressions

As we discussed above, a list of expressions, where the first one is the word 'begin' is a compound expression, whose value is the value of the last component. But, for reasons we will see in [section 3.7.2.3](#), it is important to evaluate all of the expressions, not just the last one. So, we can extend our definition of `spyEval` by adding another clause (the ellipsis below is meant to include the text from the previous `spyEval` code):

```
Spy Eval      ...
                elif form[0] == 'begin':
                    val = None
                    for entry in form[1:]:
                        val = spyEval(entry, env)
                    return val
```

3.7.2.3 Symbols

We'll use the term *symbol* to refer to any syntactic item that isn't a number or parenthesis.

What is the value of the program 'a'? All by itself, it is an error, because 'a' is undefined. So, let's consider a more complicated program:

```
(begin (set a 6)
 a)
```

Why would we ever want to evaluate expressions and throw their values away? In a *pure functional* language, the answer is that we wouldn't. But in Spy, we have assignment expressions.

They don't even have a value, but they make a change to the environment in which they are evaluated. Before we go any further, we'll have to take a slight detour, in order to understand the idea of environments.

Environments

An *environment* consists of two parts:

- a dictionary
- a parent environment (can be None)

The dictionary is a data structure that lets us associate values (numbers, functions, objects, etc.) with symbols; a Python dictionary works perfectly for this purpose.

We will say that a symbol is *bound* in an environment if it exists as a key in the environment's dictionary. We'll sometimes call the relationship between a key and a value in an environment's dictionary a *binding*.

There are three operations we can do on an environment: look up a symbol, add a binding, and extend an environment. We'll define the first two here, and return to environment extension later.

The *lookup* operation takes a symbol and an environment, and performs a slightly generalized version of a dictionary lookup, looking in a parent environment if the symbol is not in the dictionary:

- If the symbol is a key in the environment's dictionary, then it returns the associated value;
- Otherwise, if this environment has a parent, then it returns the result of looking the symbol up in the parent environment;
- Otherwise, it generates an error.

The *add binding* operation takes a symbol, an environment, and a value; it adds the value to the dictionary associated with the environment, using the symbol as the key. If there was already a value associated with that symbol in the dictionary, the old value is overwritten by the new value.

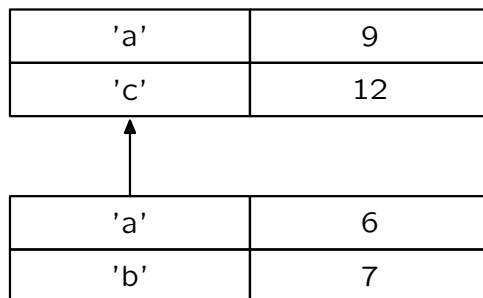


Figure 3.1 A simple environment.

Exercise 3.19. If *e* is the environment in [Figure 3.1](#), what are the results of:

- `e.lookup('a')`
- `e.lookup('c')`
- `e.lookup('d')`
- `e.add('f', 1)`
- `e.add('c', 2)`
- `e.add('a', 2)`

Where do environments come from? There is one environment that is made by the interpreter, called `global`. It contains bindings for the primitive procedures, and is the environment in which our main program is evaluated.

Assigning values

So, let's go back to the expression

```
(set a 6)
```

We'll refer to the second and third elements of this expression (the `'a'` and the `6`) as the left-hand-side (lhs) and right-hand-side (rhs) respectively.²⁵ For now, the lhs will always be a symbol. Now, to evaluate this expression in an environment, we evaluate the expression that is the rhs to get a value, and then bind the symbol that is the lhs to that value in the environment.

So, here's our augmented definition of `eval`. Note that it we have added the `env` argument, which is the environment in which we're evaluating the expression:

```
Spy Eval          elif form[0] == 'set':
                    (tag, lhs, rhs) = form
                    env.add(lhs, spyEval(rhs, env))
```

So, in our simple example, lhs is the symbol `'a'`; `expr` is `6`, which does not need further evaluation (the tokenizer has already turned the string `'6'`) into an internal Python integer `6`. More generally, though, the `expr` can be an arbitrarily complicated expression which might take a great deal of work to evaluate.

After we have evaluated that expression, our global environment will look like this:

global

'a'	6
'+'	<func>
	⋮

Evaluating symbols

Now, finally, we can evaluate the expression `'a'`. All we do to evaluate a symbol is look it up in the environment. So, the return value of this whole program

```
(begin (set a 6)
       a)
```

is the value `6`.

²⁵ That usage comes because a typical assignment statement has the form `lhs = rhs`.

So, now, we can add another clause to `spyEval`:

```

Spy Eval      ...
              elif isinstance(form, str):
                return env.lookup(form)

```

3.7.2.4 Functions

In the Spy language, any list of expressions, the first element of which is not a special symbol is treated as a function call.²⁶ Each of the expressions in the list is evaluated, yielding a function and some number of values.

Before we go any further, we have to talk about functions in some detail. There are two types of functions: *primitive functions* and *user defined functions*. Primitive functions come built into the interpreter and are, ultimately, implemented as part of the interpreter. Spy provides several primitive functions that are bound to names in the global environment. So, to call a primitive function, all we have to do is pass the values we obtained by evaluating the rest of the expressions in the function call into the primitive function.

Function definitions

Before we can talk about calling user-defined functions, we have to talk about how to define them. Here is an example definition:

```

(def fizz (x y)
  (+ x y))

```

It is a list of four elements:

1. `def`, which is a special symbol, indicating that this expression is not a function call, and therefore requires special handling;
2. the *name* of the function being defined (`'fizz'` in this case);
3. a list of symbols, called the *formal parameters* (in this example, `('x', 'y')`); and
4. a function *body*, which is a Spy expression, in this case, `'(+ x y)'`

Not very much happens at function definition time. We construct a data structure (typically an instance of a `Function` class) that simply stores three components:

- the formal parameters
- the function body
- the environment in which the function definition was evaluated

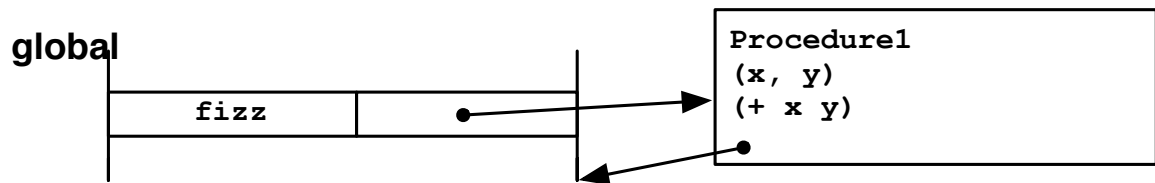
Then, we add a binding in the environment in which the function was defined from the function name to this function structure. With this understanding of function definition, we can add another clause to `spyEval`:

²⁶ So far, our special symbols are `begin` and `set`, and we'll add `def` and `if`.

Spy Eval

```
...
elif form[0] == 'def':
    (tag, name, params, body) = form
    env.add(name, Function(params, body, env))
```

Here is a picture of the global environment after the definition of `fizz` is made. Note the binding from the name `'fizz'` to a Function instance which, itself, contains a reference to the environment.



Function calls

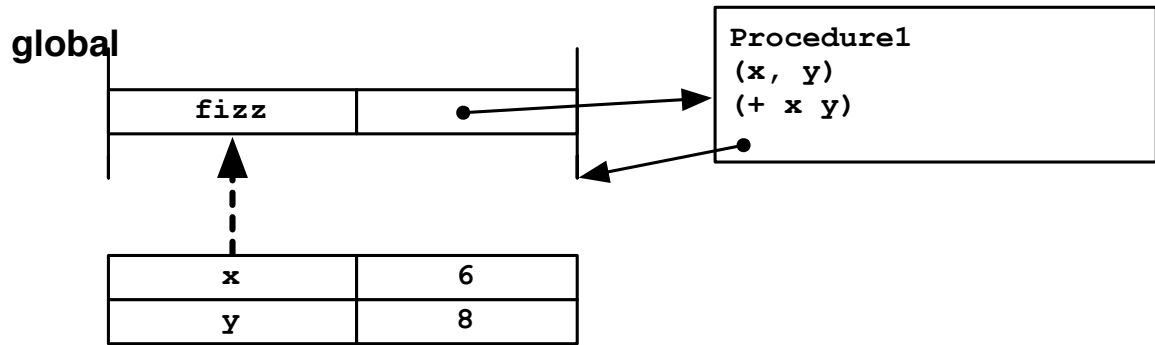
Now, we can see what happens when we actually call a function. Let's assume that the function `fizz` from the previous section has been defined, and now we want to evaluate

```
(fizz 6 8)
```

in the environment `env`. This is the trickiest part of the whole interpreter. First, we evaluate each of these expressions, getting: a Function instance (which we created when we defined the `fizz` function), the number 6 and the number 8.

Now, we make a new environment. Let's call it (imaginatively) `newEnv`. The dictionary part of `newEnv` is used to associate the symbols that are the formal parameters of the function (in the case of the `fizz` function, `'x'` and `'y'`) with the actual values being passed into the function (in this case, the numbers 6 and 8). The parent environment of `newEnv` is the environment that we stored with the function when it was defined (not the environment in which the function is being called!!). In this case, `fizz` was defined in the global environment. And so, the parent pointer is also the global environment. (Later, we may see some examples where this pointer is different, in a way that matters.)

Here is a diagram of the entire environment, at this point:



Now that we have this new environment, we evaluate the expression that is the body of the function, in this case, `'(+ x y)'` in `newEnv`. This will all work out roughly as you would expect: during the process of evaluating the body, we will have to evaluate the symbols `'+'`, `'x'` and `'y'`. We will find `'x'` and `'y'` in the dictionary of `newEnv`; and we will find `'+'` in its parent environment, which is `global`. So, we'll have the function object that is bound to `'+'`, and the number 6 and the number 8; then, we'll call the primitive function on these values, and end up with the result 14.

Here is our augmented definition of `spyEval`:

Spy Eval

```
...
elif isinstance(form, list):
    f = spyEval(form[0], env)
    return spyEval(f.body,
                   Environment(f.formal,
                               [spyEval(x, env) for x in form[1:]],
                               f.environment))
```

As required, this process evaluates the first expression in the list to get a function. Then it evaluates the body of that function in a new environment that maps the function's formal parameters to the results of evaluating all the rest of the expressions in the list, and whose parent is the environment stored in the function.

A good way to understand what is going on with `spyEval` is to show a *trace* of its behavior. So, when we evaluate the following expression:

```
(begin (def fizz (a b)
        (+ a b))
       (fizz 3 4))
```

we can print out the arguments that are passed into `spyEval` (though we have left out the environment, to keep things from being too cluttered), and the result of each call. Each time it is called recursively, we print out the arguments and results with one level more of indentation:

```
args: ['begin', ['def', 'fizz', ['a', 'b'], ['+', 'a', 'b']], ['fizz', 3, 4]]
  args: ['def', 'fizz', ['a', 'b'], ['+', 'a', 'b']]
    result: None
      args: ['fizz', 3, 4]
```

```

args: fizz
result: <__main__.Function instance at 0x7b1e18>
args: 3
result: 3
args: 4
result: 4
args: ['+', 'a', 'b']
  args: +
  result: Primitive <built-in function add>
  args: a
  result: 3
  args: b
  result: 4
result: 7
result: 7
result: 7

```

Here is another version, showing the basic part of the environment, but not the parent environment.

```

args: ['begin', ['def', 'fizz', ['a', 'b'], ['+', 'a', 'b']], ['fizz', 3, 4]] Env: global
args: ['def', 'fizz', ['a', 'b'], ['+', 'a', 'b']] Env: global
result: None
args: ['fizz', 3, 4] Env: global
  args: fizz Env: global
  result: <__main__.Function instance at 0x7b1df0>
  args: 3 Env: global
  result: 3
  args: 4 Env: global
  result: 4
  args: ['+', 'a', 'b'] Env: {'a': 3, 'b': 4}
    args: + Env: {'a': 3, 'b': 4}
    result: Primitive <built-in function add>
    args: a Env: {'a': 3, 'b': 4}
    result: 3
    args: b Env: {'a': 3, 'b': 4}
    result: 4
  result: 7
result: 7
result: 7

```

3.7.2.5 If

The last special type of expression in Spy is a conditional, of the form:

```

(if (= x 3)
  (fizz x 10)
  (+ x 4))

```

It might seem, at first, that it would be okay to implement `if` as a primitive function, similar to `+`. But there is an important reason not to do so. Consider the definition of the factorial function, in Spy:

```
(def factorial (x)
  (if (= x 1)
      1
      (* x (factorial (- x 1)))))
```

The most straightforward application of this function, `(factorial 1)` will get us into trouble. Why? We would start by evaluating `'factorial'` and `'1'`, and getting a function and 1. So far so good. Now we make a new environment, and evaluate the body in that environment. That requires evaluating all of the elements of the body in that environment. So, we'd find that `'if'` evaluates to a primitive function, that `(= x 1)` evaluates to `True`, and that 1 evaluates to 1. Then, we'd need to evaluate that last expression, which is itself a function call. That, in itself, is no problem. But we can see that eventually, we'll have to evaluate `factorial` in an environment where its argument is -1. And that will require evaluating `factorial` of -2, which will require evaluating `factorial` of -3, and so on.

The most important thing about `'if'` is that *it evaluates only one of its branches, depending on whether its condition is True or False*. Without this property, we cannot stop recursion.

So, now we can see what to do with an `'if'` expression. We'll call the second, third, and fourth elements the *condition*, *then* and *else* parts. We start by evaluating the condition part. If it is `True`, we evaluate the *then* part, otherwise we evaluate the *else* part.

It is pretty straightforward to add this to our `eval` function; but note that we had to put it before the function application clause, so that we can keep `if` expressions from being evaluated as if they were function applications.

Here is the whole interpreter.

Spy Eval

```

def spyEval(form, env=globalEnv):
    if isinstance(form, int):
        return form
    elif isinstance(form, str):
        return env.lookup(form)
    elif form[0] == 'begin':
        val = None
        for entry in form[1:]:
            val = spyEval(entry, env)
        return val
    elif form[0] == 'set':
        (tag, lhs, rhs) = form
        env.add(lhs, spyEval(rhs, env))
    elif form[0] == 'def':
        (tag, name, params, body) = form
        env.add(name, Function(params, body, env))
    elif form[0] == 'if':
        (tag, condition, ifBody, elseBody) = form
        if spyEval(condition, env):
            return spyEval(ifBody, env)
        else:
            return spyEval(elseBody, env)
    elif isinstance(form, list):
        f = spyEval(form[0], env)
        return spyEval(f.body,
                       Environment(f.formal,
                                   [spyEval(x, env) for x in form[1:]],
                                   f.environment))
    else:
        Error("Illegal expression: "+str(form))

```

Yay! Now we have a complete Spy interpreter. Well, we still need implementations of the Environment and Function classes, but that's not very much more work. We can implement those classes as exercises.

3.7.3 Object-Oriented Spy

We can add a simple object-oriented facility to Spy, which is modeled on Python's OOP facility, but is somewhat simpler. The crucial idea here is that *classes and instances are both environments*, of exactly the same kind that we have been using to support binding and function calls in basic Spy. The dictionary part of the environment supports the binding of attribute names to values within the instance or class; and the parent environment part allows an instance to be connected to its class, or a class to its superclass, so that attributes that are not defined within the instance may be found in the class or superclass.

We only have to add two new syntactic features to the language: attribute lookup and class definition.

3.7.3.1 Attribute lookup

In Python, when you want to get the value of an attribute `a` from an instance `obj`, you say `obj.a`. In OOSpy, we'll say `(attr obj a)`. Remembering that an instance is an environment, all we have to do to evaluate such an expression is to look up the symbol 'a' in the environment 'obj'.

Referring to the second part of this form as the object part and the third as the name part, we can add a clause for `attr` expressions to our interpreter. The name will always be a single symbol; but the object part can be a general expression (we might, for example, call a function `bigFun` that returns an object, yielding an expression like `(attr (bigFun 3) x)` to get the attribute named `x` from the object returned by `bigFun`). This means that we have to evaluate the object part, using our standard evaluation function. Now, we can add a clause for `attr` expressions to our `spyEval`:

```

Spy Eval
...
elif form[0] == 'attr':
    (tag, objectExpr, name) = form
    return spyEval(objectExpr, env).lookup(name)
...

```

3.7.3.2 Class definition

All the rest of our work happens at class definition time. Here is a simple OOSpy class definition:

```

(class SimpleClass None
  (begin (def init (self v)
          (set (attr self v) v))
         (def getV (self)
           (attr self v))))

```

For reference, it is roughly equivalent to the Python:

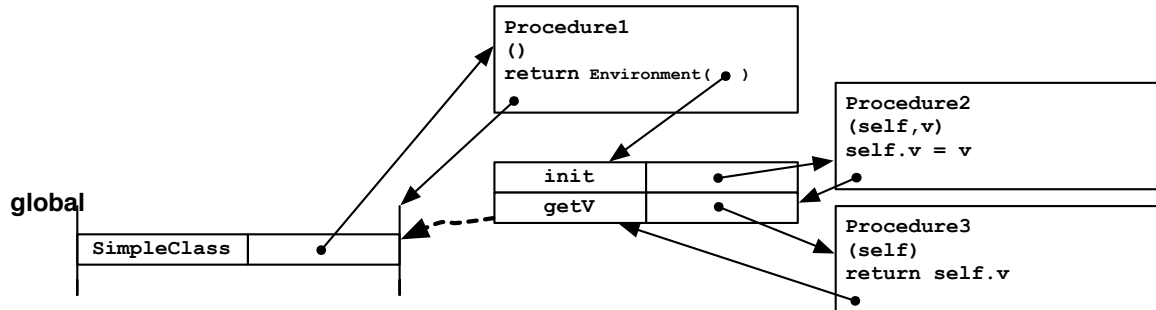
```

class SimpleClass:
    def init (self, v):
        self.v = v
    def getV (self):
        return self.v

```

It has four parts: the special symbol 'class', the class name `SimpleClass`, the name of a superclass (in this case, there is no superclass, so we write `None`), and a class body, which is a Spy expression. Our simple Spy implementation doesn't have an equivalent to Python's `__init__` facility, in which the specially-named procedure is called automatically. So we call the method `init` without underscores to make that clear.

Here is a picture of the environment that we would like to have result from this definition:



Note that Spy differs from Python here in an important way: in Python, the environment stored in a method is always the module environment; in Spy, to avoid having a special case, it is the class environment in which it is defined.

Also, rather than binding `SimpleClass` directly to the environment for the class, we will bind it to a procedure that makes a new environment whose parent is the class environment. This procedure can be called directly to create a new instance of the class.

There are three steps involved in processing this class definition.

1. *Make an environment for the class* In this step, we just make a new empty environment, and set its parent to be the class specified as the superclass of this class. If the superclass is `None`, then we use `global` as the parent environment, so that global variables and primitive function definitions will be accessible from the class body.
2. *Evaluate the class body in the class environment* In this step, we use our regular Spy evaluation mechanism, to evaluate the class body, in the new environment. In our `SimpleClass` body, we define two methods; these definitions will appear in the class environment but will not, for example, be accessible directly from the global environment.
3. *Make a constructor function* Finally, we define a new function, with the same name as the class, in the environment in which the class definition is being evaluated (not the class environment; if we did that, nobody would every be able to find the constructor!). The constructor function has the job of making a new instance of this class. What is an instance? An environment. And a brand new instance of any class is simply an empty environment, whose parent environment is the class environment.

After defining `SimpleClass`, we can use it as follows:

```
(begin
  (set a (SimpleClass))
  ((attr a init) a 37)
  ((attr a getV) a))
```

This is roughly equivalent to the Python

```
a = SimpleClass()
a.init(37)
a.getV()
```

This may look a little bit strange. Let's go step by step. First, we make an instance by calling the constructor function. That's pretty straightforward. Now, the value of variable `a` is an instance of the `SimpleClass` class, which means that it is an environment.

Now, what about this expression?

```
((attr a init) a 37)
```

Let's try to work through it the way the interpreter would. It is a list, and the first element is not a special symbol, so it must be a function call. That means we should evaluate each of the forms in the list.

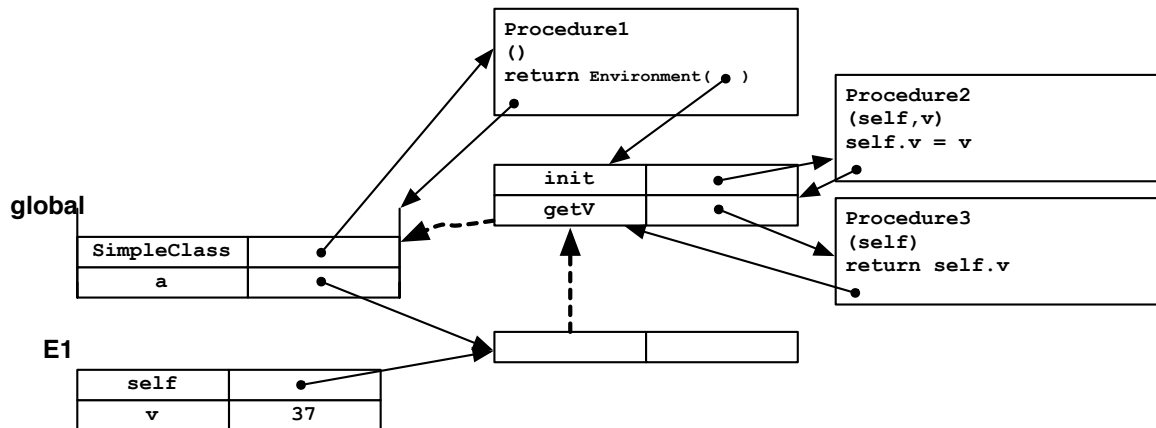
First we evaluate `(attr a init)`, which is a list starting with the special symbol `attr`, so we know it's looking up an attribute in a class. It evaluates the expression `a`, getting the `SimpleClass` instance, then looks for the `init` attribute, which yields the `init` function that we defined inside the class body.

Evaluating the second and third elements of `((attr a init) a 37)` yield our `SimpleClass` instance and the number `37`.

Now, we're ready to do a function call. Let's do it carefully. The formal parameters of the function we're calling are `('self', 'v')`. So, we make a new environment, in which those parameters are bound to the `SimpleClass` instance and to `37`, respectively, and whose parent environment is the environment in which `((attr a init) a 37)` is being evaluated. Now, we evaluate the body of that function, which is

```
(set (attr self v) v)
```

in this new environment. Let's call this environment `E1`. In order to understand what's going on here, we have to go slowly and carefully. Here's `E1`:

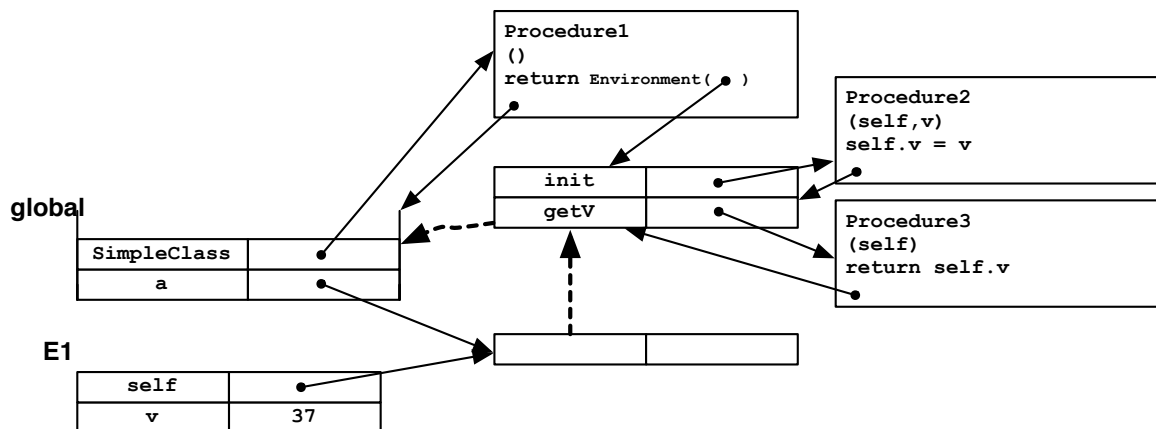


First, we recognize this as a `set` expression. But, so far, our `set` expressions have always had a single symbol as their second element. Now, we will have to generalize it, so the second element of a `set` expression can also be an `attr` expression. Let's consider this kind of expression, in general:

```
(set (attr x y) z)
```

In this form, x can be any expression that, when evaluated, yields an instance; y must be a single symbol; and z can be any expression that, when evaluated, yields a value. The result of this form is to set attribute y in the instance resulting from evaluating expression x to have the value resulting from evaluating the expression z .

So, we need to evaluate `(set (attr self v) v)` in the environment $E1$. We start by evaluating `self`, and getting the `SimpleClass` instance; we look for an attribute named `v` there and, not finding one, make a new one. Now, we evaluate the third part of the `set` expression, which is `v`, in the $E1$, and get the value `37`. Finally, we set attribute `v` of our instance to `37`. Yay. Here's the new state of the environment:



One thing to notice is that `Spy`, as we have designed it, doesn't have Python's special facility that automatically uses an object as the first argument of a method, when that method is accessed via that object. So, we have to pass the object in again, explicitly, as the first argument.

Exercise 3.20.

Add this syntactic facility to `OOSpy`.

That is, make it so that we can write `((attr a init) 37)` and `((attr a getV))`. *This is a little bit tricky.*



Evaluating the expression `((attr a getV) a)` works similarly (you should be sure you understand how), and returns the value `37`.

So, after all that, we can add the last clause to our `OOSpy` interpreter, for handling class definitions. Note that we also have to extend the code for handling `set`, so it can deal with setting attributes of instances.

Spy Eval

```

...
elif form[0] == 'set':
    (tag, lhs, expr) = form
    (targetEnv, name) = lhsEval(lhs, env)
    targetEnv.add(name, spyEval(expr, env))
elif form[0] == 'class':
    (tag, name, super, body) = form
    if super == 'None':
        super = globalEnv
    classEnv = Environment(parent = super)
    env.add(name, Primitive(lambda : Environment(parent = classEnv)))
    spyEval(body, classEnv)
...

def lhsEval(lhs, env):
    if isinstance(lhs, list):
        (tag, objectExpr, name) = lhs
        return (spyEval(objectExpr, env), name)
    else:
        return (env, lhs)

```

`Primitive` is a class that represents a primitive function in `Spy`. Its initializer just takes a procedure as an argument: in this case, it is a procedure that makes a new environment whose parent is the the class environment.

We're done with a whole object-oriented language!

Exercise 3.21. Write an `OOSpy` class definition for an `Averager` class. It should have a method `addValue` that allows you to “add” a numeric value to the set of data it as seen and method `getAverage` that returns the average of all of the values that have been added so far.

3.8 Object-Oriented Programming Examples

Here are some examples of object-oriented programming in Python.

3.8.1 A simple method example

Here's an example to illustrate the definition and use of classes, instances, and methods.

```

class Square:
    def __init__(self, initialDim):
        self.dim = initialDim

    def getArea (self):
        return self.dim * self.dim

```

```
def setArea (self, area):
    self.dim = area**0.5

def __str__(self):
    return "Square of dim " + str(self.dim)
```

This class is meant to represent a square. Squares need to store, or remember, their dimension, so we make an attribute for it, and assign it in the `__init__` method. Now, we define a method `getArea` that is intended to return the area of the square. There are a couple of interesting things going on here.

Like all methods, `getArea` has an argument, `self`, which will stand for the instance that this method is supposed to operate on. Now, the way we can find the dimension of the square is by finding the value of the `dim` attribute of `self`.

We define another method, `setArea`, which will set the area of the square to a given value. In order to change the square's area, we have to compute a new dimension and store it in the `dim` attribute of the square instance.²⁷

Now, we can experiment with instances of class `Square`.

```
>>> s = Square(6)
>>> s.getArea()
36
>>> Square.getArea(s)
36
>>> s.dim
6
>>> s.setArea(100)
>>> s.dim
10.0
>>> s1 = Square(10)
>>> s1.dim
10
>>> s1.getArea()
100
>>> s2 = Square(100)
>>> s2.getArea()
10000
>>> print s1
Square of dim 10
```

Our class `Square` has the `__str__` method defined, so it prints out nicely.

3.8.2 Superclasses and inheritance

What if we wanted to make a set of classes and instances that would help us to run a bank? We could start like this:

²⁷ We compute the square root of a value by raising to the power 0.5.

```
class Account:
    def __init__(self, initialBalance):
        self.currentBalance = initialBalance
    def balance(self):
        return self.currentBalance
    def deposit(self, amount):
        self.currentBalance = self.currentBalance + amount
    def creditLimit(self):
        return min(self.currentBalance * 0.5, 10000000)

>>> a = Account(100)
>>> b = Account(1000000)

>>> Account.balance(a)
100
>>> a.balance()
100
>>> Account.deposit(a, 100)
>>> a.deposit(100)
>>> a.balance()
300
>>> b.balance()
1000000
```

We've made an `Account` class²⁸ that maintains a balance as an attribute. There are methods for returning the balance, for making a deposit, and for returning the credit limit.

The `Account` class contains the procedures that are common to all bank accounts; the individual instances contain state in the values associated with the names in their environments. That state is *persistent*, in the sense that it exists for the lifetime of the program that is running, and doesn't disappear when a particular method call is over.

Now, imagine that the bank we're running is getting bigger, and we want to have several different kinds of accounts, with the credit limit depending on the account type. If we wanted to define another type of account as a Python class, we could do it this way:

```
class PremierAccount:
    def __init__(self, initialBalance):
        self.currentBalance = initialBalance
    def balance(self):
        return self.currentBalance
    def deposit(self, amount):
        self.currentBalance = self.currentBalance + amount
    def creditLimit(self):
        return min(self.currentBalance * 1.5, 10000000)

>>> c = PremierAccount(1000)
>>> c.creditLimit()
1500.0
```

²⁸ A note on style. It is useful to adopt some conventions for naming things, just to help your programs be more readable. We've used the convention that variables and procedure names start with lower case letters and that class names start with upper case letters. And we try to be consistent about using something called "camel caps" for writing compound words, which is to write a compound name with the successiveWordsCapitalized. An alternative is `_to_use_underscores`.

This will let people with premier accounts have larger credit limits. And, the nice thing is that we can ask for its credit limit without knowing what kind of an account it is, so we see that objects support *generic functions*, which operate on different kinds of objects by doing different, but type-appropriate operations.

However, this solution is still not satisfactory. In order to make a premier account, we had to repeat a lot of the definitions from the basic account class, violating our fundamental principle of laziness: never do twice what you could do once; instead, abstract and reuse.

Inheritance lets us make a new class that's like an old class, but with some parts overridden or new parts added. When defining a class, you can actually specify an argument, which is another class. You are saying that this new class should be exactly like the *parent class* or *superclass*, but with certain definitions added or overridden. So, for example, we can say

```
class PremierAccount(Account):
    def creditLimit(self):
        return min(self.currentBalance * 1.5, 10000000)

class EconomyAccount(Account):
    def creditLimit(self):
        return min(self.currentBalance*0.5, 20.00)

>>> a = Account(100)
>>> b = PremierAccount(100)
>>> c = EconomyAccount(100)
>>> a.creditLimit()
100.0
>>> b.creditLimit()
150.0
>>> c.creditLimit()
20.0
```

We automatically inherit the methods of our superclass (including `__init__`). So we still know how to make deposits into a premier account:

```
>>> b.deposit(100)
>>> b.balance()
200
```

The fact that instances know what class they were derived from allows us to ask each instance to do the operations appropriate to it, without having to take specific notice of what class it is. Procedures that can operate on instances of different types or classes without explicitly taking their types or classes into account are called *polymorphic*. Polymorphism is a very powerful method for capturing common patterns.

3.8.3 A data type for times

Object-oriented programming is particularly useful when we want to define a new compositional data type: that is, a representation for the data defining a class of objects and one or more operations for creating new instances of the class from old instances. We might do this with complex numbers or points in a high-dimensional space.

Here we demonstrate the basic ideas with a very simple class for representing times.

We'll start by defining an initialization method, with default values specified for the parameters.

```
class Time:
    def __init__(self, hours = 0, minutes = 0):
        self.hours = hours
        self.minutes = minutes
```

Here is a method for adding two time objects. If the summed minutes are greater than 60, it carries into the hours. If the summed hours are greater than 24, then the excess is simply thrown away. It is important that adding two Times returns a new Time object, and that it *does not change any aspect of either of the arguments*.

```
    def add(self, other):
        newMinutes = (self.minutes + other.minutes) % 60
        carryHours = (self.minutes + other.minutes) / 60
        newHours = (self.hours + other.hours + carryHours) % 24
        return Time(newHours, newMinutes)
```

Python has a special facility for making user-defined types especially cool: if, for instance, you define a method called `__add__` for your class Foo, then whenever it finds an expression of the form `<obj1> + <obj2>`, and the expression `<obj1>` evaluates to an object of class Foo, it will Foo's `__add__` method. So, here we set that up for times:

```
    def __add__(self, other):
        return self.add(other)
```

Additionally, defining methods called `__str__` and `__repr__` that convert elements of your class into strings will mean that they can be printed out nicely by the Python shell.

```
    def __str__(self):
        return str(self.hours) + ':' + str(self.minutes)
    def __repr__(self):
        return str(self)
```

Here, we make some times. In the second case, we specify only the minutes, and so the hours default to 0.

```
>>> t1 = Time(6, 30)
>>> t2 = Time(minutes = 45)
>>> t3 = Time(23, 59)
```

And now, we can do some operations on them. Note how our `__str__` method allows them to be printed nicely.

```
>>> t1
6:30
>>> t1 + t2
7:15
>>> t3 + t2
0:44
>>> t1 + t2 + t3
```



```
7:14
>>> (t1 + t2 + t3).minutes
14
```

3.8.4 Practice problem: argmax

Write a Python procedure `argmax` that takes two arguments: the first is a list of elements, and the second is a function from elements to numerical values. It should return the element of the list with the highest numerical score.

```
def argmax(elements, f):
    bestScore = None
    bestElement = None
    for e in elements:
        score = f(e)
        if bestScore == None or score > bestScore:
            bestScore = score
            bestElement = e
    return bestElement

def argmax(elements, f):
    bestElement = elements[0]
    for e in elements:
        if f(e) > f(bestElement):
            bestElement = e
    return bestElement

def argmax(elements, f):
    vals = [f(e) for e in elements]
    return elements[vals.index(max(vals))]

def argmax(elements, f):
    return max(elements, key=f)
```

There are many ways of writing this. Here are a few. It's important to keep straight in your head which variables contain scores and which contain elements. Any of these solutions would have been fine; we wouldn't expect most people to get the last one.

3.8.5 Practice problem: OOP

The following definitions have been entered into a Python shell:

```
class A:
    yours = 0
    def __init__(self, inp):
        self.yours = inp
    def give(self, amt):
        self.yours = self.yours + amt
        return self.yours
    def howmuch(self):
        return (A.yours, self.yours)
```

```

class B(A):
    yours = 0
    def give(self, amt):
        B.yours = B.yours + amt
        return A.howmuch(self)
    def take(self, amt):
        self.yours = self.yours - amt
        return self.yours
    def howmuch(self):
        return (B.yours, A.yours, self.yours)

```

Write the values of the following expressions. Write None when there is no value; write Error when an error results and explain briefly why it is an error. Assume that these expressions are evaluated one after another (all of the left column first, then right column).

test = A(5)	test = B(5)
test.take(2)	test.take(2)
test.give(6)	test.give(6)
test.howmuch()	test.howmuch()

- **None:** This makes test be equal to a new instance of A, with attribute yours equal to 5.
- **Error:** Instances of A don't have a take method.
- **11:** This adds 6 to the value of test.yours and returns its value.
- **(0, 11):** At this point, we haven't changed the class attribute A.yours, so it still has value 0.
- **None:** This makes test be equal to a new instance of B; it inherits the init method from A, so at this point test.yours is equal to 5.
- **3:** Now, since test is an instance of B, the method test.take is defined; it changes test.yours to be 3 and returns its value.
- **(0, 3):** Note that this refers to the give method defined in class B. So, first, the amount is used to increment B.yours to have value 6. Then, we return A.yours and test.yours, which are both unchanged.
- **(6, 0, 3):** This just returns the current values of all three yours attributes, one in each class definition as well as test.yours.

3.8.6 Practice problem: The Best and the Brightest

Here are some class definitions, meant to represent a collection of students making up the student body of a prestigious university.

```

class Student:
    def __init__(self, name, iq, salary, height):
        self.name = name
        self.iq = iq
        self.salary = salary
        self.height = height

```

```
class StudentBody:
    def __init__(self):
        self.students = []
    def addStudent(self, student):
        self.students.append(student)
    def nameOfSmartest(self):
        # code will go here
    def funOfBest(self, fun, feature):
        # code will go here
```

The StudentBody class is missing the code for two methods:

- `nameOfSmartest`: returns the name attribute of the student with the highest IQ
- `funOfBest`: takes a procedure `fun` and a procedure `feature` as input, and returns the result of applying procedure `fun` to the student for whom the procedure `feature` yields the highest value

For the first two problems below, assume that these methods have been implemented.

Here is a student body:

```
jody = Student('Jody', 100, 100000, 80)
chris = Student('Chris', 150, 40000, 62)
dana = Student('Dana', 120, 2000, 70)
aardvarkU = StudentBody()
aardvarkU.addStudent(jody)
aardvarkU.addStudent(chris)
aardvarkU.addStudent(dana)
```

1. What is the result of evaluating `aardvarkU.nameOfSmartest()`?

'Chris'

2. Write a Python expression that will compute the name of the person who has the greatest value of `IQ + height` in `aardvarkU` (not just for the example student body above). You can define additional procedures if you need them.

```
aardvarkU.funOfBest(lambda x: x.name, lambda x: x.iq + x.height)
```

The second lambda expression specifies how to evaluate an element (that is, to sum its `iq` and `height` attributes) and the first lambda expression says what function to apply to the element with the highest score.

3. Implement the `nameOfSmartest` method. For full credit, use `util.argmax` (defined below) or the `funOfBest` method.

If `l` is a list of items and `f` is a procedure that maps an item into a numeric score, then `util.argmax(l, f)` returns the element of `l` that has the highest score.

```
def nameOfSmartest(self):  
    return util.argmax(self.students, lambda x: x.iq).name  
  
# or  
  
def nameOfSmartest(self):  
    return funOfBest(lambda x: x.name, lambda x: x.iq)
```

Note that in the first solution, the expression `util.argmax(self.students, lambda x: x.iq)` returns the object with the highest iq attribute; then we return the name attribute of that object.

4. Implement the `funOfBest` method. For full credit, use `util.argmax`.

```
def funOfBest(self, fun, feature):
    return fun(util.argmax(self.students, feature))
```

3.8.7 Practice Problem: A Library with Class

Let's build a class to represent a library; let's call it `Library`. In this problem, we'll deal with some standard types of objects:

- A book is represented as a string – its title.
- A patron (person who uses the library) is represented as a string – his/her name.
- A date is represented by an integer – the number of days since the library opened.

The class should have an attribute called `dailyFine` that starts out as 0.25. The class should have the following methods:

- `__init__`: takes a list of books and initializes the library.
- `checkOut`: is given a book, a patron and a date on which the book is being checked out and it records this. Each book can be kept for 7 days before it becomes overdue, i.e. if checked out on day x , it becomes due on day $x + 7$ and it will be considered overdue by one day on day $x + 8$. It returns `None`.
- `checkIn`: is given a book and a date on which the book is being returned and it updates the records. It returns a number representing the fine due if the book is overdue and 0.0 otherwise. The fine is the number of days overdue times the value of the `dailyFine` attribute.
- `overdueBooks`: is given a patron and a date and returns the list of books which that patron has checked out which are overdue at the given date.

Here is an example of the operation of the library:

```
>>> lib = Library(['a', 'b', 'c', 'd', 'e', 'f'])
>>> lib.checkOut('a', 'T', 1)
>>> lib.checkOut('c', 'T', 1)
>>> lib.checkOut('e', 'T', 10)
>>> lib.overdueBooks('T', 13)
['a', 'c']
>>> lib.checkIn('a', 13)
1.25
>>> lib.checkIn('c', 18)
2.50
>>> lib.checkIn('e', 18)
0.25
```

In the boxes below, define the `Library` class as described above. Above each answer box we repeat the specification for each of the attributes and methods given above. Make sure that you enter complete definitions in the boxes, including complete `class` and `def` statements.

Use a dictionary to store the contents of the library. Do not repeat code if at all possible. You can assume that all the operations are legal, for example, all books checked out are in the library and books checked in have been previously checked out.

Class definition:

Include both the start of the class definition and the method definition for `__init__` in this first answer box.

The class should have an attribute called `dailyFine` that starts out as `0.25`.

`__init__`: takes a list of books and initializes the library.

```
class Library:
    dailyFine = 0.25
    def __init__(self, books):
        self.shelf = {}
        for book in books:
            self.shelf[book] = (None, None) # (patron, dueDate)
```

The crucial part here is deciding what information to store. Here, for each book, we store a tuple of who has checked it out, and when it is due.

`checkOut`: is given a book, a patron and a date on which the book is being checked out and it records this. Each book can be kept for 7 days before it becomes overdue, i.e. if checked out on day x , it becomes due on day $x + 7$ and it will be considered overdue by one day on day $x + 8$. It returns `None`.

```
def checkOut(self, book, patron, date):
    self.shelf[book] = (patron, date+7)
```

`checkIn`: is given a book and a date on which the book is being returned and it updates the records. It returns a number representing the fine due if the book is overdue and `0.0` otherwise. The fine is the number of days overdue times the value of the `dailyFine` attribute.

```
def checkIn(self, book, date):
    patron, due = self.shelf[book]
    self.shelf[book] = (None, None)
    return max(0.0, (date - due))*self.dailyFine
```

Note the destructuring assignment in the first line. It's not crucial, but it's nice style, and keeps us from having to refer to components like `self.shelf[book][1]`, which are ugly, long, and hard to get right. Instead of using a `max`, you could use an `if` statement to handle the case of the book being overdue or not, but this is more compact and pretty clear.

`overdueBooks`: is given a patron and a date and returns the list of books which that patron has checked out which are overdue at the given date.

```
def overdueBooks(self, patron, date):
    overdue = []
    for book in self.shelf:
        p, d = self.shelf[book]
        if p and d and p == patron and date > d:
            overdue.append(book)
    return overdue
```

It's not really necessary to check to be sure that `p` and `d` are not `None`, because `p == patron` will only be true for a real patron, and if the patron is not `None` then `d` won't be either. But this code makes it clear that we're only interested in books that have been checked out.

Define a new class called `LibraryGrace` that behaves just like the `Library` class except that it provides a grace period (some number of days after the actual due date) before fines start being accumulated. The number of days in the grace period is specified when an instance is created. See the example below.

```
>>> lib = LibraryGrace(2, ['a', 'b', 'c', 'd', 'e', 'f'])
>>> lib.checkOut('a', 'T', 1)
>>> lib.checkIn('a', 13)
0.75
```

Write the complete class definition for `LibraryGrace`. To get full credit you should not repeat any code that is already in the implementation of `Library`, in particular, you should not need to repeat the computation of the fine.

Class definition:

```
class LibraryGrace(Library):
    def __init__(self, grace, books):
        self.grace = grace
        Library.__init__(self, books)
    def checkIn(self, book, date):
        return Library.checkIn(self, book, date - self.grace)
```

The crucial things here are: remembering to call the `__init__` method of the parent class and doing something to handle the grace period. In this example, when we check a book back in, we pretend we're actually checking it in on an earlier date. Alternatively, we could have changed the checkout method to pretend that the checkout was actually happening in the future.

Chapter 4

State Machines

State machines are a method of modeling systems whose output depends on the entire history of their inputs, and not just on the most recent input. Compared to purely functional systems, in which the output is purely determined by the input, state machines have a performance that is determined by its history. State machines can be used to model a wide variety of systems, including:

- user interfaces, with typed input, mouse clicks, etc.;
- conversations, in which, for example, the meaning of a word “it” depends on the history of things that have been said;
- the state of a spacecraft, including which valves are open and closed, the levels of fuel and oxygen, etc.; and
- the sequential patterns in DNA and what they mean.

State machine models can either be *continuous time* or *discrete time*. In continuous time models, we typically assume continuous spaces for the range of values of inputs and outputs, and use differential equations to describe the system’s dynamics. This is an interesting and important approach, but it is hard to use it to describe the desired behavior of our robots, for example. The loop of reading sensors, computing, and generating an output is inherently discrete and too slow to be well-modeled as a continuous-time process. Also, our control policies are often highly non-linear and discontinuous. So, in this class, we will concentrate on discrete-time models, meaning models whose inputs and outputs are determined at specific increments of time, and which are synchronized to those specific time samples. Furthermore, in this chapter, we will make no assumptions about the form of the dependence of the output on the time-history of inputs; it can be an arbitrary function.

Generally speaking, we can think of the job of an embedded system as performing a *transduction* from a stream (infinite sequence) of input values to a stream of output values. In order to specify the behavior of a system whose output depends on the history of its inputs mathematically, you could think of trying to specify a mapping from i_1, \dots, i_t (sequences of previous inputs) to o_t (current output), but that could become very complicated to specify or execute as the history gets longer. In the state-machine approach, we try to find some set of *states* of the system, which capture the essential properties of the history of the inputs and are used to determine the current output of the system as well as its next state. It is an interesting and sometimes difficult modeling problem to find a good state-machine model with the right set of states; in this chapter we will explore how the ideas of PCAP can aid us in designing useful models.

One thing that is particularly interesting and important about state machine models is how many ways we can use them. In this class, we will use them in three fairly different ways:

1. **Synthetically:** State machines can specify a “program” for a robot or other system embedded in the world, with inputs being sensor readings and outputs being control commands.
2. **Analytically:** State machines can describe the behavior of the combination of a control system and the environment it is controlling; the input is generally a simple command to the entire system, and the output is some simple measure of the state of the system. The goal here is to analyze global properties of the coupled system, like whether it will converge to a steady state, or will oscillate, or will diverge.
3. **Predictively:** State machines can describe the way the environment works, for example, where I will end up if I drive down some road from some intersection. In this case, the inputs are control commands and the outputs are states of the external world. Such a model can be used to plan trajectories through the space of the external world to reach desirable states, by considering different courses of action and using the model to predict their results.

We will develop a single formalism, and an encoding of that formalism in Python classes, that will serve all three of these purposes.

Our strategy for building very complex state machines will be, abstractly, the same strategy we use to build any kind of complex machine. We will define a set of primitive machines (in this case, an infinite class of primitive machines) and then a collection of *combinators* that allow us to put primitive machines together to make more complex machines, which can themselves be abstracted and combined to make more complex machines.

4.1 Primitive state machines

We can specify a transducer (a process that takes as input a sequence of values which serve as inputs to the state machine, and returns as output the set of outputs of the machine for each input) as a state machine (SM) by specifying:

- a set of *states*, S ,
- a set of *inputs*, I , also called the *input vocabulary*,
- a set of *outputs*, O , also called the *output vocabulary*,
- a *next-state function*, $n(i_t, s_t) \mapsto s_{t+1}$, that maps the input at time t and the state at time t to the state at time $t + 1$,
- an *output function*, $o(i_t, s_t) \mapsto o_t$, that maps the input at time t and the state at time t to the output at time t ; and
- an *initial state*, s_0 , which is the state at time 0.

Here are a few state machines, to give you an idea of the kind of systems we are considering.

- A *tick-tock* machine that generates the sequence $1, 0, 1, 0, \dots$ is a finite-state machine that ignores its input.
- The controller for a digital watch is a more complicated finite-state machine: it transduces a sequence of inputs (combination of button presses) into a sequence of outputs (combinations of segments illuminated in the display).
- The controller for a bank of elevators in a large office building: it transduces the current set of buttons being pressed and sensors in the elevators (for position, open doors, etc.) into commands to the elevators to move up or down, and open or close their doors.

The very simplest kind of state machine is a pure function: if the machine has no state, and the output function is purely a function of the input, for example, $o_t = i_t + 1$, then we have an immediate functional relationship between inputs and outputs on the same time step. Another simple class of SMs are *finite-state machines*, for which the set of possible states is finite. The elevator controller can be thought of as a finite-state machine, if elevators are modeled as being only at one floor or another (or possibly between floors); but if the controller models the exact position of the elevator (for the purpose of stopping smoothly at each floor, for example), then it will be most easily expressed using real numbers for the state (though any real instantiation of it can ultimately only have finite precision). A different, large class of SMs are describable as *linear, time-invariant* (LTI) systems. We will discuss these in detail chapter ??.

4.1.1 Examples

Let's look at several examples of state machines, with complete formal definitions.

4.1.1.1 Language acceptor

Here is a finite-state machine whose output is *true* if the input string adheres to a simple pattern, and *false* otherwise. In this case, the pattern has to be $a, b, c, a, b, c, a, b, c, \dots$

It uses the states 0, 1, and 2 to stand for the situations in which it is expecting an $a, b,$ and $c,$ respectively; and it uses state 3 for the situation in which it has seen an input that was not the one that was expected. Once the machine goes to state 3 (sometimes called a *rejecting state*), it never exits that state.

$$\begin{aligned}
 S &= \{0, 1, 2, 3\} \\
 I &= \{a, b, c\} \\
 O &= \{true, false\} \\
 n(s, i) &= \begin{cases} 1 & \text{if } s = 0, i = a \\ 2 & \text{if } s = 1, i = b \\ 0 & \text{if } s = 2, i = c \\ 3 & \text{otherwise} \end{cases} \\
 o(s, i) &= \begin{cases} false & \text{if } n(s, i) = 3 \\ true & \text{otherwise} \end{cases} \\
 s_0 &= 0
 \end{aligned}$$

Figure 4.1 shows a state transition diagram for this state machine. Each circle represents a state. The arcs connecting the circles represent possible transitions the machine can make; the arcs are labeled with a pair i, o , which means that if the machine is in the state denoted by a circle with label s , and gets an input i , then the arc points to the next state, $n(s, i)$ and the output generated $o(s, i) = o$. Some arcs have several labels, indicating that there are many different inputs that will cause the same transition. Arcs can only be traversed in the direction of the arrow.

For a state transition diagram to be complete, there must be an arrow emerging from each state for each possible input i (if the next state is the same for some inputs, then we draw the graph more compactly by using a single arrow with multiple labels, as you will see below).

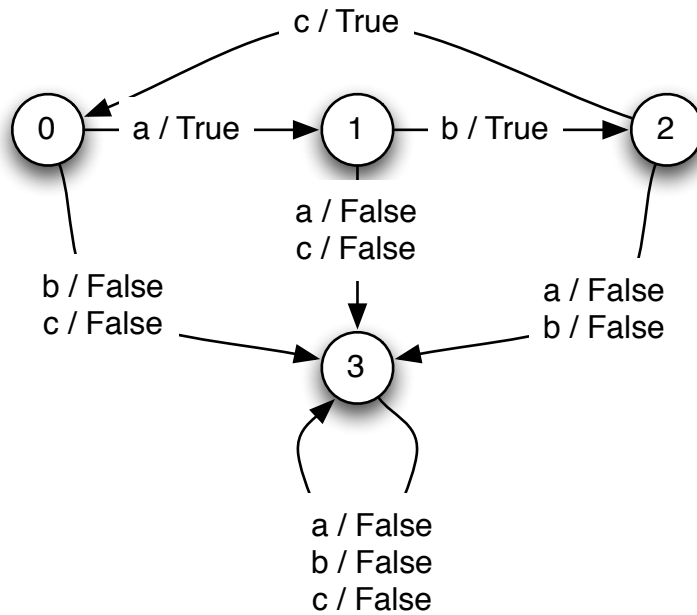


Figure 4.1 State transition diagram for language acceptor.

We will use tables like the following one to examine the evolution of a state machine:

time	0	1	2	...
input	i_0	i_1	i_2	...
state	s_0	s_1	s_2	...
output	o_1	o_2	o_3	...

For each column in the table, given the current input value and state we can use the output function o to determine the output in that column; and we use the n function applied to that input and state value to determine the state in the next column. Thus, just knowing the input sequence and s_0 , and the next-state and output functions of the machine will allow you to fill in the rest of the table.

For example, here is the state of the machine at the initial time point:

time	0	1	2	...
input	i_0			...
state	s_0			...
output				...

Using our knowledge of the next state function n , we have:

time	0	1	2	...
input	i_0			...
state	s_0	s_1		...
output				...

and using our knowledge of the output function o , we have at the next input value

time	0	1	2	...
input	i_0	i_1		...
state	s_0	s_1		...
output	o_1			...

This completes one cycle of the statement machine, and we can now repeat the process.

Here is a table showing what the language acceptor machine does with input sequence ('a', 'b', 'c', 'a', 'c', 'a', 'b'):

time	0	1	2	3	4	5	6	7
input	'a'	'b'	'c'	'a'	'c'	'a'	'b'	
state	0	1	2	0	1	3	3	3
output	True	True	True	True	False	False	False	

The output sequence is (True, True, True, True, False, False, False).

Clearly we don't want to analyze a system by considering all input sequences, but this table helps us understand the state transitions of the system model.

To learn more: Finite-state machine language acceptors can be built for a class of patterns called *regular languages*. There are many more complex patterns (such as the set of strings with equal numbers of 1's and 0's) that cannot be recognized by finite-state machines, but can be recognized by a specialized kind of infinite-state machine called a *stack machine*. To learn more about these fundamental ideas in *computability theory*, start with the Wikipedia article on **Computability_theory_(computer_science)**

4.1.1.2 Up and down counter

This machine can count up and down; its state space is the countably infinite set of integers. It starts in state 0. Now, if it gets input u , it goes to state 1; if it gets u again, it goes to state 2. If it gets d , it goes back down to 1, and so on. For this machine, the output is always the same as the next state.

$$\begin{aligned}
 S &= \text{integers} \\
 I &= \{\text{u}, \text{d}\} \\
 O &= \text{integers} \\
 n(s, i) &= \begin{cases} s + 1 & \text{if } i = \text{u} \\ s - 1 & \text{if } i = \text{d} \end{cases} \\
 o(s, i) &= n(s, i) \\
 s_0 &= 0
 \end{aligned}$$

Here is a table showing what the up and down counter does with input sequence u, u, u, d, d, u:

time	0	1	2	3	4	5	6
input	u	u	u	d	d	u	
state	0	1	2	3	2	1	2
output	1	2	3	2	1	2	

The output sequence is 1, 2, 3, 2, 1, 2.

4.1.1.3 Delay

An even simpler machine just takes the input and passes it through to the output, but with one step of delay, so the k th element of the input sequence will be the $k + 1$ st element of the output sequence. Here is the formal machine definition:

$$\begin{aligned}
 S &= \text{anything} \\
 I &= \text{anything} \\
 O &= \text{anything} \\
 n(s, i) &= i \\
 o(s, i) &= s \\
 s_0 &= 0
 \end{aligned}$$

Given an input sequence i_0, i_1, i_2, \dots , this machine will produce an output sequence $0, i_0, i_1, i_2, \dots$. The initial 0 comes because it has to be able to produce an output before it has even seen an input, and that output is produced based on the initial state, which is 0. This very simple building block will come in handy for us later on.

Here is a table showing what the delay machine does with input sequence 3, 1, 2, 5, 9:

time	0	1	2	3	4	5
input	3	1	2	5	9	
state	0	3	1	2	5	9
output	0	3	1	2	5	

The output sequence is 0, 3, 1, 2, 5.

4.1.1.4 Accumulator

Here is a machine whose output is the sum of all the inputs it has ever seen.

$$\begin{aligned}
 S &= \text{numbers} \\
 I &= \text{numbers} \\
 O &= \text{numbers} \\
 n(s, i) &= s + i \\
 o(s, i) &= n(s, i) \\
 s_0 &= 0
 \end{aligned}$$

Here is a table showing what the accumulator does with input sequence 100, -3, 4, -123, 10:

time	0	1	2	3	4	5
input	100	-3	4	-123	10	
state	0	100	97	101	-22	-12
output	100	97	101	-22	-12	

4.1.1.5 Average2

Here is a machine whose output is the average of the current input and the previous input. It stores its previous input as its state.

$$\begin{aligned}
 S &= \text{numbers} \\
 I &= \text{numbers} \\
 O &= \text{numbers} \\
 n(s, i) &= i \\
 o(s, i) &= (s + i)/2 \\
 s_0 &= 0
 \end{aligned}$$

Here is a table showing what the average2 machine does with input sequence 100, -3, 4, -123, 10:

time	0	1	2	3	4	5
input	100	-3	4	-123	10	
state	0	100	-3	4	-123	10
output	50	48.5	0.5	-59.5	-56.5	

4.1.2 State machines in Python

Now, it is time to make computational implementations of state machine models. In this section we will build up some basic Python infrastructure to make it straightforward to define primitive machines; in later sections we will see how to combine primitive machines into more complex structures.

We will use Python's object-oriented facilities to make this convenient. We have an abstract class, `SM`, which will be the superclass for all of the particular state machine classes we define. It does not make sense to make an instance of `SM`, because it does not actually specify the behavior of the machine; it just provides some utility methods. To specify a new type of state machine, you define a new class that has `SM` as a superclass.

In any subclass of `SM`, it is crucial to define an attribute `startState`, which specifies the initial state of the machine, and a method `getNextValues` which takes the state at time t and the input at time t as input, and returns the state at time $t + 1$ and the output at time t . This is a choice that we have made as designers of our state machine model; we will rely on these two pieces of information in our underlying infrastructure for simulating state machines, as we will see shortly.

Here, for example, is the Python code for an accumulator state machine, which implements the definition given in [section 4.1.1.4](#).²⁹

```
class Accumulator(SM):
    startState = 0
    def getNextValues(self, state, inp):
        return (state + inp, state + inp)
```

It is important to note that `getNextValues` *does not change the state of the machine*, in other words, it does not mutate a state variable. Its job is to be a pure function: to answer the question of what the next state and output would be if the current state were `state` and the current input were `inp`. We will use the `getNextValues` methods of state machines later in the class to make plans by considering alternative courses of action, so they *must not have any side effects*. As we noted, this is our choice as designers of the state machine infrastructure. We could have chosen to implement things differently, however this choice will prove to be very useful. Thus, in all our state machines, the function `getNextValues` will capture the transition from input and state to output and state, without mutating any stored state values.

To run a state machine, we make an instance of the appropriate state-machine class, call its `start` method (a built in method we will see shortly) to set the state to the starting state, and then ask it to take steps; each step consists of generating an output value (which is printed) and updating the state to the next state, based on the input. The abstract superclass `SM` defines the `start` and `step` methods. These methods are in charge of actually initializing and then changing the state of the machine as it is being executed. They do this by calling the `getNextValues` method for the class to which this instance belongs. The `start` method takes no arguments (but, even so, we have to put parentheses after it, indicating that we want to call the method, not to do something

²⁹ Throughout this code, we use `inp`, instead of `input`, which would be clearer. The reason is that the name `input` is used by Python as a function. Although it is legal to re-use it as the name of an argument to a procedure, doing so is a source of bugs that are hard to find (if, for instance, you misspell the name `input` in the argument list, your references to `input` later in the procedure will be legal, but will return the built-in function.)

with the method itself); the step method takes one argument, which is the input to the machine on the next step. So, here is a run of the accumulator, in which we feed it inputs 3, 4, and -2:

```
>>> a = Accumulator()
>>> a.start()
>>> a.step(3)
3
>>> a.step(4)
7
>>> a.step(-2)
5
```

The class SM specifies how state machines work in general; the class Accumulator specifies how accumulator machines work in general; and the instance a is a particular machine with a particular current state. We can make another instance of accumulator:

```
>>> b = Accumulator()
>>> b.start()
>>> b.step(10)
10
>>> b.state
10
>>> a.state
5
```

Now, we have two accumulators, a, and b, which remember, individually, in what states they exist. [Figure 4.2](#) shows the class and instance structures that will be in place after creating these two accumulators.

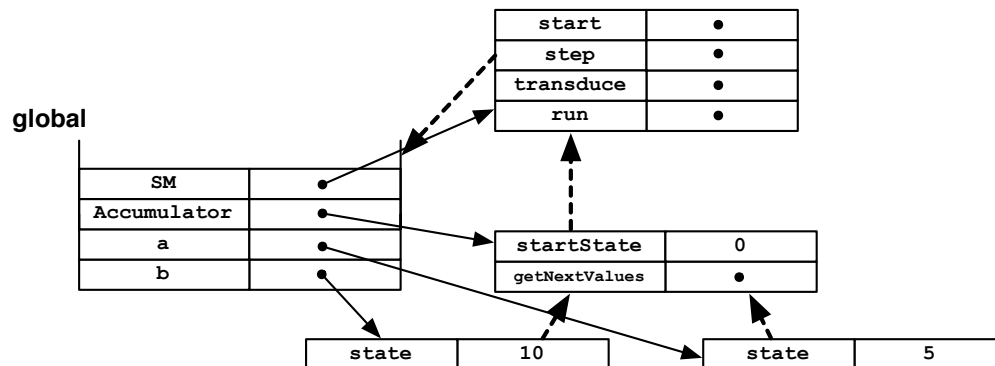


Figure 4.2 Classes and instances for an Accumulator SM. All of the dots represent procedure objects.

4.1.2.1 Defining a type of SM

Let's go back to the definition of the Accumulator class, and study it piece by piece.

First, we define the class and say that it is a subclass of SM:

```
class Accumulator(SM):
```


Next, we define an attribute of the class, `startState`, which is the starting state of the machine. In this case, our accumulator starts up with the value 0.

```
startState = 0
```

Note that `startState` is required by the underlying SM class, so we must either define it in our subclass definition or use a default value from the SM superclass.

The next method defines both the next-state function and the output function, by taking the current state and input as arguments and returning a tuple containing both the next state and the output.

For our accumulator, the next state is just the sum of the previous state and the input; and the output is the same thing:

```
def getNextValues(self, state, inp):
    return (state + inp, state + inp)
```

It is crucial that `getNextValues` be a pure function; that is, that it not change the state of the object (by assigning to any attributes of `self`). It must simply compute the necessary values and return them. We do not promise anything about how many times this method will be called and in what circumstances.

Sometimes, it is convenient to arrange it so that the class really defines a range of machines with slightly different behavior, which depends on some parameters that we set at the time we create an instance. So, for example, if we wanted to specify the initial value for an accumulator at the time the machine is created, we could add an `__init__` method³⁰ to our class, which takes an initial value as a parameter and uses it to set an attribute called `startState` of the instance.³¹

```
class Accumulator(SM):
    def __init__(self, initialValue):
        self.startState = initialValue
    def getNextValues(self, state, inp):
        return state + inp, state + inp
```

Now we can make an accumulator and run it like this:

```
>>> c = Accumulator(100)
>>> c.start()
>>> c.step(20)
120
>>> c.step(2)
122
```

³⁰ Remember that the `__init__` method is a special feature of the Python object-oriented system, which is called whenever an instance of the associated class is created.

³¹ Note that in the original version of `Accumulator`, `startState` was an attribute of the class, since it was the same for every instance of the class; now that we want it to be different for different instances, we need `startState` to be an attribute of the instance rather than the class, which is why we assign it in the `__init__` method, which modifies the already-created instance.

4.1.2.2 The SM Class

The SM class contains generally useful methods that apply to all state machines. A state machine is an instance of any subclass of SM, that has defined the attribute `startState` and the method `getNextValues`, as we did for the `Accumulator` class. Here we examine these methods in more detail.

Running a machine

The first group of methods allows us to run a state machine. To run a machine is to provide it with a sequence of inputs and then sequentially go forward, computing the next state and generating the next output, as if we were filling in a state table.

To run a machine, we have to start by calling the `start` method. All it does is create an attribute of the instance, called `state`, and assign to it the value of the `startState` attribute. It is crucial that we have both of these attributes: if we were to just modify `startState`, then if we wanted to run this machine again, we would have permanently forgotten what the starting state should be. Note that `state` becomes a repository for the state of **this** instance; however we should not mutate it directly. This variable becomes the internal representation of state for each instance of this class.

```
class SM:
    def start(self):
        self.state = self.startState
```

Once it has started, we can ask it to take a step, using the `step` method, which, given an input, computes the output and updates the internal state of the machine, and returns the output value.

```
    def step(self, inp):
        (s, o) = self.getNextValues(self.state, inp)
        self.state = s
        return o
```

To run a machine on a whole sequence of input values, we can use the `transduce` method, which will return the sequence of output values that results from feeding the elements of the list `inputs` into the machine in order.

```
    def transduce(self, inputs):
        self.start()
        return [self.step(inp) for inp in inputs]
```

Here are the results of running `transduce` on our accumulator machine. We run it twice, first with a simple call that does not generate any debugging information, and simply returns the result. The second time, we ask it to be verbose, resulting in a print-out of what is happening on the intermediate steps.³²

³² In fact, the second machine trace, and all the others in this section were generated with a call like:

```
>>> a.transduce([100, -3, 4, -123, 10], verbose = True, compact = True)
```

See the *Infrastructure Guide* for details on different debugging options. To simplify the code examples we show in these notes, we have omitted parts of the code that are responsible for debugging printouts.

```

a = Accumulator()
>>> a.transduce([100, -3, 4, -123, 10])
[100, 97, 101, -22, -12]
>>> a.transduce([100, -3, 4, -123, 10], verbose = True)
Start state: 0
In: 100 Out: 100 Next State: 100
In: -3 Out: 97 Next State: 97
In: 4 Out: 101 Next State: 101
In: -123 Out: -22 Next State: -22
In: 10 Out: -12 Next State: -12
[100, 97, 101, -22, -12]

```

Some machines do not take any inputs; in that case, we can simply call the SM `run` method, which is equivalent to doing `transduce` on an input sequence of `[None, None, None, ...]`.

```

def run(self, n = 10):
    return self.transduce([None]*n)

```

4.1.2.2.1 Default methods

This section is optional.

In order to make the specifications for the simplest machine types as short as possible, we have also provided a set of default methods in the SM class. These default methods say that, unless they are overridden in a subclass, as they were when we defined `Accumulator`, we will assume that a machine starts in state `None` and that its output is the same as its next state.

```

startState = None
def getNextValues(self, state, inp):
    nextState = self.getNextState(state, inp)
    return (nextState, nextState)

```

Because these methods are provided in SM, we can define, for example, a state machine whose output is always `k` times its input, with this simple class definition, which defines a `getNextState` procedure that simply returns a value that is treated as both the next state and the output.

```

class Gain(SM):
    def __init__(self, k):
        self.k = k
    def getNextState(self, state, inp):
        return inp * self.k

```

We can use this class as follows:

```

>>> g = Gain(3)
>>> g.transduce([1.1, -2, 100, 5])
[3.3000000000000003, -6, 300, 15]

```

The parameter `k` is specified at the time the instance is created. Then, the output of the machine is always just `k` times the input.

We can also use this strategy to write the `Accumulator` class even more succinctly:

```
class Accumulator(SM):
    startState = 0
    def getNextState(self, state, inp):
        return state + inp
```

The output of the `getNextState` method will be treated both as the output and the next state of the machine, because the inherited `getNextValues` function uses it to compute both values.

4.1.2.3 More examples

Here are Python versions of the rest of the machines we introduced in the first section.

Language acceptor

Here is a Python class for a machine that “accepts” the language that is any prefix of the infinite sequence `['a', 'b', 'c', 'a', 'b', 'c', ...]`.

```
class ABC(SM):
    startState = 0
    def getNextValues(self, state, inp):
        if state == 0 and inp == 'a':
            return (1, True)
        elif state == 1 and inp == 'b':
            return (2, True)
        elif state == 2 and inp == 'c':
            return (0, True)
        else:
            return (3, False)
```

It behaves as we would expect. As soon as it sees a character that deviates from the desired sequence, the output is `False`, and it will remain `False` for ever after.

```
>>> abc = ABC()
>>> abc.transduce(['a', 'a', 'a'], verbose = True)
Start state: 0
In: a Out: True Next State: 1
In: a Out: False Next State: 3
In: a Out: False Next State: 3
[True, False, False]

>>> abc.transduce(['a', 'b', 'c', 'a', 'c', 'a', 'b'], verbose = True)
Start state: 0
In: a Out: True Next State: 1
In: b Out: True Next State: 2
In: c Out: True Next State: 0
In: a Out: True Next State: 1
In: c Out: False Next State: 3
In: a Out: False Next State: 3
In: b Out: False Next State: 3
[True, True, True, True, False, False, False]
```

Count up and down

This is a direct translation of the machine defined in [section 4.1.1.2](#).

```
class UpDown(SM):
    startState = 0
    def getNextState(self, state, inp):
        if inp == 'u':
            return state + 1
        else:
            return state - 1
```

We take advantage of the default `getNextValues` method to make the output the same as the next state.

```
>>> ud = UpDown()
>>> ud.transduce(['u','u','u','d','d','u'], verbose = True)
Start state: 0
In: u Out: 1 Next State: 1
In: u Out: 2 Next State: 2
In: u Out: 3 Next State: 3
In: d Out: 2 Next State: 2
In: d Out: 1 Next State: 1
In: u Out: 2 Next State: 2
[1, 2, 3, 2, 1, 2]
```

Delay

In order to make a machine that delays its input stream by one time step, we have to specify what the first output should be. We do this by passing the parameter, `v0`, into the `__init__` method of the `Delay` class. The state of a `Delay` machine is just the input from the previous step, and the output is the state (which is, therefore, the input from the previous time step).

```
class Delay(SM):
    def __init__(self, v0):
        self.startState = v0
    def getNextValues(self, state, inp):
        return (inp, state)

>>> d = Delay(7)
>>> d.transduce([3, 1, 2, 5, 9], verbose = True)
Start state: 7
In: 3 Out: 7 Next State: 3
In: 1 Out: 3 Next State: 1
In: 2 Out: 1 Next State: 2
In: 5 Out: 2 Next State: 5
In: 9 Out: 5 Next State: 9
[7, 3, 1, 2, 5]

>>> d100 = Delay(100)
>>> d100.transduce([3, 1, 2, 5, 9], verbose = True)
Start state: 100
In: 3 Out: 100 Next State: 3
```

```
In: 1 Out: 3 Next State: 1
In: 2 Out: 1 Next State: 2
In: 5 Out: 2 Next State: 5
In: 9 Out: 5 Next State: 9
[100, 3, 1, 2, 5]
```

We will use this machine so frequently that we put its definition in the `sm` module (file), along with the class.

R

We can use `R` as another name for the `Delay` class of state machines. It will be an important primitive in a compositional system of *linear time-invariant systems*, which we explore in the next chapter.

Average2

Here is a state machine whose output at time t is the average of the input values from times $t - 1$ and t .

```
class Average2(SM):
    startState = 0
    def getNextValues(self, state, inp):
        return (inp, (inp + state) / 2.0)
```

It needs to remember the previous input, so the next state is equal to the input. The output is the average of the current input and the state (because the state is the previous input).

```
>>> a2 = Average2()
>>> a2.transduce([10, 5, 2, 10], verbose = True, compact = True)
Start state: 0
In: 10 Out: 5.0 Next State: 10
In: 5 Out: 7.5 Next State: 5
In: 2 Out: 3.5 Next State: 2
In: 10 Out: 6.0 Next State: 10
[5.0, 7.5, 3.5, 6.0]
```

Sum of last three inputs

Here is an example of a state machine where the state is actually a list of values. Generally speaking, the state can be anything (a dictionary, an array, a list); but it is important to be sure that the `getNextValues` method does not make direct changes to components of the state, instead returning a *new copy* of the state with appropriate changes. We may make several calls to the `getNextValues` function on one step (or, later in our work, call the `getNextValues` function with several different inputs to see what would happen under different choices); these function calls are made to find out a value of the next state, but if they actually change the state, then the same call with the same arguments may return a different value the next time.

This machine generates as output at time t the sum of i_{t-2} , i_{t-1} and i_t ; that is, of the last three inputs. In order to do this, it has to remember the values of two previous inputs; so the state is a pair of numbers. We have defined it so that the initial state is $(0, 0)$. The `getNextValues`

method gets rid of the oldest value that it has been remembering, and remembers the current input as part of the state; the output is the sum of the current input with the two old inputs that are stored in the state. Note that the first line of the `getNextValues` procedure is a structured assignment (see [section 3.3](#)).

```
class SumLast3 (SM):
    startState = (0, 0)
    def getNextValues(self, state, inp):
        (previousPreviousInput, previousInput) = state
        return ((previousInput, inp),
                previousPreviousInput + previousInput + inp)

>>> sl3 = SumLast3()
>>> sl3.transduce([2, 1, 3, 4, 10, 1, 2, 1, 5], verbose = True)
Start state: (0, 0)
In: 2 Out: 2 Next State: (0, 2)
In: 1 Out: 3 Next State: (2, 1)
In: 3 Out: 6 Next State: (1, 3)
In: 4 Out: 8 Next State: (3, 4)
In: 10 Out: 17 Next State: (4, 10)
In: 1 Out: 15 Next State: (10, 1)
In: 2 Out: 13 Next State: (1, 2)
In: 1 Out: 4 Next State: (2, 1)
In: 5 Out: 8 Next State: (1, 5)
[2, 3, 6, 8, 17, 15, 13, 4, 8]
```

Selector

A simple functional machine that is very useful is the `Select` machine. You can make many different versions of this, but the simplest one takes an input that is a stream of lists or tuples of several values (or structures of values) and generates the stream made up only of the k th elements of the input values. Which particular component this machine is going to select is determined by the value k , which is passed in at the time the machine instance is initialized.

```
class Select (SM):
    def __init__(self, k):
        self.k = k
    def getNextState(self, state, inp):
        return inp[self.k]
```

4.1.3 Simple parking gate controller

As one more demonstration, here is a simple example of a finite-state controller for a gate leading out of a parking lot.

The gate has three sensors:

- `gatePosition` has one of three values 'top', 'middle', 'bottom', signifying the position of the arm of the parking gate.
- `carAtGate` is True if a car is waiting to come through the gate and False otherwise.
- `carJustExited` is True if a car has just passed through the gate; it is true for only one step before resetting to False.

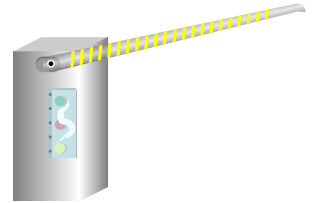


Image by MIT OpenCourseWare.

Pause to try 4.1. How many possible inputs are there?

A: 12.

The gate has three possible outputs (think of them as controls to the motor for the gate arm): 'raise', 'lower', and 'nop'. (Nop means "no operation.")

Roughly, here is what the gate needs to do:

- If a car wants to come through, the gate needs to raise the arm until it is at the top position.
- Once the gate is at the top position, it has to stay there until the car has driven through the gate.
- After the car has driven through the gate needs to lower the arm until it reaches the bottom position.

So, we have designed a simple finite-state controller with a state transition diagram as shown in [figure 4.3](#). The machine has four possible states: 'waiting' (for a car to arrive at the gate), 'raising' (the arm), 'raised' (the arm is at the top position and we're waiting for the car to drive through the gate), and 'lowering' (the arm). To keep the figure from being too cluttered, we do not label each arc with every possible input that would cause that transition to be made: instead, we give a condition (think of it as a Boolean expression) that, if true, would cause the transition to be followed. The conditions on the arcs leading out of each state cover all the possible inputs, so the machine remains completely well specified.

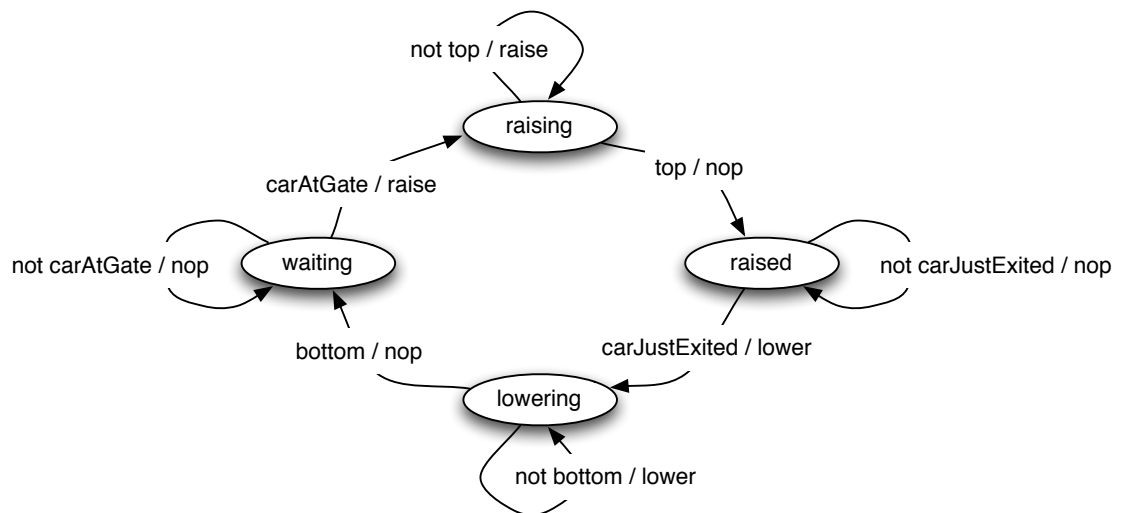


Figure 4.3 State transition diagram for parking gate controller.

Here is a simple state machine that implements the controller for the parking gate. For compactness, the getNextValues method starts by determining the next state of the gate. Then, depending on the next state, the generateOutput method selects the appropriate output.

```
class SimpleParkingGate (SM):
    startState = 'waiting'

    def generateOutput(self, state):
        if state == 'raising':
            return 'raise'
        elif state == 'lowering':
            return 'lower'
        else:
            return 'nop'

    def getNextValues(self, state, inp):
        (gatePosition, carAtGate, carJustExited) = inp
        if state == 'waiting' and carAtGate:
            nextState = 'raising'
        elif state == 'raising' and gatePosition == 'top':
            nextState = 'raised'
        elif state == 'raised' and carJustExited:
            nextState = 'lowering'
        elif state == 'lowering' and gatePosition == 'bottom':
            nextState = 'waiting'
        else:
            nextState = state
        return (nextState, self.generateOutput(nextState))
```

In the situations where the state does not change (that is, when the arcs lead back to the same state in the diagram), we do not explicitly specify the next state in the code: instead, we cover it in the else clause, saying that unless otherwise specified, the state stays the same. So, for example, if the state is raising but the gatePosition is not yet top, then the state simply stays raising until the top is reached.

```
>>> spg = SimpleParkingGate()
>>> spg.transduce(testInput, verbose = True)
Start state: waiting
In: ('bottom', False, False) Out: nop Next State: waiting
In: ('bottom', True, False) Out: raise Next State: raising
In: ('bottom', True, False) Out: raise Next State: raising
In: ('middle', True, False) Out: raise Next State: raising
In: ('middle', True, False) Out: raise Next State: raising
In: ('middle', True, False) Out: raise Next State: raising
In: ('top', True, False) Out: nop Next State: raised
In: ('top', True, False) Out: nop Next State: raised
In: ('top', True, False) Out: nop Next State: raised
In: ('top', True, True) Out: lower Next State: lowering
In: ('top', True, True) Out: lower Next State: lowering
In: ('top', True, False) Out: lower Next State: lowering
In: ('middle', True, False) Out: lower Next State: lowering
In: ('middle', True, False) Out: lower Next State: lowering
In: ('middle', True, False) Out: lower Next State: lowering
```

```

In: ('bottom', True, False) Out: nop Next State: waiting
In: ('bottom', True, False) Out: raise Next State: raising
['nop', 'raise', 'raise', 'raise', 'raise', 'raise', 'nop', 'nop', 'nop', 'lower', 'lower',
'lower', 'lower', 'lower', 'lower', 'nop', 'raise']

```

Exercise 4.1. What would the code for this machine look like if it were written without using the `generateOutput` method?

4.2 Basic combination and abstraction of state machines

In the previous section, we studied the definition of a primitive state machine, and saw a number of examples. State machines are useful for a wide variety of problems, but specifying complex machines by explicitly writing out their state transition functions can be quite tedious. Ultimately, we will want to build large state-machine descriptions compositionally, by specifying primitive machines and then combining them into more complex systems. We will start here by looking at ways of combining state machines.

We can apply our PCAP (primitive, combination, abstraction, pattern) methodology, to build more complex SMs out of simpler ones. In the rest of this section we consider “dataflow” compositions, where inputs and outputs of primitive machines are connected together; after that, we consider “conditional” compositions that make use of different sub-machines depending on the input to the machine, and finally “sequential” compositions that run one machine after another.

4.2.1 Cascade composition

In cascade composition, we take two machines and use the output of the first one as the input to the second, as shown in [figure 4.4](#). The result is a new composite machine, whose input vocabulary is the input vocabulary of the first machine and whose output vocabulary is the output vocabulary of the second machine. It is, of course, crucial that the output vocabulary of the first machine be the same as the input vocabulary of the second machine.

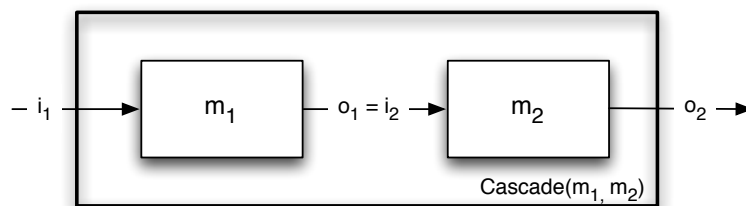


Figure 4.4 Cascade composition of state machines

Recalling the `Delay` machine from the previous section, let’s see what happens if we make the cascade composition of two delay machines. Let m_1 be a delay machine with initial value 99 and m_2 be a delay machine with initial value 22. Then $Cascade(m_1, m_2)$ is a new state machine, constructed by making the output of m_1 be the input of m_2 . Now, imagine we feed a sequence of values, 3, 8, 2, 4, 6, 5, into the composite machine, m . What will come out? Let’s try to understand this by making a table of the states and values at different times:

time	0	1	2	3	4	5	6
m ₁ input	3	8	2	4	6	5	
m ₁ state	99	3	8	2	4	6	5
m ₁ output	99	3	8	2	4	6	
m ₂ input	99	3	8	2	4	6	
m ₂ state	22	99	3	8	2	4	6
m ₂ output	22	99	3	8	2	4	

The output sequence is 22, 99, 3, 8, 2, 4, which is the input sequence, delayed by two time steps.

Another way to think about cascade composition is as follows. Let the input to m_1 at time t be called $i_1[t]$ and the output of m_1 at time t be called $o_1[t]$. Then, we can describe the workings of the delay machine in terms of an equation:

$$o_1[t] = i_1[t - 1] \text{ for all values of } t > 0;$$

$$o_1[0] = \text{init}_1$$

that is, that the output value at every time t is equal to the input value at the previous time step. You can see that in the table above. The same relation holds for the input and output of m_2 :

$$o_2[t] = i_2[t - 1] \text{ for all values of } t > 0.$$

$$o_2[0] = \text{init}_2$$

Now, since we have connected the output of m_1 to the input of m_2 , we also have that $i_2[t] = o_1[t]$ for all values of t . This lets us make the following derivation:

$$o_2[t] = i_2[t - 1]$$

$$= o_1[t - 1]$$

$$= i_1[t - 2]$$

This makes it clear that we have built a “delay by two” machine, by cascading two single delay machines.

As with all of our systems of combination, we will be able to form the cascade composition not only of two primitive machines, but of any two machines that we can make, through any set of compositions of primitive machines.

Here is the Python code for an Increment machine. It is a pure function whose output at time t is just the input at time t plus the constant `incr`. The `safeAdd` function is the same as addition, if the inputs are numbers. We will see, later, why it is important.

```
class Increment(SM):
    def __init__(self, incr):
        self.incr = incr
    def getNextState(self, state, inp):
        return safeAdd(inp, self.incr)
```

Exercise 4.2. Derive what happens when you cascade two delay-by-two machines?

Exercise 4.3. What is the difference between these two machines?

```
>>> foo1 = sm.Cascade(sm.Delay(100), Increment(1))
>>> foo2 = sm.Cascade(Increment(1), sm.Delay(100))
```

Demonstrate by drawing a table of their inputs, states, and outputs, over time.

4.2.2 Parallel composition

In parallel composition, we take two machines and run them “side by side”. They both take the same input, and the output of the composite machine is the pair of outputs of the individual machines. The result is a new composite machine, whose input vocabulary is the same as the input vocabulary of the component machines (which is the same for both machines) and whose output vocabulary is pairs of elements, the first from the output vocabulary of the first machine and the second from the output vocabulary of the second machine. [Figure 4.5](#) shows two types of parallel composition; in this section we are talking about the first type.

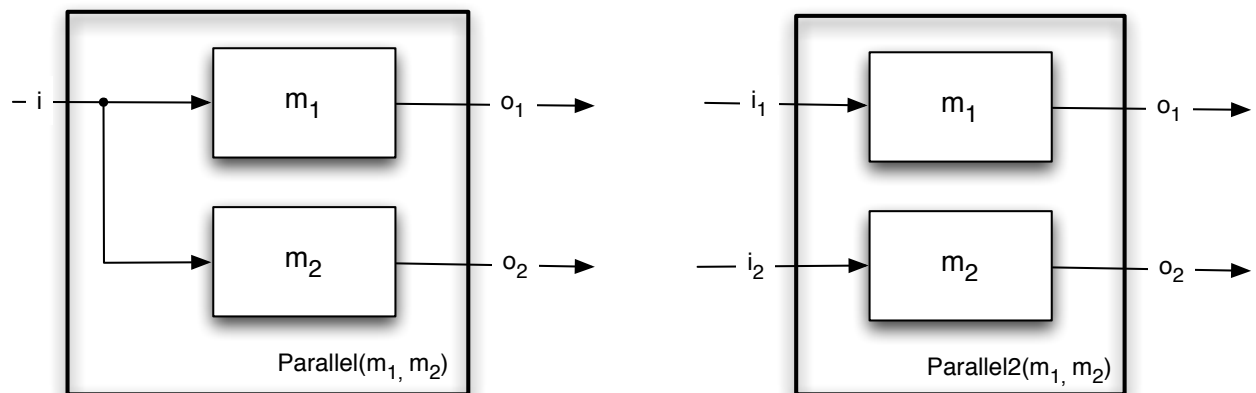


Figure 4.5 Parallel and Parallel2 compositions of state machines.

In Python, we can define a new class of state machines, called `Parallel`, which is a subclass of `SM`. To make an instance of `Parallel`, we pass two `SM`s of any type into the initializer. The state of the parallel machine is a pair consisting of the states of the constituent machines. So, the starting state is the pair of the starting states of the constituents.

```
class Parallel (SM):
    def __init__(self, sm1, sm2):
        self.m1 = sm1
        self.m2 = sm2
        self.startState = (sm1.startState, sm2.startState)
```

To get a new state of the composite machine, we just have to get new states for each of the constituents, and return the pair of them; similarly for the outputs.

```
def getNextValues(self, state, inp):
    (s1, s2) = state
    (newS1, o1) = self.m1.getNextValues(s1, inp)
    (newS2, o2) = self.m2.getNextValues(s2, inp)
    return ((newS1, newS2), (o1, o2))
```

Parallel2

Sometimes we will want a variant on parallel combination, in which rather than having the input be a single item which is fed to both machines, the input is a pair of items, the first of which is fed to the first machine and the second to the second machine. This composition is shown in the second part of [figure 4.5](#).

Here is a Python class that implements this two-input parallel composition. It can inherit the `__init__` method from `Parallel`, so use `Parallel` as the superclass, and we only have to define two methods.

```
class Parallel2(Parallel):
    def getNextValues(self, state, inp):
        (s1, s2) = state
        (i1, i2) = splitValue(inp)
        (newS1, o1) = self.m1.getNextValues(s1, i1)
        (newS2, o2) = self.m2.getNextValues(s2, i2)
        return ((newS1, newS2), (o1, o2))
```

Later, when dealing with feedback systems ([section 4.2.3](#)), we will need to be able to deal with 'undefined' as an input. If the `Parallel2` machine gets an input of 'undefined', then we want to pass 'undefined' into the constituent machines. We make our code more beautiful by defining the helper function below, which is guaranteed to return a pair, if its argument is either a pair or 'undefined'.³³

```
def splitValue(v):
    if v == 'undefined':
        return ('undefined', 'undefined')
    else:
        return v
```

ParallelAdd

The `ParallelAdd` state machine combination is just like `Parallel`, except that it has a single output whose value is the sum of the outputs of the constituent machines. It is straightforward to define:

³³ We are trying to make the code examples we show here as simple and clear as possible; if we were writing code for actual deployment, we would check and generate error messages for all sorts of potential problems (in this case, for instance, if `v` is neither `None` nor a two-element list or tuple.)

```

class ParallelAdd (Parallel):
    def getNextValues(self, state, inp):
        (s1, s2) = state
        (newS1, o1) = self.m1.getNextValues(s1, inp)
        (newS2, o2) = self.m2.getNextValues(s2, inp)
        return ((newS1, newS2), o1 + o2)

```

4.2.3 Feedback composition

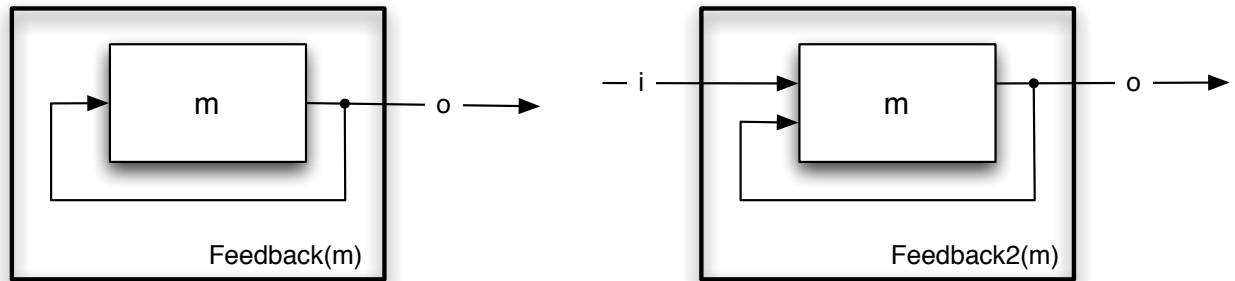


Figure 4.6 Two forms of feedback composition.

Another important means of combination that we will use frequently is the feedback combinator, in which the output of a machine is fed back to be the input of the same machine at the next step, as shown in [figure 4.6](#). The first value that is fed back is the output associated with the initial state of the machine on which we are operating. It is crucial that the input and output vocabularies of the machine are the same (because the output at step t will be the input at step $t + 1$). Because we have fed the output back to the input, this machine does not consume any inputs; but we will treat the feedback value as an output of this machine.

Here is an example of using feedback to make a machine that counts. We can start with a simple machine, an incrementer, that takes a number as input and returns that same number plus 1 as the output. By itself, it has no memory. Here is its formal description:

$$\begin{aligned}
 S &= \text{numbers} \\
 I &= \text{numbers} \\
 O &= \text{numbers} \\
 n(s, i) &= i + 1 \\
 o(s, i) &= n(s, i) \\
 s_0 &= 0
 \end{aligned}$$

What would happen if we performed the feedback operation on this machine? We can try to understand this in terms of the input/output equations. From the definition of the increment machine, we have

$$o[t] = i[t] + 1 .$$

And if we connect the input to the output, then we will have

$$i[t] = o[t] .$$

And so, we have a problem; these equations cannot be satisfied.

A crucial requirement for applying feedback to a machine is: *that machine must not have a direct dependence of its output on its input.*

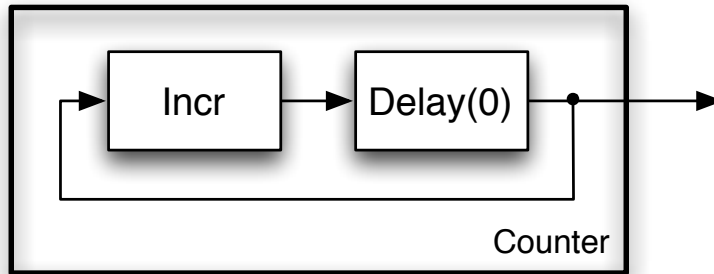


Figure 4.7 Counter made with feedback and serial combination of an incrementer and a delay.

We have already explored a Delay machine, which delays its output by one step. We can delay the result of our incrementer, by cascading it with a Delay machine, as shown in [figure 4.7](#). Now, we have the following equations describing the system:

$$o_i[t] = i_i[t] + 1$$

$$o_d[t] = i_d[t - 1]$$

$$i_i[t] = o_d[t]$$

$$i_d[t] = o_i[t]$$

The first two equations describe the operations of the increment and delay boxes; the second two describe the wiring between the modules. Now we can see that, in general,

$$o_i[t] = i_i[t] + 1$$

$$o_i[t] = o_d[t] + 1$$

$$o_i[t] = i_d[t - 1] + 1$$

$$o_i[t] = o_i[t - 1] + 1$$

that is, that the output of the incrementer is going to be one greater on each time step.

Exercise 4.4. How could you use feedback and a *negation* primitive machine (which is a pure function that takes a Boolean as input and returns the negation of that Boolean) to make a machine whose output alternates between *true* and *false*.

4.2.3.1 Python Implementation

Following is a Python implementation of the feedback combinator, as a new subclass of SM that takes, at initialization time, a state machine.

```
class Feedback (SM):
    def __init__(self, sm):
        self.m = sm
        self.startState = self.m.startState
```

The starting state of the feedback machine is just the state of the constituent machine.

Generating an output for the feedback machine is interesting: by our hypothesis that the output of the constituent machine cannot depend directly on the current input, it means that, for the purposes of generating the output, we can actually feed an explicitly undefined value into the machine as input. Why would we do this? The answer is that we do not know what the input value should be (in fact, it is defined to be the output that we are trying to compute).

We must, at this point, add an extra condition on our getNextValues methods. They have to be prepared to accept 'undefined' as an input. If they get an undefined input, they should return 'undefined' as an output. For convenience, in our files, we have defined the procedures safeAdd and safeMul to do addition and multiplication, but passing through 'undefined' if it occurs in either argument.

So: if we pass 'undefined' into the constituent machine's getNextValues method, we must not get 'undefined' back as output; if we do, it means that there is an immediate dependence of the output on the input. Now we know the output *o* of the machine.

To get the next state of the machine, we get the next state of the constituent machine, by taking the feedback value, *o*, that we just computed and using it as input for getNextValues. This will generate the next state of the feedback machine. (Note that throughout this process *inp* is ignored—a feedback machine has no input.)

```
def getNextValues(self, state, inp):
    (ignore, o) = self.m.getNextValues(state, 'undefined')
    (newS, ignore) = self.m.getNextValues(state, o)
    return (newS, o)
```

Now, we can construct the counter we designed. The Increment machine, as we saw in its definition, uses a safeAdd procedure, which has the following property: if either argument is 'undefined', then the answer is 'undefined'; otherwise, it is the sum of the inputs.

```
def makeCounter(init, step):
    return sm.Feedback(sm.Cascade(Increment(step), sm.Delay(init)))
```

```
>>> c = makeCounter(3, 2)
>>> c.run(verbose = True)
Start state: (None, 3)
Step: 0
Feedback_96
  Cascade_97
    Increment_98 In: 3 Out: 5 Next State: 5
```



```

    Delay_99 In: 5 Out: 3 Next State: 5
Step: 1
  Feedback_96
    Cascade_97
      Increment_98 In: 5 Out: 7 Next State: 7
      Delay_99 In: 7 Out: 5 Next State: 7
Step: 2
  Feedback_96
    Cascade_97
      Increment_98 In: 7 Out: 9 Next State: 9
      Delay_99 In: 9 Out: 7 Next State: 9
Step: 3
  Feedback_96
    Cascade_97
      Increment_98 In: 9 Out: 11 Next State: 11
      Delay_99 In: 11 Out: 9 Next State: 11
Step: 4
  Feedback_96
    Cascade_97
      Increment_98 In: 11 Out: 13 Next State: 13
      Delay_99 In: 13 Out: 11 Next State: 13
...

```

[3, 5, 7, 9, 11, 13, 15, 17, 19, 21]

(The numbers, like 96 in `Feedback_96` are not important; they are just tags generated internally to indicate different instances of a class.)

Exercise 4.5. Draw state tables illustrating whether the following machines are different, and if so, how:

```

m1 = sm.Feedback(sm.Cascade(sm.Delay(1), Increment(1)))
m2 = sm.Feedback(sm.Cascade(Increment(1), sm.Delay(1)))

```

4.2.3.2 Fibonacci

Now, we can get very fancy. We can generate the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21, etc), in which the first two outputs are 1, and each subsequent output is the sum of the two previous outputs, using a combination of very simple machines. Basically, we have to arrange for the output of the machine to be fed back into a parallel combination of elements, one of which delays the value by one step, and one of which delays by two steps. Then, those values are added, to compute the next output. [Figure 4.8](#) shows a diagram of one way to construct this system.

The corresponding Python code is shown below. First, we have to define a new component machine. An `Adder` takes pairs of numbers (appearing simultaneously) as input, and immediately generates their sum as output.

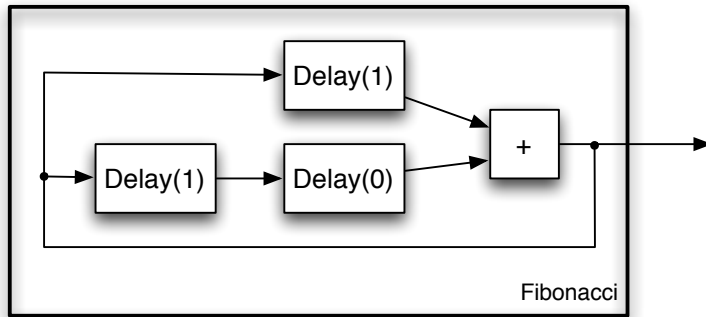


Figure 4.8 Machine to generate the Fibonacci sequence.

```
class Adder(SM):
    def getNextState(self, state, inp):
        (i1, i2) = splitValue(inp)
        return safeAdd(i1, i2)
```

Now, we can define our `fib` machine. It is a great example of building a complex machine out of very nearly trivial components. In fact, we will see in the next module that there is an interesting and important class of machines that can be constructed with cascade and parallel compositions of delay, adder, and gain machines. It is crucial for the delay machines to have the right values (as shown in the figure) in order for the sequence to start off correctly.

```
>>> fib = sm.Feedback(sm.Cascade(sm.Parallel(sm.Delay(1),
                                             sm.Cascade(sm.Delay(1), sm.Delay(0))),
                                             Adder()))
```

```
>>> fib.run(verbose = True)
Start state: ((1, (1, 0)), None)
Step: 0
Feedback_100
  Cascade_101
    Parallel_102
      Delay_103 In: 1 Out: 1 Next State: 1
      Cascade_104
        Delay_105 In: 1 Out: 1 Next State: 1
        Delay_106 In: 1 Out: 0 Next State: 1
      Adder_107 In: (1, 0) Out: 1 Next State: 1
Step: 1
Feedback_100
  Cascade_101
    Parallel_102
      Delay_103 In: 2 Out: 1 Next State: 2
      Cascade_104
        Delay_105 In: 2 Out: 1 Next State: 2
        Delay_106 In: 1 Out: 1 Next State: 1
      Adder_107 In: (1, 1) Out: 2 Next State: 2
Step: 2
Feedback_100
  Cascade_101
    Parallel_102
      Delay_103 In: 3 Out: 2 Next State: 3
```

```

    Cascade_104
      Delay_105 In: 3 Out: 2 Next State: 3
      Delay_106 In: 2 Out: 1 Next State: 2
    Adder_107 In: (2, 1) Out: 3 Next State: 3
Step: 3
Feedback_100
  Cascade_101
    Parallel_102
      Delay_103 In: 5 Out: 3 Next State: 5
      Cascade_104
        Delay_105 In: 5 Out: 3 Next State: 5
        Delay_106 In: 3 Out: 2 Next State: 3
      Adder_107 In: (3, 2) Out: 5 Next State: 5
...

```

[1, 2, 3, 5, 8, 13, 21, 34, 55, 89]

Exercise 4.6. What would we have to do to this machine to get the sequence [1, 1, 2, 3, 5, ...]?

Exercise 4.7. Define `fib` as a composition involving only two delay components and an adder. You might want to use an instance of the `Wire` class. A `Wire` is the completely passive machine, whose output is always instantaneously equal to its input. It is not very interesting by itself, but sometimes handy when building things.

```

class Wire(SM):
    def getNextState(self, state, inp):
        return inp

```

Exercise 4.8. Use feedback and a multiplier (analogous to `Adder`) to make a machine whose output doubles on every step.

Exercise 4.9. Use feedback and a multiplier (analogous to `Adder`) to make a machine whose output squares on every step.

4.2.3.3 Feedback2

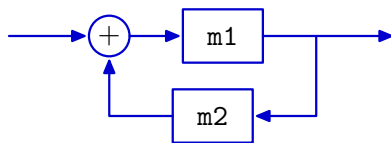
The second part of [figure 4.6](#) shows a combination we call *feedback2*: it assumes that it takes a machine with two inputs and one output, and connects the output of the machine to the second input, resulting in a machine with one input and one output.

Feedback2 is very similar to the basic feedback combinator, but it gives, as input to the constituent machine, the pair of the input to the machine and the feedback value.

```
class Feedback2 (Feedback):
    def getNextValues(self, state, inp):
        (ignore, o) = self.m.getNextValues(state, (inp, 'undefined'))
        (newS, ignore) = self.m.getNextValues(state, (inp, o))
        return (newS, o)
```

4.2.3.4 FeedbackSubtract and FeedbackAdd

In *feedback addition* composition, we take two machines and connect them as shown below:



If m_1 and m_2 are state machines, then you can create their feedback addition composition with

```
newM = sm.FeedbackAdd(m1, m2)
```

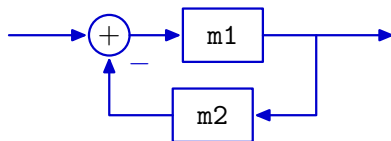
Now $newM$ is itself a state machine. So, for example,

```
newM = sm.FeedbackAdd(sm.R(0), sm.Wire())
```

makes a machine whose output is the sum of all the inputs it has ever had (remember that $sm.R$ is shorthand for $sm.Delay$). You can test it by feeding it a sequence of inputs; in the example below, it is the numbers 0 through 9:

```
>>> newM.transduce(range(10))
[0, 0, 1, 3, 6, 10, 15, 21, 28, 36]
```

Feedback subtraction composition is the same, except the output of m_2 is *subtracted* from the input, to get the input to m_1 .



Note that if you want to apply one of the feedback operators in a situation where there is only one machine, you can use the $sm.Gain(1.0)$ machine (defined [section 4.1.2.2.1](#)), which is essentially a wire, as the other argument.

4.2.3.5 Factorial

We will do one more tricky example, and illustrate the use of `Feedback2`. What if we wanted to generate the sequence of numbers $\{1!, 2!, 3!, 4!, \dots\}$ (where $k! = 1 \cdot 2 \cdot 3 \dots k$)? We can do so by multiplying the previous value of the sequence by a number equal to the



“index” of the sequence. **Figure 4.9** shows the structure of a machine for solving this problem. It uses a counter (which is, as we saw before, made with feedback around a delay and increment) as the input to a machine that takes a single input, and multiplies it by the output value of the machine, fed back through a delay.

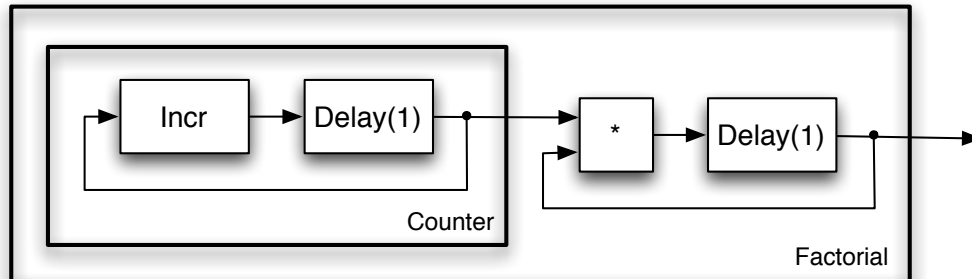


Figure 4.9 Machine to generate the Factorial sequence.

Here is how to do it in Python; we take advantage of having defined counter machines to abstract away from them and use that definition here without thinking about its internal structure. The initial values in the delays get the series started off in the right place. What would happen if we started at 0?

```
fact = sm.Cascade(makeCounter(1, 1),
                  sm.Feedback2(sm.Cascade(Multiplier(), sm.Delay(1))))
```

```
>>> fact.run(verbose = True)
Start state: ((None, 1), (None, 1))
Step: 0
  Cascade_1
    Feedback_2
      Cascade_3
        Increment_4 In: 1 Out: 2 Next State: 2
        Delay_5 In: 2 Out: 1 Next State: 2
      Feedback2_6
        Cascade_7
          Multiplier_8 In: (1, 1) Out: 1 Next State: 1
          Delay_9 In: 1 Out: 1 Next State: 1
Step: 1
  Cascade_1
    Feedback_2
      Cascade_3
        Increment_4 In: 2 Out: 3 Next State: 3
        Delay_5 In: 3 Out: 2 Next State: 3
      Feedback2_6
        Cascade_7
          Multiplier_8 In: (2, 1) Out: 2 Next State: 2
          Delay_9 In: 2 Out: 1 Next State: 2
Step: 2
  Cascade_1
    Feedback_2
      Cascade_3
        Increment_4 In: 3 Out: 4 Next State: 4
        Delay_5 In: 4 Out: 3 Next State: 4
```

```

Feedback2_6
  Cascade_7
    Multiplier_8 In: (3, 2) Out: 6 Next State: 6
    Delay_9 In: 6 Out: 2 Next State: 6
Step: 3
  Cascade_1
    Feedback_2
      Cascade_3
        Increment_4 In: 4 Out: 5 Next State: 5
        Delay_5 In: 5 Out: 4 Next State: 5
      Feedback2_6
        Cascade_7
          Multiplier_8 In: (4, 6) Out: 24 Next State: 24
          Delay_9 In: 24 Out: 6 Next State: 24
...

[1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880]

```

It might bother you that we get a 1 as the zeroth element of the sequence, but it is reasonable as a definition of $0!$, because 1 is the multiplicative identity (and is often defined that way by mathematicians).

4.2.4 Plants and controllers

One common situation in which we combine machines is to simulate the effects of coupling a controller and a so-called “plant”. A plant is a factory or other external environment that we might wish to control. In this case, we connect two state machines so that the output of the plant (typically thought of as sensory observations) is input to the controller, and the output of the controller (typically thought of as actions) is input to the plant. This is shown schematically in [figure 4.10](#). For example, when you build a Soar brain that interacts with the robot, the robot (and the world in which it is operating) is the “plant” and the brain is the controller. We can build a coupled machine by first connecting the machines in a cascade and then using feedback on that combination.

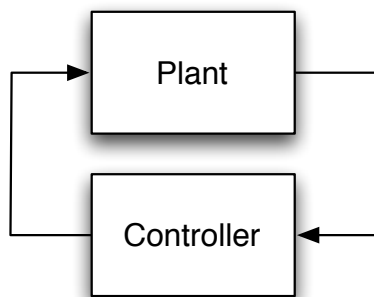


Figure 4.10 Two coupled machines.

As a concrete example, let’s think about a robot driving straight toward a wall. It has a distance sensor that allows it to observe the distance to the wall at time t , $d[t]$, and it desires to stop at some distance $d_{desired}$. The robot can execute velocity commands, and we program it to use the following rule to set its velocity at time t , based on its most recent sensor reading:

$$v[t] = K(d_{desired} - d[t - 1]) .$$

This controller can also be described as a state machine, whose input sequence is the observed values of d and whose output sequence is the values of v .

$$\begin{aligned} S &= \text{numbers} \\ I &= \text{numbers} \\ O &= \text{numbers} \\ n(s, i) &= K(d_{desired} - i) \\ o(s) &= s \\ s_0 &= d_{init} \end{aligned}$$

Now, we can think about the “plant”; that is, the relationship between the robot and the world. The distance of the robot to the wall changes at each time step depending on the robot’s forward velocity and the length of the time steps. Let δT be the length of time between velocity commands issued by the robot. Then we can describe the world with the equation:

$$d[t] = d[t - 1] - \delta T v[t - 1] .$$

which assumes that a positive velocity moves the robot *toward* the wall (and therefore decreases the distance). This system can be described as a state machine, whose input sequence is the values of the robot’s velocity, v , and whose output sequence is the values of its distance to the wall, d .

Finally, we can couple these two systems, as for a simulator, to get a single state machine with no inputs. We can observe the sequence of internal values of d and v to understand how the system is behaving.

In Python, we start by defining the controller machine; the values k and $dDesired$ are constants of the whole system.

```
k = -1.5
dDesired = 1.0
class WallController(SM):
    def getNextState(self, state, inp):
        return safeMul(k, safeAdd(dDesired, safeMul(-1, inp)))
```

The output being generated is actually $k * (dDesired - inp)$, but because this method is going to be used in a feedback machine, it might have to deal with ‘undefined’ as an input. It has no delay built into it.

Think about why we want k to be negative. What happens when the robot is closer to the wall than desired? What happens when it is farther from the wall than desired?

Now, we can define a class that describes the behavior of the “plant”:

```
deltaT = 0.1
class WallWorld(SM):
    startState = 5
    def getNextValues(self, state, inp):
        return (state - deltaT * inp, state)
```

Setting `startState = 5` means that the robot starts 5 meters from the wall. Note that the output of this machine does not depend instantaneously on the input; so there *is* a delay in it.

Now, we can define a general combinator for coupling two machines, as in a plant and controller:

```
def coupledMachine(m1, m2):
    return sm.Feedback(sm.Cascade(m1, m2))
```

We can use it to connect our controller to the world, and run it:

```
>>> wallSim = coupledMachine(WallController(), WallWorld())
>>> wallSim.run(30)
[5, 4.4000000000000004, 3.8900000000000001, 3.4565000000000001,
 3.088025, 2.77482125, 2.5085980624999999, 2.2823083531249999,
 2.0899621001562498, 1.9264677851328122, 1.7874976173628905,
 1.6693729747584569, 1.5689670285446884, 1.483621974262985,
 1.4110786781235374, 1.3494168764050067, 1.2970043449442556,
 1.2524536932026173, 1.2145856392222247, 1.1823977933388909,
 1.1550381243380574, 1.1317824056873489, 1.1120150448342465,
 1.0952127881091096, 1.0809308698927431, 1.0687912394088317,
 1.058472553497507, 1.049701670472881, 1.0422464199019488,
 1.0359094569166565]
```

Because `WallWorld` is the second machine in the cascade, its output is the output of the whole machine; so, we can see that the distance from the robot to the wall is converging monotonically to `dDesired` (which is 1).

Exercise 4.10. What kind of behavior do you get with different values of `k`?

4.2.5 Conditionals

We might want to use different machines depending on something that is happening in the outside world. Here we describe three different conditional combinators, that make choices, at the run-time of the machine, about what to do.

4.2.6 Switch

We will start by considering a conditional combinator that runs two machines in parallel, but decides *on every input* whether to send the input into one machine or the other. So, only one of the parallel machines has its state updated on each step. We will call this *switch*, to emphasize the fact that the decision about which machine to execute is being re-made on every step.

Implementing this requires us to maintain the states of both machines, just as we did for parallel combination. The `getNextValues` method tests the condition and then gets a new state and output from the appropriate constituent machine; it also has to be sure to pass through the old state for the constituent machine that was not updated this time.


```

class Switch (SM):
    def __init__(self, condition, sm1, sm2):
        self.m1 = sm1
        self.m2 = sm2
        self.condition = condition
        self.startState = (self.m1.startState, self.m2.startState)

    def getNextValues(self, state, inp):
        (s1, s2) = state
        if self.condition(inp):
            (ns1, o) = self.m1.getNextValues(s1, inp)
            return ((ns1, s2), o)
        else:
            (ns2, o) = self.m2.getNextValues(s2, inp)
            return ((s1, ns2), o)

```

Multiplex

The `switch` combinator takes care to only update one of the component machines; in some other cases, we want to update both machines on every step and simply use the condition to select the output of one machine or the other to be the current output of the combined machine.

This is a very small variation on `Switch`, so we will just implement it as a subclass.

```

class Mux (Switch):
    def getNextValues(self, state, inp):
        (s1, s2) = state
        (ns1, o1) = self.m1.getNextValues(s1, inp)
        (ns2, o2) = self.m2.getNextValues(s2, inp)
        if self.condition(inp):
            return ((ns1, ns2), o1)
        else:
            return ((ns1, ns2), o2)

```

Exercise 4.11. What is the result of running these two machines

```

m1 = Switch(lambda inp: inp > 100,
            Accumulator(),
            Accumulator())
m2 = Mux(lambda inp: inp > 100,
         Accumulator(),
         Accumulator())

```

on the input

```
[2, 3, 4, 200, 300, 400, 1, 2, 3]
```

Explain why they are the same or are different.

If

Feel free to skip this example; it is only useful in fairly complicated contexts.



The *If* combinator. It takes a *condition*, which is a function from the input to *true* or *false*, and two machines. It evaluates the condition on the first input it receives. If the value is *true* then it executes the first machine forever more; if it is *false*, then it executes the second machine.

This can be straightforwardly implemented in Python; we will work through a slightly simplified version of our code below. We start by defining an initializer that remembers the conditions and the two constituent state machines.

```
class If (SM):
    startState = ('start', None)
    def __init__(self, condition, sm1, sm2):
        self.sm1 = sm1
        self.sm2 = sm2
        self.condition = condition
```

Because this machine does not have an input available at start time, it can not decide whether it is going to execute *sm1* or *sm2*. Ultimately, the state of the *If* machine will be a pair of values: the first will indicate which constituent machine we are running and the second will be the state of that machine. But, to start, we will be in the state ('start', None), which indicates that the decision about which machine to execute has not yet been made.

Now, when it is time to do a state update, we have an input. We destructure the state into its two parts, and check to see if the first component is 'start'. If so, we have to make the decision about which machine to execute. The method *getFirstRealState* first calls the condition on the current input, to decide which machine to run; then it returns the pair of a symbol indicating which machine has been selected and the starting state of that machine. Once the first real state is determined, then that is used to compute a transition into an appropriate next state, based on the input.

If the machine was already in a non-start state, then it just updates the constituent state, using the already-selected constituent machine. Similarly, to generate an output, we have use the output function of the appropriate machine, with special handling of the start state.

```
startState = ('start', None)

def __init__(self, condition, sm1, sm2):
    self.sm1 = sm1
    self.sm2 = sm2
    self.condition = condition

def getFirstRealState(self, inp):
    if self.condition(inp):
        return ('runningM1', self.sm1.startState)
    else:
        return ('runningM2', self.sm2.startState)

def getNextValues(self, state, inp):
    (ifState, smState) = state
    if ifState == 'start':
        (ifState, smState) = self.getFirstRealState(inp)
```

```

if ifState == 'runningM1':
    (newS, o) = self.sm1.getNextValues(smState, inp)
    return (('runningM1', newS), o)
else:
    (newS, o) = self.sm2.getNextValues(smState, inp)
    return (('runningM2', newS), o)

```

4.3 Terminating state machines and sequential compositions

So far, all the machines we have discussed run forever; or, at least, until we quit giving them inputs. But in some cases, it is particularly useful to think of a process as consisting of a sequence of processes, one executing until termination, and then another one starting. For example, you might want to robot to clean first room A, **and then** clean room B; or, for it to search in an area **until** it finds a person **and then** sound an alarm.

Temporal combinations of machines form a new, different PCAP system for state machines. Our primitives will be state machines, as described above, but with one additional property: they will have a termination or *done* function, $d(s)$, which takes a state and returns *true* if the machine has finished execution and *false* otherwise.

Rather than defining a whole new class of state machines (though we could do that), we will just augment the SM class with a default method, which says that, by default, machines do not terminate.

```

def done(self, state):
    return False

```

Then, in the definition of any subclass of SM, you are free to implement your own *done* method that will override this base one. The *done* method is used by state machine combinators that, for example, run one machine until it is done, and then switch to running another one.

Here is an example *terminating state machine* (TSM) that consumes a stream of numbers; its output is *None* on the first four steps and then on the fifth step, it generates the sum of the numbers it has seen as inputs, and then terminates. It looks just like the state machines we have seen before, with the addition of a *done* method. Its state consists of two numbers: the first is the number of times the machine has been updated and the second is the total input it has accumulated so far.

```

class ConsumeFiveValues(SM):
    startState = (0, 0)      # count, total

    def getNextValues(self, state, inp):
        (count, total) = state
        if count == 4:
            return ((count + 1, total + inp), total + inp)
        else:
            return ((count + 1, total + inp), None)

    def done(self, state):
        (count, total) = state
        return count == 5

```

Here is the result of running a simple example. We have modified the `transduce` method of SM to stop when the machine is done.

```
>>> c5 = ConsumeFiveValues()
>>> c5.transduce([1, 2, 3, 4, 5, 6, 7, 8, 9, 10], verbose = True)
Start state: (0, 0)
In: 1 Out: None Next State: (1, 1)
In: 2 Out: None Next State: (2, 3)
In: 3 Out: None Next State: (3, 6)
In: 4 Out: None Next State: (4, 10)
In: 5 Out: 15 Next State: (5, 15)
[None, None, None, None, 15]
```

Now we can define a new set of combinators that operate on TSMs. Each of these combinators assumes that its constituent machines are terminating state machines, and are, themselves, terminating state machines. We have to respect certain rules about TSMs when we do this. In particular, it is not legal to call the `getNextValues` method on a TSM that says it is done. This may or may not cause an actual Python error, but it is never a sensible thing to do, and may result in meaningless answers.

4.3.1 Repeat

The simplest of the TSM combinators is one that takes a terminating state machine `sm` and repeats it `n` times. In the Python method below, we give a default value of `None` for `n`, so that if no value is passed in for `n` it will repeat forever.

```
class Repeat (SM):
    def __init__(self, sm, n = None):
        self.sm = sm
        self.startState = (0, self.sm.startState)
        self.n = n
```

The state of this machine will be the number of times the constituent machine has been executed to completion, together with the current state of the constituent machine. So, the starting state is a pair consisting of 0 and the starting state of the constituent machine.

Because we are going to, later, ask the constituent machine to generate an output, we are going to adopt a convention that the constituent machine is never left in a state that is done, unless the whole `Repeat` is itself done. If the constituent machine is done, then we will increment the counter for the number of times we have repeated it, and restart it. Just in case the constituent machine “wakes up” in a state that is done, we use a `while` loop here, instead of an `if`: we will keep restarting this machine until the count runs out. Why? Because we promised not to leave our constituent machine in a done state (so, for example, nobody asks for its output, when its done), unless the whole repeat machine is done as well.

```
def advanceIfDone(self, counter, smState):
    while self.sm.done(smState) and not self.done((counter, smState)):
        counter = counter + 1
        smState = self.sm.startState
    return (counter, smState)
```

To get the next state, we start by getting the next state of the constituent machine; then, we check to see if the counter needs to be advanced and the machine restarted; the `advanceIfDone` method handles this situation and returns the appropriate next state. The output of the `Repeat` machine is just the output of the constituent machine. We just have to be sure to destructure the state of the overall machine and pass the right part of it into the constituent.

```
def getNextValues(self, state, inp):
    (counter, smState) = state
    (smState, o) = self.sm.getNextValues(smState, inp)
    (counter, smState) = self.advanceIfDone(counter, smState)
    return ((counter, smState), o)
```

We know the whole `Repeat` is done if the counter is equal to `n`.

```
def done(self, state):
    (counter, smState) = state
    return counter == self.n
```

Now, we can see some examples of `Repeat`. As a primitive, here is a silly little example TSM. It takes a character at initialization time. Its state is a Boolean, indicating whether it is done. It starts up in state `False` (not done). Then it makes its first transition into state `True` and stays there. Its output is always the character it was initialized with; it completely ignores its input.

```
class CharTSM (SM):
    startState = False
    def __init__(self, c):
        self.c = c
    def getNextValues(self, state, inp):
        return (True, self.c)
    def done(self, state):
        return state
```

```
>>> a = CharTSM('a')
>>> a.run(verbose = True)
Start state: False
In: None Out: a Next State: True
['a']
```

See that it terminates after one output. But, now, we can repeat it several times.

```
>>> a4 = sm.Repeat(a, 4)
>>> a4.run()
['a', 'a', 'a', 'a']
```

Exercise 4.12. Would it have made a difference if we had executed:

```
>>> sm.Repeat(CharTSM('a'), 4).run()
```

Exercise 4.13. Monty P. thinks that the following call

```
>>> sm.Repeat(ConsumeFiveValues(), 3).transduce(range(100))
```

will generate a sequence of 14 Nones followed by the sum of the first 15 integers (starting at 0). R. Reticulatis disagrees. Who is right and why?

4.3.2 Sequence

Another useful thing to do with TSMs is to execute several different machines sequentially. That is, take a list of TSMs, run the first one until it is done, start the next one and run it until it is done, and so on. This machine is similar in style and structure to a Repeat TSM. Its state is a pair of values: an index that says which of the constituent machines is currently being executed, and the state of the current constituent.

Here is a Python class for creating a Sequence TSM. It takes as input a list of state machines; it remembers the machines and number of machines in the list.

```
class Sequence (SM):
    def __init__(self, smList):
        self.smList = smList
        self.startState = (0, self.smList[0].startState)
        self.n = len(smList)
```

The initial state of this machine is the value 0 (because we start by executing the 0th constituent machine on the list) and the initial state of that constituent machine.

The method for advancing is also similar that for Repeat. The only difference is that each time, we start the next machine in the list of machines, until we have finished executing the last one.

```
def advanceIfDone(self, counter, smState):
    while self.smList[counter].done(smState) and counter + 1 < self.n:
        counter = counter + 1
        smState = self.smList[counter].startState
    return (counter, smState)
```

To get the next state, we ask the current constituent machine for its next state, and then, if it is done, advance the state to the next machine in the list that is not done when it wakes up. The output of the composite machine is just the output of the current constituent.

```
def getNextValues(self, state, inp):
    (counter, smState) = state
    (smState, o) = self.smList[counter].getNextValues(smState, inp)
    (counter, smState) = self.advanceIfDone(counter, smState)
    return ((counter, smState), o)
```

We have constructed this machine so that it always advances past any constituent machine that is done; if, in fact, the current constituent machine is done, then the whole machine is also done.

```
def done(self, state):
    (counter, smState) = state
    return self.smList[counter].done(smState)
```

We can make good use of the CharTSM to test our sequential combinator. First, we will try something simple:

```
>>> m = sm.Sequence([CharTSM('a'), CharTSM('b'), CharTSM('c')])
>>> m.run()
Start state: (0, False)
In: None Out: a Next State: (1, False)
In: None Out: b Next State: (2, False)
In: None Out: c Next State: (2, True)
['a', 'b', 'c']
```

Even in a test case, there is something unsatisfying about all that repetitive typing required to make each individual CharTSM. If we are repeating, we should abstract. So, we can write a function that takes a string as input, and returns a sequential TSM that will output that string. It uses a list comprehension to turn each character into a CharTSM that generates that character, and then uses that sequence to make a Sequence.

```
def makeTextSequenceTSM(str):
    return sm.Sequence([CharTSM(c) for c in str])

>>> m = makeTextSequenceTSM('Hello World')
>>> m.run(20, verbose = True)
Start state: (0, False)
In: None Out: H Next State: (1, False)
In: None Out: e Next State: (2, False)
In: None Out: l Next State: (3, False)
In: None Out: l Next State: (4, False)
In: None Out: o Next State: (5, False)
In: None Out:   Next State: (6, False)
In: None Out: W Next State: (7, False)
In: None Out: o Next State: (8, False)
In: None Out: r Next State: (9, False)
In: None Out: l Next State: (10, False)
In: None Out: d Next State: (10, True)
['H', 'e', 'l', 'l', 'o', ' ', 'W', 'o', 'r', 'l', 'd']
```

We can also see that sequencing interacts well with the Repeat combinator.

```
>>> m = sm.Repeat(makeTextSequenceTSM('abc'), 3)
>>> m.run(verbose = True)
Start state: (0, (0, False))
In: None Out: a Next State: (0, (1, False))
In: None Out: b Next State: (0, (2, False))
In: None Out: c Next State: (1, (0, False))
In: None Out: a Next State: (1, (1, False))
In: None Out: b Next State: (1, (2, False))
In: None Out: c Next State: (2, (0, False))
In: None Out: a Next State: (2, (1, False))
In: None Out: b Next State: (2, (2, False))
```

```
In: None Out: c Next State: (3, (0, False))
['a', 'b', 'c', 'a', 'b', 'c', 'a', 'b', 'c']
```

It is interesting to understand the state here. The first value is the number of times the constituent machine of the Repeat machine has finished executing; the second value is the index of the sequential machine into its list of machines, and the last Boolean is the state of the CharTSM that is being executed, which is an indicator for whether it is done or not.

4.3.3 RepeatUntil and Until

In order to use Repeat, we need to know in advance how many times we want to execute the constituent TSM. Just as in ordinary programming, we often want to terminate when a particular condition is met in the world. For this purpose, we can construct a new TSM combinator, called RepeatUntil. It takes, at initialization time, a condition, which is a function from an input to a Boolean, and a TSM. It runs the TSM to completion, then tests the condition on the input; if the condition is true, then the RepeatUntil terminates; if it is false, then it runs the TSM to completion again, tests the condition, etc.

Here is the Python code for implementing RepeatUntil. The state of this machine has two parts: a Boolean indicating whether the condition is true, and the state of the constituent machine.

```
class RepeatUntil (SM):
    def __init__(self, condition, sm):
        self.sm = sm
        self.condition = condition
        self.startState = (False, self.sm.startState)

    def getNextValues(self, state, inp):
        (condTrue, smState) = state
        (smState, o) = self.sm.getNextValues(smState, inp)
        condTrue = self.condition(inp)
        if self.sm.done(smState) and not condTrue:
            smState = self.sm.getStartState()
        return ((condTrue, smState), o)

    def done(self, state):
        (condTrue, smState) = state
        return self.sm.done(smState) and condTrue
```

One important thing to note is that, in the RepeatUntil TSM the condition is only evaluated when the constituent TSM is done. This is appropriate in some situations; but in other cases, we would like to terminate the execution of a TSM if a condition becomes true *at any single step of the machine*. We could easily implement something like this in any particular case, by defining a special-purpose TSM class that has a done method that tests the termination condition. But, because this structure is generally useful, we can define a general-purpose combinator, called Until. This combinator also takes a condition and a constituent machine. It simply executes the constituent machine, and terminates either when the condition becomes true, or when the constituent machine terminates. As before, the state includes the value of the condition on the last input and the state of the constituent machine.

Note that this machine will never execute the constituent machine more than once; it will either run it once to completion (if the condition never becomes true), or terminate it early.

Here are some examples of using `RepeatUntil` and `Until`. First, we run the `ConsumeFiveValues` machine until the input is greater than 10. Because it only tests the condition when `ConsumeFiveValues` is done, and the condition only becomes true on the 11th step, the `ConsumeFiveValues` machine is run to completion three times.

```
def greaterThan10 (x):
    return x > 10
>>> m = sm.RepeatUntil(greaterThan10, ConsumeFiveValues())
>>> m.transduce(range(20), verbose = True)
Start state: (0, 0)
In: 0 Out: None Next State: (1, 0)
In: 1 Out: None Next State: (2, 1)
In: 2 Out: None Next State: (3, 3)
In: 3 Out: None Next State: (4, 6)
In: 4 Out: 10 Next State: (0, 0)
In: 5 Out: None Next State: (1, 5)
In: 6 Out: None Next State: (2, 11)
In: 7 Out: None Next State: (3, 18)
In: 8 Out: None Next State: (4, 26)
In: 9 Out: 35 Next State: (0, 0)
In: 10 Out: None Next State: (1, 10)
In: 11 Out: None Next State: (2, 21)
In: 12 Out: None Next State: (3, 33)
In: 13 Out: None Next State: (4, 46)
In: 14 Out: 60 Next State: (5, 60)
[None, None, None, None, 10, None, None, None, None, 35, None, None, None, None, 60]
```

If we do `Until` on the basic `ConsumeFiveValues` machine, then it just runs `ConsumeFiveValues` until it terminates normally, because the condition never becomes true during this time.

```
>>> m = sm.Until(greaterThan10, ConsumeFiveValues())
>>> m.transduce(range(20), verbose = True)
Start state: (False, (0, 0))
In: 0 Out: None Next State: (False, (1, 0))
In: 1 Out: None Next State: (False, (2, 1))
In: 2 Out: None Next State: (False, (3, 3))
In: 3 Out: None Next State: (False, (4, 6))
In: 4 Out: 10 Next State: (False, (5, 10))
[None, None, None, None, 10]
```

However, if we change the termination condition, the execution will be terminated early. Note that we can use a lambda expression directly as an argument; sometimes this is actually clearer than defining a function with `def`, but it is fine to do it either way.

```
>>> m = sm.Until(lambda x: x == 2, ConsumeFiveValues())
>>> m.transduce(range(20), verbose = True)
Start state: (False, (0, 0))
In: 0 Out: None Next State: (False, (1, 0))
In: 1 Out: None Next State: (False, (2, 1))
In: 2 Out: None Next State: (True, (3, 3))
[None, None, None]
```

If we actually want to keep repeating `ConsumeFiveValues()` until the condition becomes true, we can combine `Until` with `Repeat`. Now, we see that it executes the constituent machine multiple times, but terminates as soon as the condition is satisfied.

```
>>> m = sm.Until(greaterThan10, sm.Repeat(ConsumeFiveValues()))
>>> m.transduce(range(20), verbose = True)
Start state: (False, (0, (0, 0)))
In: 0 Out: None Next State: (False, (0, (1, 0)))
In: 1 Out: None Next State: (False, (0, (2, 1)))
In: 2 Out: None Next State: (False, (0, (3, 3)))
In: 3 Out: None Next State: (False, (0, (4, 6)))
In: 4 Out: 10 Next State: (False, (1, (0, 0)))
In: 5 Out: None Next State: (False, (1, (1, 5)))
In: 6 Out: None Next State: (False, (1, (2, 11)))
In: 7 Out: None Next State: (False, (1, (3, 18)))
In: 8 Out: None Next State: (False, (1, (4, 26)))
In: 9 Out: 35 Next State: (False, (2, (0, 0)))
In: 10 Out: None Next State: (False, (2, (1, 10)))
In: 11 Out: None Next State: (True, (2, (2, 21)))
[None, None, None, None, 10, None, None, None, None, 35, None, None]
```

4.4 Using a state machine to control the robot

This section gives an overview of how to control the robot with a state machine. For a much more detailed description, see the *Infrastructure Guide*, which documents the `io` and `util` modules in detail. The `io` module provides procedures and methods for interacting with the robot; the `util` module provides procedures and methods for doing computations that are generally useful (manipulating angles, dealing with coordinate frames, etc.)

We can implement a robot controller as a state machine whose inputs are instances of class `io.SensorInput`, and whose outputs are instances of class `io.Action`.

Here is Python code for a brain that is controlled by the most basic of state machines. This machine always emits the default action, `io.Action()`, which sets all of the output values to zero. When the brain is set up, we create a “behavior”, which is a name we will use for a state machine that transduces a stream of `io.SensorInputs` to a stream of `io.Actions`. Finally, we ask the behavior to start.

Then, all we do in the `step` method of the robot is:

- Read the sensors, by calling `io.SensorInput()` to get an instance that contains sonar and odometry readings;
- Feed that sensor input to the brain state machine, by calling its `step` method with that as input; and
- Take the `io.Action` that is generated by the brain as output, and call its `execute` method, which causes it to actually send motor commands to the robot.

You can set the `verbose` flag to `True` if you want to see a lot of output on each step for debugging.

Inside a Soar brain, we have access to an object `robot`, which persists during the entire execution of the brain, and gives us a place to store important objects (like the state machine that will be doing all the work).

```

import sm
import io

class StopSM(sm.SM):
    def getNextValues(self, state, inp):
        return (None, io.Action())

def setup():
    robot.behavior = StopSM()
    robot.behavior.start()

def step():
    robot.behavior.step(io.SensorInput(), verbose = False).execute()

```

In the following sections we will develop two simple machines for controlling a robot to move a fixed distance or turn through a fixed angle. Then we will put them together and explore why it can be useful to have the starting state of a machine depend on the input.

4.4.1 Rotate

Imagine that we want the robot to rotate a fixed angle, say 90 degrees, to the left of where it is when it starts to run a behavior. We can use the robot's odometry to measure approximately where it is, in an arbitrary coordinate frame; but to know how much it has moved since we started, we have to store some information in the state.

Here is a class that defines a `Rotate` state machine. It takes, at initialization time, a desired change in heading.

```

class RotateTSM (SM):
    rotationalGain = 3.0
    angleEpsilon = 0.01
    startState = 'start'

    def __init__(self, headingDelta):
        self.headingDelta = headingDelta

```

When it is time to start this machine, we would like to look at the robot's current heading (`theta`), add the desired change in heading, and store the result in our state as the desired heading. Then, in order to test whether the behavior is done, we want to see whether the current heading is close enough to the desired heading. Because the `done` method does not have access to the input of the machine (it is a property only of states), we need to include the current `theta` in the state. So, the state of the machine is (`thetaDesired`, `thetaLast`).

Thus, the `getNextValues` method looks at the state; if it is the special symbol `'start'`, it means that the machine has not previously had a chance to observe the input and see what its current heading is, so it computes the desired heading (by adding the desired change to the current heading, and then calling a utility procedure to be sure the resulting angle is between plus and minus π), and returns it and the current heading. Otherwise, we keep the `thetaDesired` component of the state, and just get a new value of `theta` out of the input. We generate an action with a

rotational velocity that will rotate toward the desired heading with velocity proportional to the magnitude of the angular error.

```
def getNextValues(self, state, inp):
    currentTheta = inp.odometry.theta
    if state == 'start':
        thetaDesired = \
            util.fixAnglePlusMinusPi(currentTheta + self.headingDelta)
    else:
        (thetaDesired, thetaLast) = state
    newState = (thetaDesired, currentTheta)
    action = io.Action(rvel = self.rotationalGain * \
        util.fixAnglePlusMinusPi(thetaDesired - currentTheta))

    return (newState, action)
```

Finally, we have to say which states are done. Clearly, the 'start' state is not done; but we are done if the most recent theta from the odometry is within some tolerance, `self.angleEpsilon`, of the desired heading.

```
def done(self, state):
    if state == 'start':
        return False
    else:
        (thetaDesired, thetaLast) = state
        return util.nearAngle(thetaDesired, thetaLast, self.angleEpsilon)
```

Exercise 4.14. Change this machine so that it rotates *through* an angle, so you could give it 2 pi or minus 2 pi to have it rotate all the way around.

4.4.2 Forward

Moving the robot forward a fixed distance is similar. In this case, we remember the robot's x and y coordinates when it starts, and drive straight forward until the distance between the initial position and the current position is close to the desired distance. The state of the machine is the robot's starting position and its current position.

```
class ForwardTSM (SM):
    forwardGain = 1.0
    distTargetEpsilon = 0.01
    startState = 'start'

    def __init__(self, delta):
        self.deltaDesired = delta

    def getNextValues(self, state, inp):
        currentPos = inp.odometry.point()
        if state == 'start':
            print "Starting forward", self.deltaDesired
```

```

        startPos = currentPos
    else:
        (startPos, lastPos) = state
    newState = (startPos, currentPos)
    error = self.deltaDesired - startPos.distance(currentPos)
    action = io.Action(fvel = self.forwardGain * error)
    return (newState, action)

def done(self, state):
    if state == 'start':
        return False
    else:
        (startPos, lastPos) = state
        return util.within(startPos.distance(lastPos),
                           self.deltaDesired,
                           self.distTargetEpsilon)

```

4.4.3 Square Spiral

Imagine we would like to have the robot drive in a square spiral, similar to the one shown in [figure 4.11](#). One way to approach this problem is to make a “low-level” machine that can consume a goal point and the sensor input and drive (in the absence of obstacles) to the goal point; and then make a “high-level” machine that will keep track of where we are in the figure and feed goal points to the low-level machine.

4.4.3.1 XYDriver

Here is a class that describes a machine that takes as input a series of pairs of goal points (expressed in the robot’s odometry frame) and sensor input structures. It generates as output a series of actions. This machine is very nearly a pure function machine, which has the following basic control structure:

- If the robot is headed toward the goal point, move forward.
- If it is not headed toward the goal point, rotate toward the goal point.

This decision is made on every step, and results in a robust ability to drive toward a point in two-dimensional space.

For many uses, this machine does not need any state. But the modularity is nicer, in some cases, if it has a meaningful done method, which depends only on the state. So, we will let the state of this machine be whether it is done or not. It needs several constants to govern rotational and forward speeds, and tolerances for deciding whether it is pointed close enough toward the target and whether it has arrived close enough to the target.

```

class XYDriver(SM):
    forwardGain = 2.0
    rotationGain = 2.0
    angleEps = 0.05
    distEps = 0.02
    startState = False

```

The getNextValues method embodies the control structure described above.

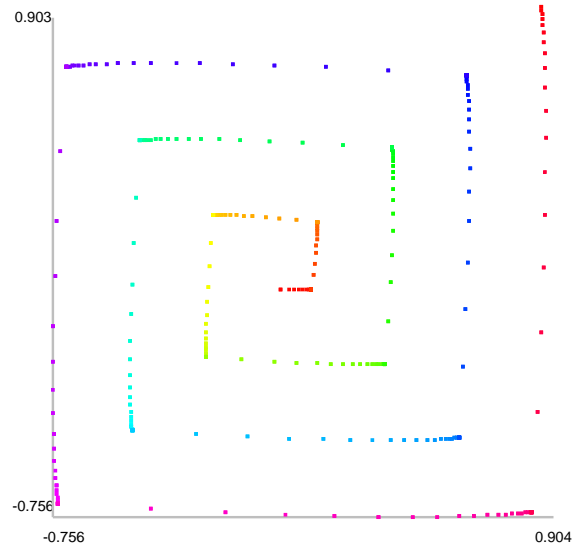


Figure 4.11 Square spiral path of the robot using the methods in this section.

```

def getNextValues(self, state, inp):
    (goalPoint, sensors) = inp
    robotPose = sensors.odometry
    robotPoint = robotPose.point()
    robotTheta = robotPose.theta

    if goalPoint == None:
        return (True, io.Action())

    headingTheta = robotPoint.angleTo(goalPoint)
    if util.nearAngle(robotTheta, headingTheta, self.angleEps):
        # Pointing in the right direction, so move forward
        r = robotPoint.distance(goalPoint)
        if r < self.distEps:
            # We're there
            return (True, io.Action())
        else:
            return (False, io.Action(fvel = r * self.forwardGain))
    else:
        # Rotate to point toward goal
        headingError = util.fixAnglePlusMinusPi(\

```

```

        headingTheta - robotTheta)
    return (False, io.Action(rvel = headingError * self.rotationGain))

```

The state of the machine is just a boolean indicating whether we are done.

```

def done(self, state):
    return state

```

4.4.3.2 Cascade approach

We make a spiral by building a machine that takes `SensorInput` objects as input and generates pairs of subgoals and sensorinputs; such a machine can be cascaded with `XYDriver` to generate a spiral.

Our implementation of this is a class called `SpyroGyra`. It takes the increment (amount that each new side is larger than the previous) at initialization time. Its state consists of three components:

- `direction`: one of 'north', 'south', 'east', or 'west', indicating which way the robot is traveling
- `length`: length in meters of the current line segment being followed
- `subGoal`: the point in the robot's odometry frame that defines the end of the current line segment

It requires a tolerance to decide when the current subgoal point has been reached.

```

class SpyroGyra(SM):
    distEps = 0.02
    def __init__(self, incr):
        self.incr = incr
        self.startState = ('south', 0, None)

```

If the robot is close enough to the subgoal point, then it is time to change the state. We increment the side length, pick the next direction (counter clockwise around the cardinal compass directions), and compute the next subgoal point. The output is just the subgoal and the sensor input, which is what the driver needs.

```

def getNextValues(self, state, inp):
    (direction, length, subGoal) = state
    robotPose = inp.odometry
    robotPoint = robotPose.point()
    if subGoal == None:
        subGoal = robotPoint
    if robotPoint.isNear(subGoal, self.distEps):
        # Time to change state
        length = length + self.incr
        if direction == 'east':
            direction = 'north'
            subGoal.y += length
        elif direction == 'north':
            direction = 'west'
            subGoal.x -= length
        elif direction == 'west':
            direction = 'south'
            subGoal.y -= length

```

```
    else: # south
        direction = 'east'
        subGoal.x += length
    print 'new:', direction, length, subGoal
    return ((direction, length, subGoal),
            (subGoal, inp))
```

Finally, to make the spiral, we just cascade these two machines together.

```
def spiroFlow(incr):
    return sm.Cascade(SpyroGyra(incr), XYDriver())
```

Exercise 4.15. What explains the rounded sides of the path in [figure 4.11](#)?

4.5 Conclusion

State machines

State machines are such a general formalism, that a huge class of discrete-time systems can be described as state machines. The system of defining primitive machines and combinations gives us one discipline for describing complex systems. It will turn out that there are some systems that are conveniently defined using this discipline, but that for other kinds of systems, other disciplines would be more natural. As you encounter complex engineering problems, your job is to find the PCAP system that is appropriate for them, and if one does not exist already, invent one.

State machines are such a general class of systems that although it is a useful framework for implementing systems, we cannot generally analyze the behavior of state machines. That is, we can't make much in the way of generic predictions about their future behavior, except by running them to see what will happen.

In the next module, we will look at a restricted class of state machines, whose state is representable as a bounded history of their previous states and previous inputs, and whose output is a linear function of those states and inputs. This is a *much* smaller class of systems than all state machines, but it is nonetheless very powerful. The important lesson will be that restricting the form of the models we are using will allow us to make stronger claims about their behavior.

Knuth on Elevator Controllers

Donald E. Knuth is a computer scientist who is famous for, among other things, his series of textbooks (as well as for $T_{E}X$, the typesetting system we use to make all of our handouts), and a variety of other contributions to theoretical computer science.

“It is perhaps significant to note that although the author had used the elevator system for years and thought he knew it well, it wasn't until he attempted to write this section that he realized there were quite a few facts about the elevator's system of choosing directions that he did not know. He went back to experiment with the elevator six separate times, each time believing he

had finally achieved a complete understanding of its *modus operandi*. (Now he is reluctant to ride it for fear some new facet of its operation will appear, contradicting the algorithms given.) We often fail to realize how little we know about a thing until we attempt to simulate it on a computer.”

The Art of Computer Programming, Donald E., Knuth, Vol 1. page 295. On the elevator system in the Mathematics Building at Cal Tech. First published in 1968

4.6 Examples

4.6.1 Practice problem: Things

Consider the following program

```
def thing(inputList):
    output = []
    i = 0
    for x in range(3):
        y = 0
        while y < 100 and i < len(inputList):
            y = y + inputList[i]
            output.append(y)
            i = i + 1
    return output
```

A. What is the value of

```
thing([1, 2, 3, 100, 4, 9, 500, 51, -2, 57, 103, 1, 1, 1, 1, -10, 207, 3, 1])
```

```
[1, 3, 6, 106, 4, 13, 513, 51, 49, 106]
```

It’s important to understand the loop structure of the Python program: It goes through (at most) three times, and adds up the elements of the input list, generating a partial sum as output on each step, and terminating the inner loop when the sum becomes greater than 100.

B. Write a single state machine class `MySM` such that `MySM().transduce(inputList)` gives the same result as `thing(inputList)`, if `inputList` is a list of numbers. Remember to include a `done` method, that will cause it to terminate at the same time as `thing`.

```

class MySM(sm.SM):
    startState = (0,0)
    def getNextValues(self, state, inp):
        (x, y) = state
        y += inp
        if y >= 100:
            return ((x + 1, 0), y)
        return ((x, y), y)
    def done(self, state):
        (x, y) = state
        return x >= 3

```

The most important step, conceptually, is deciding what the state of the machine will be. Looking at the original Python program, we can see that we had to keep track of how many times we had completed the outer loop, and then what the current partial sum was of the inner loop.

The `getNextValues` method first increments the partial sum by the input value, and then checks to see whether it's time to reset. If so, it increments the 'loop counter' (`x`) component of the state and resets the partial sum to 0. It's important to remember that the output of the `getNextValues` method is a pair, containing the next state and the output.

The `done` method just checks to see whether we have finished three whole iterations.

- C. Recall the definition of `sm.Repeat(m, n)`: Given a terminating state machine `m`, it returns a new terminating state machine that will execute the machine `m` to completion `n` times, and then terminate.

Use `sm.Repeat` and a very simple state machine that you define to create a new state machine `MyNewSM`, such that `MyNewSM` is equivalent to an instance of `MySM`.

```

class Sum(sm.SM):
    startState = 0
    def getNextValues(self, state, inp):
        return (state + inp, state + inp)
    def done(self, state):
        return state > 100

myNewSM = sm.Repeat(Sum(), 3)

```

4.6.2 Practice problem: Inheritance and State Machines

Recall that we have defined a Python class `sm.SM` to represent state machines. Here we consider a special type of state machine, whose states are always integers that start at 0 and increment by 1 on each transition. We can represent this new type of state machines as a Python subclass of `sm.SM` called `CountingStateMachine`.

We wish to use the `CountingStateMachine` class to define new subclasses that each provide a single new method `getOutput(self, state, inp)` which returns *just the output* for that state

and input; the `CountingStateMachine` will take care of managing and incrementing the state, so its subclasses don't have to worry about it.

Here is an example of a subclass of `CountingStateMachine`.

```
class CountMod5(CountingStateMachine):
    def getOutput(self, state, inp):
        return state % 5
```

Instances of `CountMod5` generate output sequences of the form 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0, ...

Part a. Define the `CountingStateMachine` class. Since `CountingStateMachine` is a subclass of `sm.SM`, you will have to provide definitions of the `startState` instance variable and `getNextValues` method, just as we have done for other state machines. You can assume that every subclass of `CountingStateMachine` will provide an appropriate `getOutput` method.

```
class CountingStateMachine(sm.SM):
    def __init__(self):
        self.startState = 0
    def getNextState(self, state, inp):
        return(state + 1, self.getOutput(state, inp))
```

Part b. Define a subclass of `CountingStateMachine` called `AlternateZeros`. Instances of `AlternateZeros` should be state machines for which, on even steps, the output is the same as the input, and on odd steps, the output is 0. That is, given inputs, $i_0, i_1, i_2, i_3, \dots$, they generate outputs, $i_0, 0, i_2, 0, \dots$

```
class AlternateZeros(CountingStateMachine):
    def getOutput(self, state, inp):
        if not state % 2:
            return inp
        return 0
```

Chapter 5

Signals and Systems

Imagine that you are asked to design a system to steer a car straight down the middle of a lane. It seems easy, right? You can figure out some way to sense the position of the car within its lane. Then, if the car is right of center, turn the steering wheel to the left. As the car moves so that it is less to the right, turn the steering wheel less to the left. If it is left of center, turn the steering wheel to the right. This sort of *proportional controller* works well for many applications – but not for steering, as can be seen below.

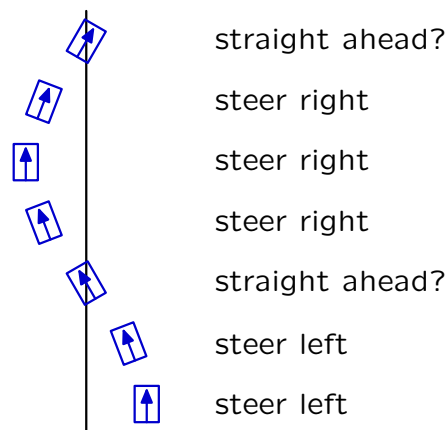


Figure 5.1 Simple (but poor) algorithm for steering: steer to the left in proportion to how far the car is to the right, and vice versa.

It is relatively easy to describe better algorithms in terms that humans would understand: e.g., *Stop turning back and forth!* It is not so easy to specify exactly what one might mean by that, in a way that it could be automated.

In this chapter, we will develop a *Signals and Systems* framework to facilitate reasoning about the dynamic behaviors of systems. This framework will enable construction of simple mathematical models that are useful in both analysis and design of a wide range of systems, including the car-steering system.

5.1 The signals and systems abstraction

To think about dynamic behaviors of systems, we need to think not only about how to describe the system but also about how to describe the *signals* that characterize the inputs and outputs of the system, as illustrated below.

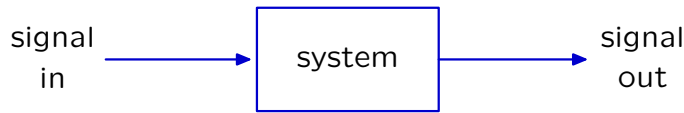


Figure 5.2 Signals and Systems: the system transforms an input signal into an output signal.

This diagram represents a system with one input and one output. Both the input and output are *signals*. A signal is a mathematical function with an independent variable (most often it will be *time* for the problems that we will study) and a dependent variable (that depends on the independent variable). The *system* is described by the way that it transforms the input signal into the output signal. In the simplest case, we might imagine that the input signal is the time sequence of steering-wheel angles (assuming constant speed) and that the output signal is the time sequence of distances between the center of the car and the midline of the lane.

Representing a system with a single input signal and a single output signal seems too simplistic for any real application. For example, the car in the steering example ([figure 5.1](#)) surely has more than one possible output signal.

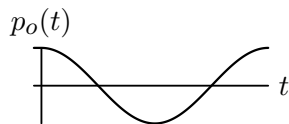
Exercise 5.1. List at least four possible output signals for the car-steering problem.

Possible output signals include

- its three-dimensional position (which could be represented by a 3D vector $\hat{p}(t)$ or by three scalar functions of time),
- its angular position,
- the rotational speeds of the wheels,
- the temperature of the tires, and many other possibilities.

The important point is that the first step in using the signals and systems representation is *abstraction*: we must choose the output(s) that are most relevant to the problem at hand and abstract away the rest.

To understand the steering of a car, one vital output signal is the lateral position $p_o(t)$ within the lane, where $p_o(t)$ represents the distance (in meters) from the center of the lane. That signal alone tells us a great deal about how well we are steering. Consider a plot of $p_o(t)$ that corresponds to [figure 5.1](#), as follows.



The oscillations in $p_o(t)$ as a function of time correspond to the oscillations of the car within its lane. Thus, this signal clearly represents an important failure mode of our car steering system.

Is $p_o(t)$ the only important output signal from the car-steering system? The answer to this question depends on your goals. Analyzing a system with this single output is likely to give important insights into some systems (e.g., low-speed robotic steering) but not others (e.g., NASCAR). More

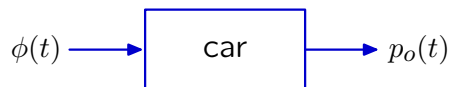
complicated applications may require more complicated models. But all useful models focus on the most relevant signals and ignore those of lesser significance.³⁴

Throughout this chapter, we will focus on systems with one input signal and one output signal (as illustrated [figure 5.2](#)). When multiple output signals are important for understanding a problem, we will find that it is possible to generalize the methods and results developed here for single-input and single-output systems to systems with multiple inputs and outputs.

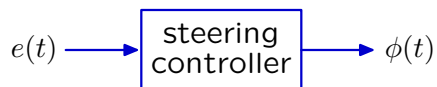
The signals and systems approach has very broad applicability: it can be applied to mechanical systems (such as mass-spring systems), electrical systems (such as circuits and radio transmissions), financial systems (such as markets), and biological systems (such as insulin regulation or population dynamics). The fundamental notion of signals applies no matter what physical substrate supports them: it could be sound or electromagnetic waves or light or water or currency value or blood sugar levels.

5.1.1 Modularity, primitives, and composition

The car-steering system can be analyzed by thinking of it as the combination of car and steering sub-systems. The input to the car is the angle of the steering wheel. Let's call that angle $\phi(t)$. The output of the car is its position in the lane, $p_o(t)$, measured as the lateral distance to the center of the lane.



The steering controller turns the steering wheel to compensate for differences between our desired position in the lane, $p_i(t)$ (which is zero since we would like to be in the center of the lane), and our actual position in the lane $p_o(t)$. Let $e(t) = p_i(t) - p_o(t)$. Thus we can think about the steering controller as having an input $e(t)$ and output $\phi(t)$.



In the composite system (in [figure 5.3](#)), the steering controller determines $\phi(t)$, which is the input to the car. The car generates $p_o(t)$, which is subtracted from $p_i(t)$ to get $e(t)$ (which is the input to the steering controller). The triangular component is called a *gain* or *scale* of -1 : its output is equal to -1 times its input. More generally, we will use a triangle symbol to indicate that we are multiplying all the values of the signal by a numerical constant, which is shown inside the triangle.

The dashed-red box in [figure 5.3](#) illustrates *modularity* of the signals and systems abstraction. Three single-input, single-output sub-systems (steering controller, car, and inverter) and an adder (two inputs and 1 output) are combined to generate a new single-input ($p_i(t)$), single-output ($p_o(t)$) system. By abstraction, we could treat this new system as a primitive (represented by a single-input single-output box) and combine it with other subsystems to create a new, more

³⁴ There are always unimportant outputs. Think about the number of moving parts in a car. They are not all important for steering!

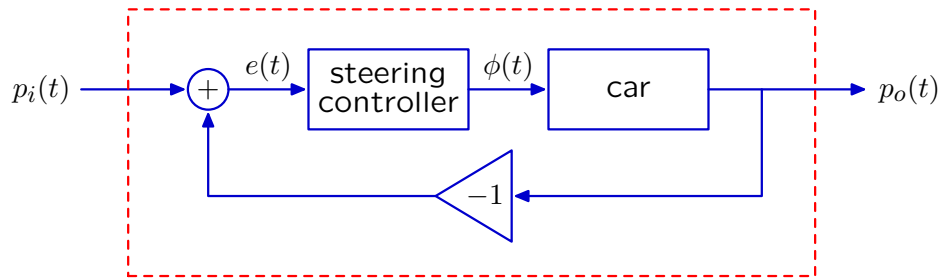


Figure 5.3 Modularity of systems

complex, system. A principal goal of this chapter is to develop methods of analysis for the sub-systems that can be *combined* to analyze the overall system.

5.1.2 Discrete-time signals and systems

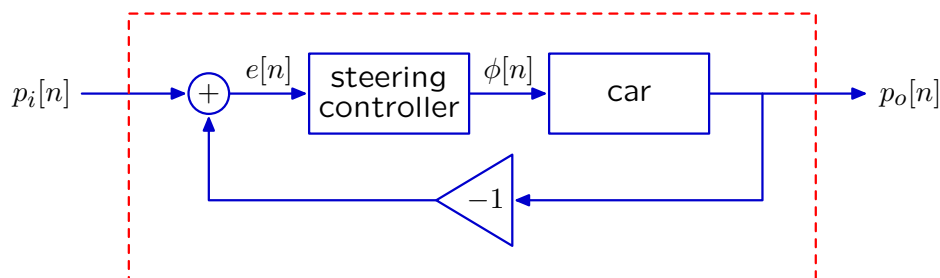
This chapter focuses on signals whose independent variables are discrete (e.g., take on only integer values). Some such signals are found in nature. For example, the primary structure of DNA is described by a *sequence* of base-pairs. However, we are primarily interested in discrete-time signals, not so much because they are found in nature, but because they are found in computers. Even though we focus on interactions with the real world, these interactions will primarily occur at discrete instants of time. For example, the difference between our desired position $p_i(t)$ and our actual position $p_o(t)$ is an error signal $e(t)$, which is a function of continuous time t . If the controller only observes this signal at regular sampling intervals T , then its input could be regarded as a sequence of values $x[n]$ that is indexed by the integer n . The relation between the discrete-time sequence $x[n]$ (note square brackets) and the continuous signal $x(t)$ (note round brackets) is given by

$$x[n] = x(nT) ,$$

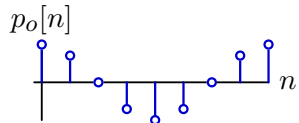
which we call the *sampling* relation. *Sampling* converts a signal of continuous domain to one of discrete domain.

While our primary focus will be on time signals, sampling works equally well in other domains. For example, images are typically represented as arrays of pixels accessed by integer-valued rows and columns, rather than as continuous brightness fields, indexed by real-valued spatial coordinates.

If the car-steering problem in [figure 5.1](#) were modeled in discrete time, we could describe the system with a diagram that is very similar to the continuous-time diagram in [figure 5.3](#). However, only discrete time instants are considered



and the output position is now only defined at discrete times, as shown below.



5.1.3 Linear time-invariant systems

We already have a great way of specifying systems that operate on discrete-time signals: a state machine transduces a discrete-time input signal into a discrete-time output signal. State machines, as we have defined them, allow us to specify *any* discrete-time system whose output is computable from its history of previous inputs.

The representation of systems as state machines allows us to execute a machine on any input we'd like, in order to see what happens. Execution lets us examine the behavior of the system for any particular input for any particular finite amount of time, but it does not let us characterize any general properties of the system or its long-term behavior. Computer programs are such a powerful specification language that we cannot, in general, predict what a program will do (or even whether it will ever stop and return a value) without running it. In the rest of this chapter, we will concentrate on a small but powerful subclass of the whole class of state machines, called discrete-time *linear time-invariant (LTI) systems*, which will allow deeper forms of analysis.

In an LTI system:

- Inputs and outputs are real numbers;
- The state is some fixed number of previous inputs to the system as well as a fixed number of previous outputs of the system; and
- The output is a fixed, linear function of the current input and any of the elements of the state.

In general, each input could be a fixed-length vector of numbers, and each output could also be a fixed-length vector of numbers; we will restrict our attention to the case where the input is a single real number and the output is a single real number.

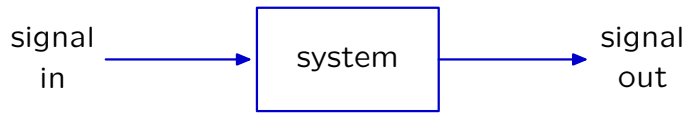
We are particularly interested in LTI systems because they can be analyzed mathematically, in a way that lets us characterize some properties of their output signal for *any* possible input signal. This is a much more powerful kind of insight than can be gained by trying a machine out with several different inputs.

Another important property of LTI systems is that they are compositional: the cascade, parallel, and feedback combinations (introduced in section 4.2) of LTI systems are themselves LTI systems.

5.2 Discrete-time signals

In this section, we will work through the PCAP system for discrete time signals, by introducing a primitive and three methods of composition, and the ability to abstract by treating composite signals as if they themselves were primitive.

A *signal* is an infinite sequence of *sample* values at discrete time steps. We will use the following common notational conventions: A capital letter X stands for the whole input signal and $x[n]$ stands for the value of signal X at time step n . It is conventional, if there is a single system under discussion, to use X for the input signal to that system and Y for the output signal.



We will say that systems *transduce* input signals into output signals.

5.2.1 Unit sample signal

We will work with a single primitive, called the *unit sample signal*, Δ . It is defined on all positive and negative integer indices as follows³⁵:

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} .$$

That is, it has value 1 at index $n = 0$ and 0 otherwise, as shown below:

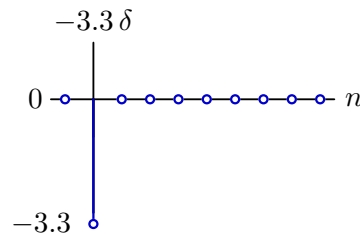
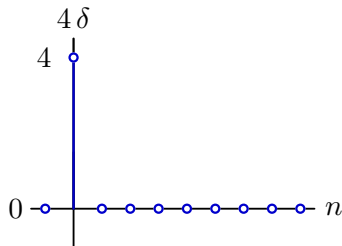


5.2.2 Signal combinatorics

Our first operation will be *scaling*, or *multiplication by a scalar*. A scalar is any real number. The result of multiplying any signal X by a scalar c is a signal, so that,

$$\text{if } Y = c \cdot X \text{ then } y[n] = c \cdot x[n] .$$

That is, the resulting signal has a value at every index n that is c times the value of the original signal at that location. Here are the signals 4Δ and -3.3Δ .

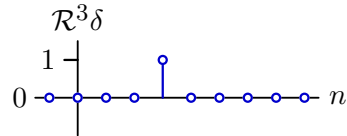
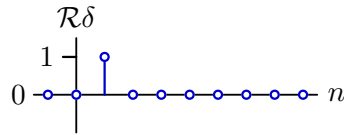


The next operation is the *delay* operation. The result of delaying a signal X is a new signal $\mathcal{R}X$ such that:

$$\text{if } Y = \mathcal{R}X \text{ then } y[n] = x[n - 1] .$$

That is, the resulting signal has the same values as the original signal, but delayed by one step in time. You can also think of this, graphically, as shifting the signal one step to the \mathcal{R} ight. Here is the unit sample delayed by 1 and by 3 steps. We can describe the second signal as $\mathcal{R}\mathcal{R}\mathcal{R}\Delta$, or, using shorthand, as $\mathcal{R}^3\Delta$.

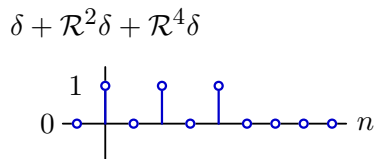
³⁵ Note that δ is the lowercase version of Δ , both of which are the Greek letter 'delta'.



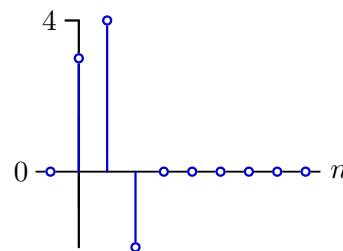
Finally, we can *add* two signals together. Addition of signals is accomplished component-wise, so that

$$\text{if } Y = X_1 + X_2 \text{ then } y[n] = x_1[n] + x_2[n] .$$

That is, the value of the composite signal at step n is the sum of the values of the component signals. Here are some new signals constructed by summing, scaling, and delaying the unit sample.

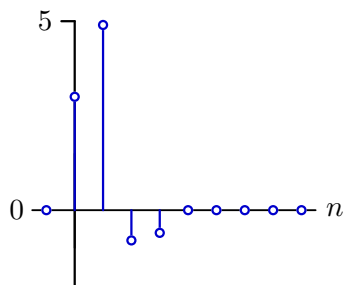


$$3\delta + 4R\delta - 2R^2\delta$$



Note that, because each of our operations returns a signal, we can use their results again as elements in new combinations, showing that our system has true compositionality. In addition, we can abstract, by naming signals. So, for example, we might define $Y = 3\Delta + 4R\Delta - 2R^2\Delta$, and then make a new signal $Z = Y + 0.3RY$, which would look like this:

$$Z = Y + 0.3RY$$



Be sure you understand how the heights of the spikes are determined by the definition of Z .

Exercise 5.2. Draw a picture of samples -1 through 4 of $Y - RY$.

It is important to remember that, because signals are infinite objects, these combination operations are abstract mathematical operations. You could never somehow 'make' a new signal by calculating its value at every index. It is possible, however, to calculate the value at any particular index, as it is required.

Advancing

If we allow ourselves one more operation, that of ‘advancing’ the signal one step (just like delaying, but in the other direction, written \mathcal{L} for left-shift), then *any* signal can be composed from the unit sample, using a (possibly infinite) number of these operations. We can demonstrate this claim by construction: to define a signal V with value v_n at index n , for any set of integer n , we simply set

$$V = v_0\Delta + \sum_{n=1}^{\infty} v_n\mathcal{R}^n\Delta + \sum_{n=1}^{\infty} v_{-n}\mathcal{L}^n\Delta ,$$

where \mathcal{R}^n and \mathcal{L}^n are shorthand for applying \mathcal{R} and \mathcal{L} , respectively, n times.

If n represents time, then physical systems are always *causal*: inputs that arrive after time n_0 cannot affect the output before time n_0 . Such systems cannot advance signals: they can be written without \mathcal{L} .

5.2.3 Algebraic properties of operations on signals

Adding and scaling satisfy the familiar algebraic properties of addition and multiplication: addition is commutative and associative, scaling is commutative (in the sense that it doesn’t matter whether we pre- or post-multiply) and scaling distributes over addition:

$$c \cdot (X_1 + X_2) = c \cdot X_1 + c \cdot X_2 ,$$

which can be verified by defining $Y = c \cdot (X_1 + X_2)$ and $Z = c \cdot X_1 + c \cdot X_2$ and checking that $y[n] = z[n]$ for all n :

$$y[n] = z[n]$$

$$c \cdot (x_1[n] + x_2[n]) = (c \cdot x_1[n]) + (c \cdot x_2[n])$$

which clearly holds based on algebraic properties of arithmetic on real numbers.

In addition, \mathcal{R} distributes over addition and scaling, so that:

$$\mathcal{R}(X_1 + X_2) = \mathcal{R}X_1 + \mathcal{R}X_2$$

$$\mathcal{R}(c \cdot X) = c \cdot \mathcal{R}X .$$

Exercise 5.3. Verify that \mathcal{R} distributes over addition and multiplication by checking that the appropriate relations hold at some arbitrary step n .

These algebraic relationships mean that we can take any finite expression involving Δ , \mathcal{R} , $+$ and \cdot and convert it into the form

$$(a_0 + a_1\mathcal{R}^1 + a_2\mathcal{R}^2 + \dots + a_N\mathcal{R}^N)\Delta .$$

That is, we can express the entire signal as a *polynomial in \mathcal{R}* , applied to the unit sample.

In our previous example, it means that we can rewrite $3\Delta + 4\mathcal{R}\Delta - 2\mathcal{R}^2\Delta$ as $(3 + 4\mathcal{R} - 2\mathcal{R}^2)\Delta$.

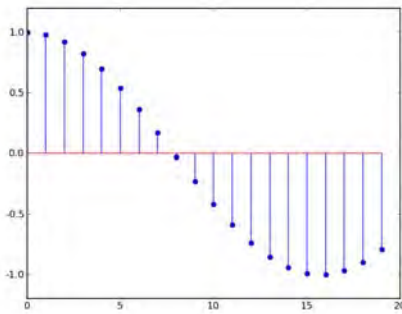
5.2.4 Sinusoidal primitives

We just saw how to construct complicated signals by summing unit sample signals that are appropriately scaled and shifted. We could similarly start with a family of discretely-sampled sinusoids as our primitives, where

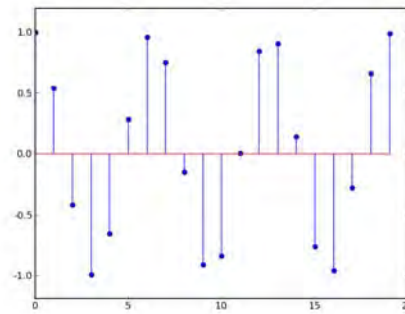


$$x[n] = \cos(\Omega n) .$$

Here are plots of two primitives in this family:



$\cos(0.2n)$



$\cos(1.0n)$

The second plot may seem confusing, but it is just a sparsely sampled sinusoid. Note that signals constructed from even a single sinusoid have non-zero values defined at an infinity of steps; this is in contrast to signals constructed from a finite sum of scaled and shifted unit samples.

Exercise 5.4. If $x[n] = \cos(0.2n)$, what would be the values of $\mathcal{R}X$ at steps -3 and 5 ?

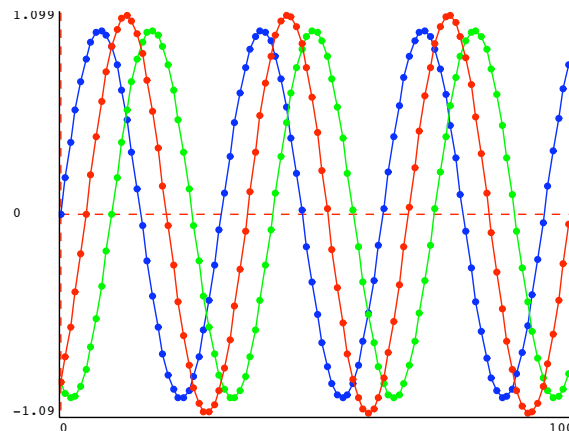
Here are two sinusoids and their sum, made as follows:

$$s_1[n] = \cos(0.2n - \pi/2)$$

$$S_2 = \mathcal{R}^{10}S_1$$

$$S_3 = S_1 + S_2$$

The blue line is the S_1 , the green line is the same signal, delayed by 10, which is S_2 , and the red line is their sum.



5.3 Feedforward systems

We will start by looking at a subclass of discrete-time LTI system, which are exactly those that can be described as performing some combination of scaling, delay, and addition operations on the input signal. We will develop several ways of representing such systems, and see how to combine them to get more complex systems in this same class.

5.3.1 Representing systems

We can represent systems using operator equations, difference equations, block diagrams, and Python state machines. Each makes some things clearer and some operations easier. It is important to understand how to convert between the different representations.

Operator equation

An operator equation is a description of how signals are related to one another, using the operations of scaling, delay, and addition on whole signals.

Consider a system that has an input signal X , and whose output signal is $X - \mathcal{R}X$. We can describe that system using the operator equation

$$Y = X - \mathcal{R}X .$$

Using the algebraic properties of operators on signals described in section 5.2.3, we can rewrite this as

$$Y = (1 - \mathcal{R})X ,$$

which clearly expresses a relationship between input signal X and output signal Y , whatever X may be.

Feedforward systems can always be described using an operator equation of the form

$$Y = \Phi X ,$$

where Φ is a polynomial in \mathcal{R} .

Difference Equation

An alternative representation of the relationship between signals is a *difference equation*. A difference equation describes a relationship that holds among samples (values at particular times) of signals. We use an index n in the difference equation to refer to a particular time index, but the specification of the corresponding system is that the difference equation hold for *all* values of n .

The operator equation

$$Y = X - \mathcal{R}X .$$

can be expressed as this equivalent difference equation:

$$y[n] = x[n] - x[n - 1] .$$

The operation of delaying a signal can be seen here as referring to a sample of that signal at time step $n - 1$.

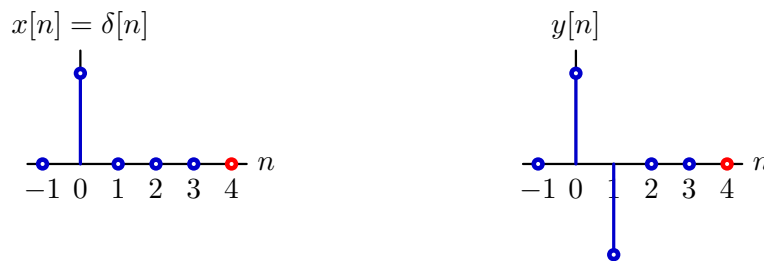
Difference equations are convenient for step-by-step analysis, letting us compute the value of an output signal at any time step, given the values of the input signal.

So, if the input signal X is the unit sample signal,

$$x[n] = \delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} .$$

then using a difference equation, we can compute individual values of the output signal Y :

$$\begin{aligned} y[n] &= x[n] - x[n - 1] \\ y[0] &= x[0] - x[-1] = 1 - 0 = 1 \\ y[1] &= x[1] - x[0] = 0 - 1 = -1 \\ y[2] &= x[2] - x[1] = 0 - 0 = 0 \\ y[3] &= x[3] - x[2] = 0 - 0 = 0 \\ &\dots \end{aligned}$$



Block diagrams

Another way of describing a system is by drawing a *block diagram*, which is made up of components, connected by lines with arrows on them. The lines represent signals; all lines that are connected to one another (not going through a round, triangular, or circular component) represent the same signal.

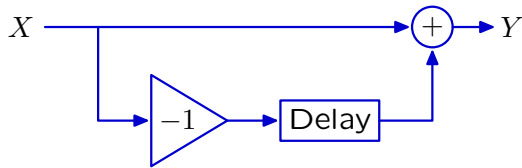
The components represent systems. There are three primitive components corresponding to our operations on signals:

- *Delay* components are drawn as rectangles, labeled Delay, with two lines connected to them, one with an arrow coming in and one going out. If X is the signal on the line coming into the delay, then the signal coming out is $\mathcal{R}X$.
- *Scale* (or *gain*) components are drawn as triangles, labeled with a positive or negative number c , with two lines connected to them, one with an arrow coming in and one going out. If X is the signal on the line coming into the gain component, then the signal coming out is $c \cdot X$.
- *Adder* components are drawn as circles, labeled with $+$, three lines connected to them, two with arrows coming in and one going out. If X_1 and X_2 are the signals on the lines point into the adder, then the signal coming out is $X_1 + X_2$.

The system

$$Y = X - \mathcal{R}X$$

can be represented with this block diagram.



State machines

Of course, since feedforward LTI systems are a type of state machine, we can make an equivalent definition using our Python state-machine specification language. So, our system

$$Y = X - \mathcal{R}X$$

can be specified in Python as a state machine by:

```
class Diff(sm.SM):
    def __init__(self, previousInput):
        self.startState = previousInput
    def getNextValues(self, state, inp):
        return (inp, inp-state)
```

Here, the state is the value of the previous input. One important thing to notice is that, since we have to be able to run a state machine and generate outputs, it has to start with a value for its internal state, which is the input signal's value at time -1 . If we were to run:

```
Diff(0).transduce([1, 0, 0, 0])
```

we would get the result

```
[1, -1, 0, 0]
```

This same state machine can also be expressed as a combination of primitive state machines (as defined in sections 4.1.2 and 4.2).

```
diff = sm.ParallelAdd(sm.Wire(),
                    sm.Cascade(sm.Gain(-1), sm.R(0)))
```

Note that `sm.R` is another name for `sm.Delay` and that the desired initial output value for the system appears as the initialization argument to the `sm.R` machine.

5.3.2 Combinations of systems

To combine LTI systems, we will use the same cascade and parallel-add operations as we had for state machines.

Cascade multiplication

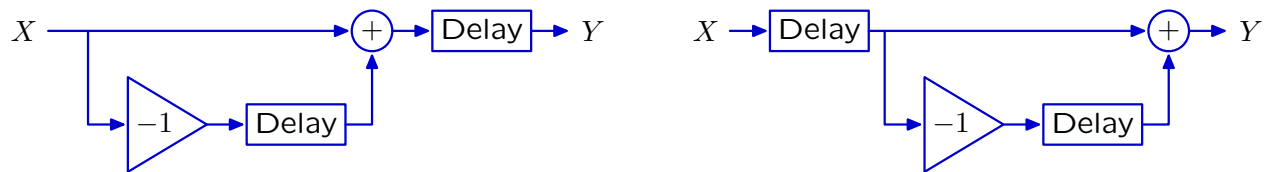
When we make a cascade combination of two systems, we let the output of one system be the input of another. So, if the system M_1 has operator equation $Y = \Phi_1 X$ and system M_2 has operator equation $Z = \Phi_2 W$, and then we compose M_1 and M_2 in cascade by setting $Y = W$, then we have a new system, with input signal X , output signal Z , and operator equation $Z = (\Phi_2 \cdot \Phi_1)X$.

The product of polynomials is another polynomial, so $\Phi_2 \cdot \Phi_1$ is a polynomial in \mathcal{R} . Furthermore, because polynomial multiplication is commutative, cascade combination is commutative as well (as long as the systems are *at rest*, which means that their initial states are 0).

So, for example,

$$\mathcal{R}(1 - \mathcal{R})X = (1 - \mathcal{R})\mathcal{R}X$$

and these two corresponding block diagrams are equivalent (the algebraic equivalence justifies the diagram equivalence):



Cascade combination, because it results in multiplication, is also associative, which means that any grouping of cascade operations on systems has the same result.

Exercise 5.5.

Remembering that the condition on commutativity of cascading is that the systems start at rest, explain why machines `m3` and `m4` do not generate the same output sequence in response to the unit sample signal as input.

```
m1 = sm.ParallelAdd(sm.Wire(), sm.Cascade(sm.Gain(-1), sm.R(2)))
m2 = sm.R(3)
m3 = sm.Cascade(m1, m2)
m4 = sm.Cascade(m2, m1)
```

Parallel addition

When we make a parallel addition combination of two systems, the output signal is the sum of the output signals that would have resulted from the individual systems. So, if the system M_1 has system function $Y = \Phi_1 X$ and system M_2 has system function $Z = \Phi_2 X$, and then we compose M_1 and M_2 with parallel addition by setting output $W = Y + Z$, then we have a new system, with input signal X , output signal W , and operator equation $W = (\Phi_1 + \Phi_2)X$.

Because addition of polynomials is associative and commutative, then so is parallel addition of feed-forward linear systems.

Combining cascade and parallel operations

Finally, the distributive law applies for cascade and parallel combination, *for systems at rest*, in the same way that it applies for multiplication and addition of polynomials, so that if we have three systems, with operator equations:

$$Y = \Phi_1 X$$

$$U = \Phi_2 V$$

$$W = \Phi_3 Z ,$$

and we form a cascade combination of the sum of the first two, with the third, then we have a system describable as:

$$B = (\Phi_3 \cdot (\Phi_1 + \Phi_2))A .$$

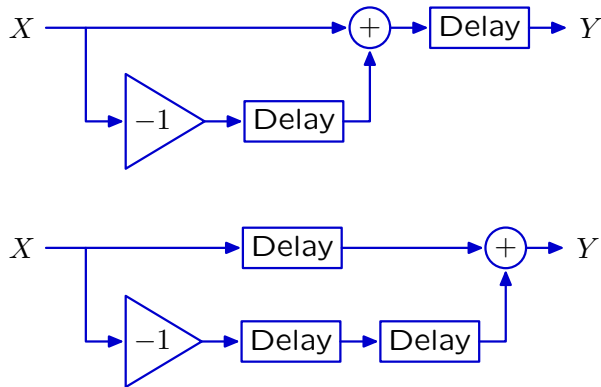
We can rewrite this, using the distributive law, as:

$$B = ((\Phi_3 \cdot \Phi_1) + (\Phi_3 \cdot \Phi_2))A .$$

So, for example,

$$\mathcal{R}(1 - \mathcal{R}) = \mathcal{R} - \mathcal{R}^2 ,$$

and these two corresponding block diagrams are equivalent:



Exercise 5.6. The first machine in the diagram above can be described, for certain initial output values as:

```
m1 = sm.Cascade(sm.ParallelAdd(sm.Wire(),
                                sm.Cascade(sm.Gain(-1), Delay(2))),
                sm.Delay(3))
```

The second machine can be described as:

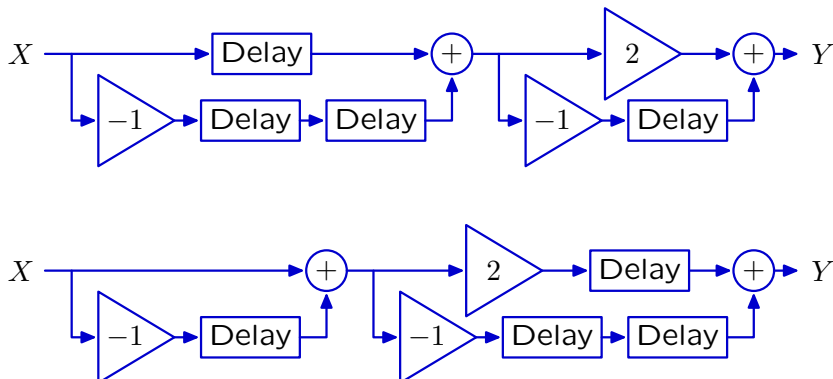
```
m2 = sm.ParallelAdd(sm.Delay(d1),
                    sm.Cascade(sm.Cascade(sm.Gain(-1),
                                           sm.Cascade(sm.Delay(d2), sm.Delay(d3))))))
```

Provide values of $d1$, $d2$, and $d3$ that will cause $m2$ to generate the same output sequence as $m1$ in response to the unit sample signal as input.

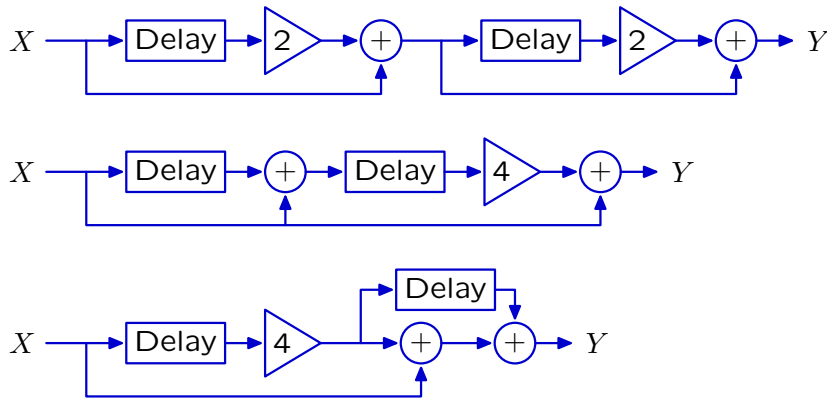
Here is another example of two equivalent operator equations

$$(\mathcal{R} - \mathcal{R}^2)(2 - \mathcal{R})X = (1 - \mathcal{R})(2\mathcal{R} - \mathcal{R}^2)X$$

and these two corresponding block diagrams are equivalent if the systems start at rest:



Exercise 5.7. Convince yourself that all of these systems are equivalent. One strategy is to convert them all to operator equation representation.

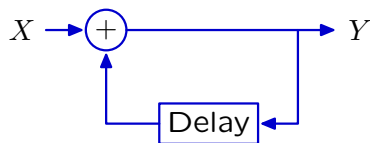


5.4 Feedback Systems

So far, all of our example systems have been *feedforward*: the dependencies have all flowed from the input through to the output, with no dependence of an output on previous output values. In this section, we will extend our representations and analysis to handle the general class of LTI systems in which the output can depend on any finite number of previous input or output values.

5.4.1 Accumulator example

Consider this block diagram, of an *accumulator*:



It's reasonably straightforward to look at this block diagram and see that the associated difference equation is

$$y[n] = x[n] + y[n - 1] ,$$

because the output on any give step is the sum of the the input on that step and the output from the previous step.

Let's use the difference equation to understand what the output of this system is when the input is the unit sample signal. To compute the output at step n , we need to evaluate

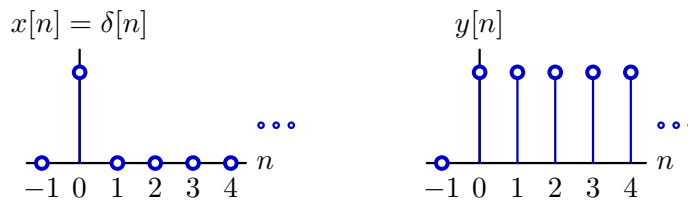
$$y[n] = x[n] + y[n - 1] .$$

We immediately run up against a question: what is the value of $y[n - 1]$? The answer clearly has a profound effect on the output of the system. In our treatment of feedback systems, we will

generally assume that they start 'at rest', which means that all values of the inputs and outputs at steps less than 0 are 0. That assumption lets us fill in the following table:

$$\begin{aligned} y[n] &= x[n] + y[n - 1] \\ y[0] &= x[0] + y[-1] = 1 + 0 = 1 \\ y[1] &= x[1] + y[0] = 0 + 1 = 1 \\ y[2] &= x[2] + y[1] = 0 + 1 = 1 \\ &\dots \end{aligned}$$

Here are plots of the input signal X and the output signal Y :



This result may be somewhat surprising! In feedforward systems, we saw that the output was always a finite sum of scaled and delayed versions of the input signal; so that if the input signal was transient (had a finite number of non-zero samples) then the output signal would be transient as well. But, in this feedback system, we have a transient input with a persistent (infinitely many non-zero samples) output.

We can also look at the operator equation for this system. Again, reading it off of the block diagram, it seems like it should be

$$Y = X + \mathcal{R}Y .$$

It's a well-formed equation, but it isn't immediately clear how to use it to determine Y . Using what we already know about operator algebra, we can rewrite it as:

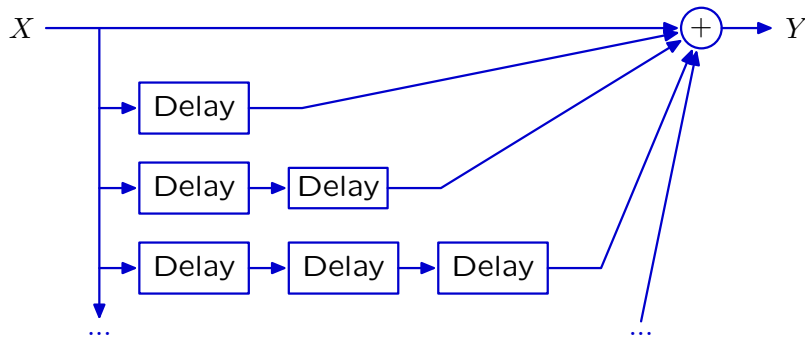
$$Y(1 - \mathcal{R}) = X ,$$

which defines Y to be the signal such that the difference between Y and $\mathcal{R}Y$ is X . But how can we find that Y ?

We will now show that we can think of the accumulator system as being equivalent to another system, in which the output is the sum of infinitely many feedforward paths, each of which delays the input by a different, fixed value. This system has an operator equation of the form

$$Y = (1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \dots) X$$

and can be represented with a block diagram of the form:



These systems are equivalent in the sense that if each is initially at rest, they will produce identical outputs from the same input. We can see this by taking the original definition and repeatedly substituting in the definition of Y in for its occurrence on the right hand side:

$$Y = X + \mathcal{R}Y$$

$$Y = X + \mathcal{R}(X + \mathcal{R}Y)$$

$$Y = X + \mathcal{R}(X + \mathcal{R}(X + \mathcal{R}Y))$$

$$Y = X + \mathcal{R}(X + \mathcal{R}(X + \mathcal{R}(X + \mathcal{R}Y)))$$

$$Y = (1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \dots)X$$

Now, we can informally derive a 'definition' of the reciprocal of $1 - \mathcal{R}$ (the mathematical details underlying this are subtle and not presented here),

$$\frac{1}{1 - \mathcal{R}} = 1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \dots$$

In the following it will help to remind ourselves of the derivation of the formula for the sum of an infinite geometric series:

$$S = 1 + x + x^2 + x^3 + \dots$$

$$Sx = x + x^2 + x^3 + x^4 + \dots$$

Subtracting the second equation from the first we get

$$S(1 - x) = 1$$

And so, provided $|x| < 1$,

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \dots$$

Similarly, we can consider the system O , where

$$\begin{aligned} O &= 1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \dots \\ OR &= \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \dots \end{aligned}$$

So

$$O(1 - \mathcal{R}) = 1$$

And so,

$$\frac{1}{1 - \mathcal{R}} = 1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \dots$$

Exercise 5.8. Check this derivation by showing that

$$(1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \dots)(1 - \mathcal{R}) = 1$$

So, we can rewrite the operator equation for the accumulator as

$$Y = \frac{1}{1 - \mathcal{R}} X .$$

We don't have an intuitive way to interpret this relationship between X and Y yet, but we will spend most of the rest of this chapter on developing intuition and analysis for systems with feedback.

5.4.2 General form of LTI systems

We can now treat the general case of LTI systems, including feedback. In general, LTI systems can be described by difference equations of the form:

$$\begin{aligned} y[n] &= c_0 y[n - 1] + c_1 y[n - 2] + \dots + c_{k-1} y[n - k] \\ &\quad + d_0 x[n] + d_1 x[n - 1] + \dots + d_j x[n - j] . \end{aligned}$$

The state of this system consists of the k previous output values and j previous input values. The output $y[n]$ is a linear combination of the k previous output values, $y[n - 1], \dots, y[n - k]$, j previous input values, $x[n - 1], \dots, x[n - j]$, and the current input, $x[n]$.

This class of state machines can be represented, in generality, in Python, using the `LTISM` class. The state is a tuple, containing a list of the j previous input values and a list of the k previous output values.

```
class LTISM (sm.SM):
    def __init__(self, dCoeffs, cCoeffs):
        j = len(dCoeffs) - 1
        k = len(cCoeffs)

        self.cCoeffs = cCoeffs
        self.dCoeffs = dCoeffs
        self.startState = ([0.0]*j, [0.0]*k)
```

```

def getNextValues(self, state, input):
    (inputs, outputs) = state
    inputs = [input] + inputs

    currentOutput = util.dotProd(outputs, self.cCoeffs) + \
        util.dotProd(inputs, self.dCoeffs)

    return ((inputs[:-1], ([currentOutput] + outputs[:-1]),
        currentOutput)

```

The `util.dotProd` method takes two equal-length lists of numbers and returns the sum of their elementwise products (the dot-product of the two vectors). To keep this code easy to read, we do not handle correctly the case where `dCoeffs` is empty, though it is handled properly in our library implementation.

5.4.3 System functions

Now, we are going to engage in a shift of perspective. We started by defining a new signal Y in terms of an old signal X , much as we might, in algebra, define $y = x + 6$. Sometimes, however, we want to speak of the relationship between x and y in the general case, without a specific x or y in mind. We do this by defining a function f : $f(x) = x + 6$. We can do the same thing with LTI systems, by defining *system functions*.

If we take the general form of an LTI system given in the previous section and write it as an operator equation, we have

$$\begin{aligned}
 Y &= c_0 \mathcal{R}Y + c_1 \mathcal{R}^2 Y + \dots + c_{k-1} \mathcal{R}^k Y + d_0 X + d_1 \mathcal{R}X + \dots + d_j \mathcal{R}^j X \\
 &= (c_0 \mathcal{R} + c_1 \mathcal{R}^2 + \dots + c_{k-1} \mathcal{R}^k) Y + (d_0 + d_1 \mathcal{R} + \dots + d_j \mathcal{R}^j) X .
 \end{aligned}$$

We can rewrite this as

$$(1 - c_0 \mathcal{R} - c_1 \mathcal{R}^2 - \dots - c_{k-1} \mathcal{R}^k) Y = (d_0 + d_1 \mathcal{R} + \dots + d_j \mathcal{R}^j) X ,$$

so

$$\frac{Y}{X} = \frac{d_0 + d_1 \mathcal{R} + d_2 \mathcal{R}^2 + d_3 \mathcal{R}^3 + \dots}{1 - c_0 \mathcal{R} - c_1 \mathcal{R}^2 - c_2 \mathcal{R}^3 - \dots} ,$$

which has the form

$$\frac{Y}{X} = \frac{N(\mathcal{R})}{D(\mathcal{R})} ,$$

where $N(\mathcal{R})$, the numerator, is a polynomial in \mathcal{R} , and $D(\mathcal{R})$, the denominator, is also a polynomial in \mathcal{R} . We will refer to Y/X as the *system function*: it characterizes the operation of a system, independent of the particular input and output signals involved.

The system function is most typically written in the form

$$\frac{Y}{X} = \frac{b_0 + b_1 \mathcal{R} + b_2 \mathcal{R}^2 + b_3 \mathcal{R}^3 + \dots}{a_0 + a_1 \mathcal{R} + a_2 \mathcal{R}^2 + a_3 \mathcal{R}^3 + \dots} ,$$

where $c_i = -a_{i+1}/a_0$ and $d_i = b_i/a_0$. It can be completely characterized by the coefficients of the denominator polynomial, a_i , and the coefficients of the numerator polynomial, b_i . It is always possible to rewrite this in a form in which $a_0 = 1$.

Feedforward systems have no dependence on previous values of Y , so they have $D(\mathcal{R}) = 1$. Feedback systems have persistent behavior, which is determined by $D(\mathcal{R})$. We will study this dependence in detail in section 5.5.

5.4.4 Primitive systems

Just as we had a PCAP system for signals, we have one for LTI system, in terms of system functions, as well. We can specify system functions for each of our system primitives.

A *gain* element is governed by operator equation $Y = kX$, for constant k , so its system function is

$$H = \frac{Y}{X} = k .$$

A *delay* element is governed by operator equation $Y = \mathcal{R}X$, so its system function is

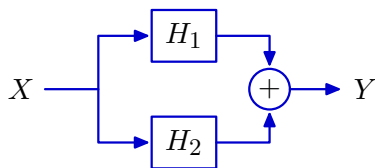
$$H = \frac{Y}{X} = \mathcal{R} .$$

5.4.5 Combining system functions

We have three basic composition operations: sum, cascade, and feedback. This PCAP system, as our previous ones have been, is *compositional*, in the sense that whenever we make a new system function out of existing ones, it is a system function in its own right, which can be an element in further compositions.

Addition

The system function of the *sum* of two systems is the sum of their system functions. So, given two systems with system functions H_1 and H_2 , connected like this:



and letting

$$Y_1 = H_1X \quad \text{and} \quad Y_2 = H_2X ,$$

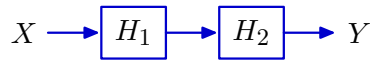
we have

$$\begin{aligned} Y &= Y_1 + Y_2 \\ &= H_1X + H_2X \\ &= (H_1 + H_2)X \\ &= HX , \end{aligned}$$

where $H = H_1 + H_2$.

Cascade

The system function of the *cascade* of two systems is the product of their system functions. So, given two systems with system functions H_1 and H_2 , connected like this:



and letting

$$W = H_1 X \quad \text{and} \quad Y = H_2 W ,$$

we have

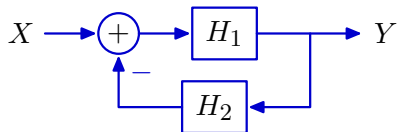
$$\begin{aligned} Y &= H_2 W \\ &= H_2 H_1 X \\ &= H X , \end{aligned}$$

where $H = H_2 H_1$. And note that, as was the case with purely feedforward systems, cascade combination is still commutative, so it doesn't matter whether H_1 or H_2 comes first in the cascade. This surprising fact holds because we are only considering LTI systems *starting at rest*; for more general classes of systems, such as the general class of state machines we have worked with before, the ordering of a cascade *does* matter.

Feedback

There are several ways of connecting systems in feedback. Here we study a particular case of *negative feedback* combination, which results in a classical formula called *Black's formula*.

Consider two systems connected like this



and *pay careful attention to the negative sign on the feedback input to the addition*. It is really just shorthand; the negative sign could be replaced with a gain component with value -1 . This negative feedback arrangement is frequently used to model a case in which X is a desired value for some signal and Y is its actual value; thus the input to H_1 is the difference between the desired and actual values, often called an *error signal*. We can simply write down the operator equation governing this system and use standard algebraic operations to determine the system function:

$$\begin{aligned}
 Y &= H_1(X - H_2Y) \\
 Y + H_1H_2Y &= H_1X \\
 Y(1 + H_1H_2) &= H_1X \\
 Y &= \frac{H_1}{1 + H_1H_2}X \\
 Y &= HX \text{ ,}
 \end{aligned}$$

where

$$H = \frac{H_1}{1 + H_1H_2} \text{ .}$$

Armed with this set of primitives and composition methods, we can specify a large class of machines. Ultimately, we will want to construct systems with multiple inputs and outputs; such systems are specified with a matrix of basic system functions, describing how each output depends on each input.

5.5 Predicting system behavior

We have seen how to construct complex discrete-time LTI systems; in this section we will see how we can use properties of the system function to predict how the system will behave, in the long term, and for any input. We will start by analyzing simple systems and then move to more complex ones.

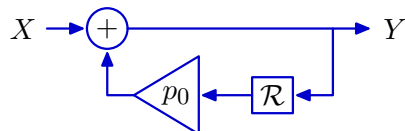
We can provide a general characterization of the long-term behavior of the output, as increasing or decreasing, with constant or alternating sign, for any finite input to the system. We will begin by studying the *unit-sample* response of systems, and then generalize to more general input signals; similarly, we will begin by studying simple systems and generalize to more complex ones.

5.5.1 First-order systems

Systems that only have forward connections can only have a finite response; that means that if we put in a unit sample (or other signal with only a finite number of non-zero samples) then the output signal will only have a finite number of non-zero samples.

Systems with feedback have a surprisingly different character. Finite inputs can result in *persistent response*; that is, in output signals with infinitely many non-zero samples. Furthermore, the qualitative long-term behavior of this output is generally independent of the particular input given to the system, for any finite input. In this section, we will consider the class of *first-order* systems, in which the denominator of the system function is a first-order polynomial (that is, it only involves \mathcal{R} , but not \mathcal{R}^2 or other higher powers of \mathcal{R} .)

Let's consider this very simple system



for which we can write an operator equation

$$Y = X + p_0 \mathcal{R}Y$$

$$(1 - p_0 \mathcal{R})Y = X$$

$$Y = \frac{X}{1 - p_0 \mathcal{R}}$$

and derive a system function

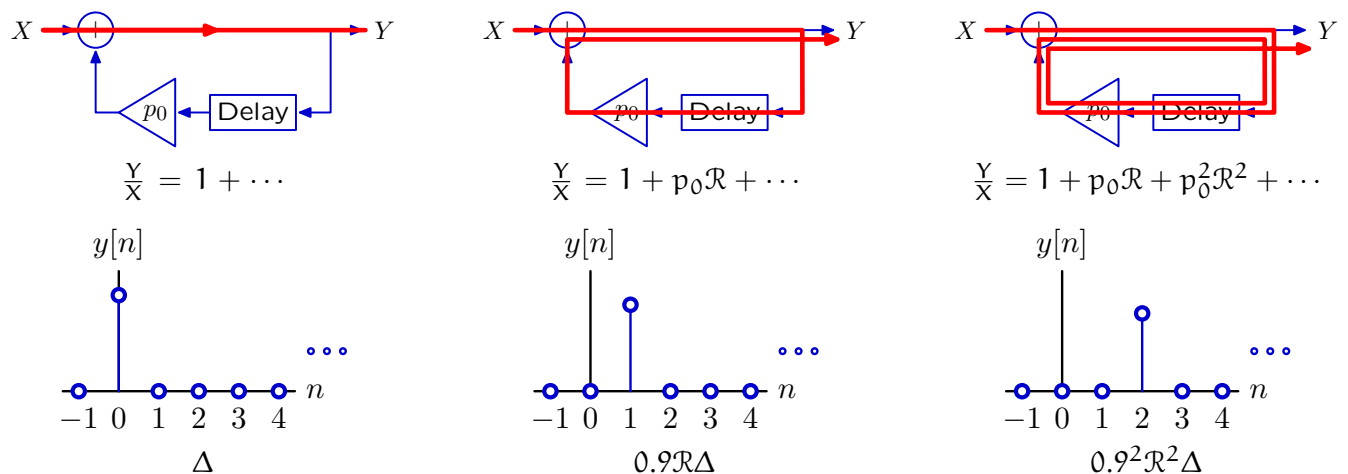
$$H = \frac{Y}{X} = \frac{1}{1 - p_0 \mathcal{R}} .$$

Recall the infinite series representation of this system function (derived in section 5.4.1):

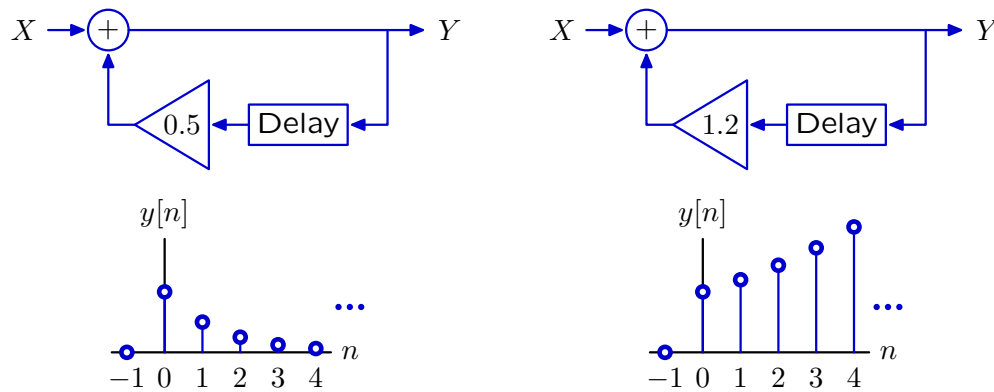
$$\frac{1}{1 - p_0 \mathcal{R}} = 1 + p_0 \mathcal{R} + p_0^2 \mathcal{R}^2 + p_0^3 \mathcal{R}^3 + p_0^4 \mathcal{R}^4 + \dots .$$

We can make intuitive sense of this by considering how the signal flows through the system. On each step, the output of the system is being fed back into the input. Consider the simple case where the input is the unit sample ($X = \Delta$). Then, after step 0, when the input is 1, there is no further input, and the system continues to respond.

In this table, we see that the whole output signal is a sum of scaled and delayed copies of the input signal; the bottom row of figures shows the first three terms in the infinite sum of signals, for the case where $p_0 = 0.9$.



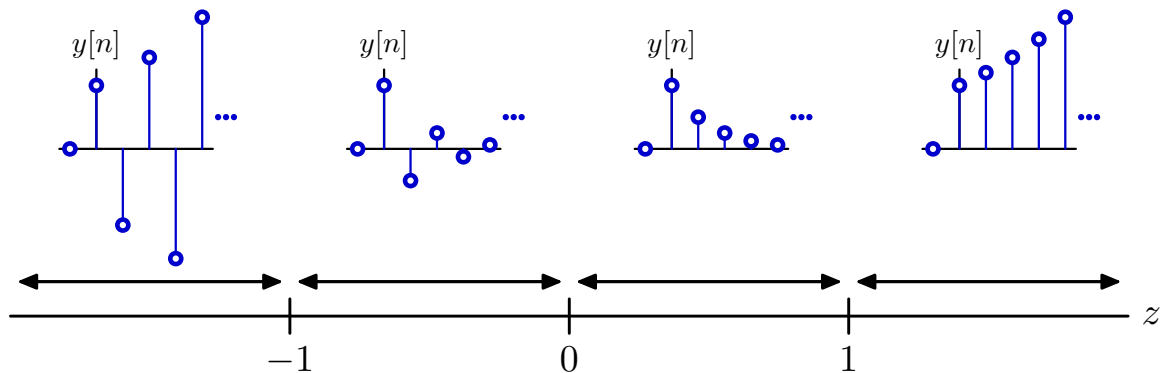
If traversing the cycle decreases or increases the magnitude of the signal, then the sample values will decay or grow, respectively, as time increases.



For the first system, the unit sample response is $y[n] = (0.5)^n$; for the second, it's $y[n] = (1.2)^n$.

These system responses can be characterized by a single number, called the *pole*, which is the base of the geometric sequence. The value of the pole, p_0 , determines the nature and rate of growth.

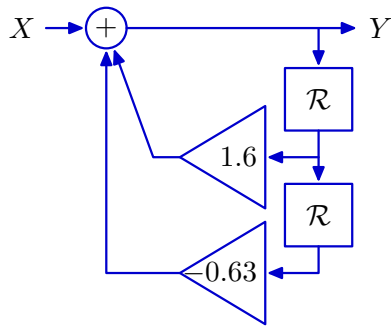
- If $p_0 < -1$, the magnitude increases to infinity and the sign alternates.
- If $-1 < p_0 < 0$, the magnitude decreases and the sign alternates.
- If $0 < p_0 < 1$, the magnitude decreases monotonically.
- If $p_0 > 1$, the magnitude increases monotonically to infinity.



5.5.2 Second-order systems

We will call these persistent long-term behaviors of a signal (and, hence, of the system that generates such signals) *modes*. For a fixed p_0 , the first-order system only exhibited one mode (but different values of p_0 resulted in very different modes). As we build more complex systems, they will have multiple modes, which manifest as more complex behavior. Second-order systems are characterized by a system function whose denominator polynomial is second order; they will generally exhibit two modes.

Consider this system



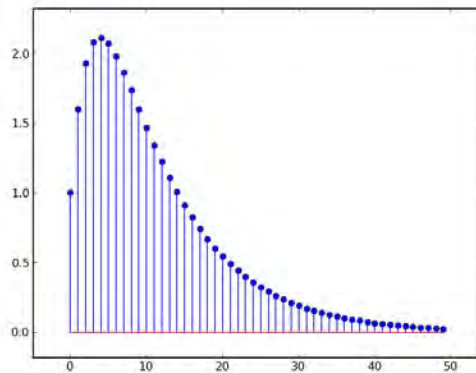
We can describe it with the operator equation

$$Y = 1.6\mathcal{R}Y - 0.63\mathcal{R}^2Y + X ,$$

so the system function is

$$H = \frac{1}{1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2} .$$

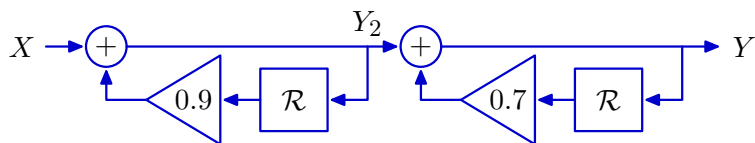
Here is its response to a unit sample signal:

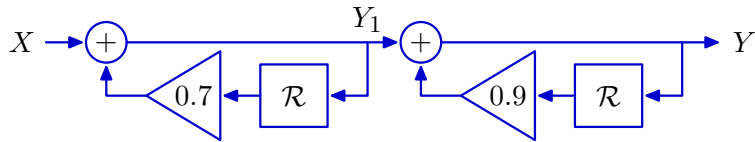


We can try to understand its behavior by decomposing it in different ways. First, let's see if we can see it as a cascade of two systems. To do so, we need to find H_1 and H_2 such that $H_1 H_2 = H$. We can do that by factoring H to get

$$H_1 = \frac{1}{1 - 0.7\mathcal{R}} \quad \text{and} \quad H_2 = \frac{1}{1 - 0.9\mathcal{R}} .$$

So, we have two equivalent version of this system, describable as cascades of two systems, one with $p_0 = 0.9$ and one with $p_0 = 0.7$:





This decomposition is interesting, but it does not yet let us understand the behavior of the system as the combination of the behaviors of two subsystems.

5.5.2.1 Additive decomposition

Another way to try to decompose the system is as the *sum* of two simpler systems. In this case, we seek H_1 and H_2 such that $H_1 + H_2 = H$. We can do a partial fraction decomposition (don't worry if you don't remember the process for doing this...we won't need to solve problems like this in detail). We start by factoring, as above, and then figure out how to decompose into additive terms:

$$\begin{aligned} H &= \frac{1}{1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2} \\ &= \frac{1}{(1 - 0.9\mathcal{R})(1 - 0.7\mathcal{R})} \\ &= \frac{A}{1 - 0.9\mathcal{R}} + \frac{B}{1 - 0.7\mathcal{R}} \\ &= H_1 + H_2 . \end{aligned}$$

To find values for A and B , we start with

$$\frac{1}{(1 - 0.9\mathcal{R})(1 - 0.7\mathcal{R})} = \frac{A}{1 - 0.9\mathcal{R}} + \frac{B}{1 - 0.7\mathcal{R}} ,$$

multiply through by $1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2$ to get

$$1 = A(1 - 0.7\mathcal{R}) + B(1 - 0.9\mathcal{R}) ,$$

and collect like terms:

$$1 = (A + B) - (0.7A + 0.9B)\mathcal{R} .$$

Equating the terms that involve equal powers of \mathcal{R} (including constants as terms that involve \mathcal{R}^0), we have:

$$1 = A + B$$

$$0 = 0.7A + 0.9B .$$

Solving, we find $A = 4.5$ and $B = -3.5$, so

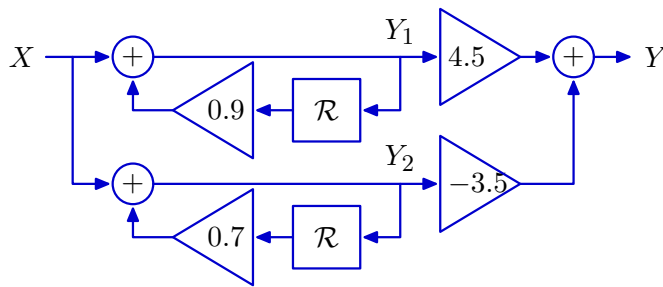
$$\frac{Y}{X} = \frac{4.5}{1 - 0.9\mathcal{R}} + \frac{-3.5}{1 - 0.7\mathcal{R}} ,$$

where

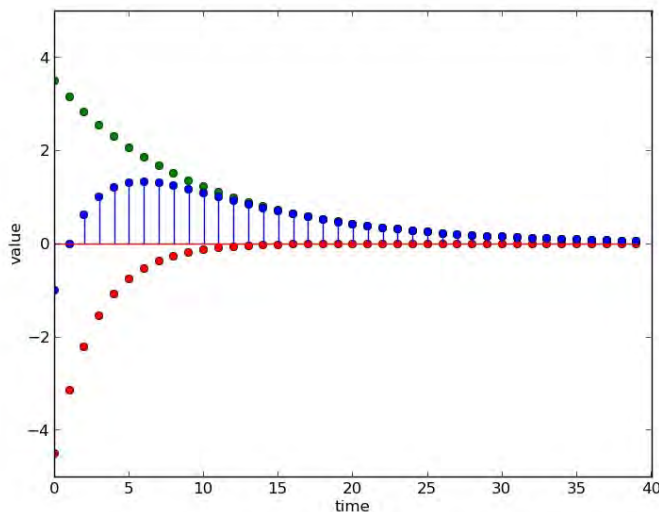
$$H_1 = \frac{4.5}{1 - 0.9\mathcal{R}} \quad \text{and} \quad H_2 = \frac{-3.5}{1 - 0.7\mathcal{R}} .$$

Exercise 5.9. Verify that $H_1 + H_2 = H$.

Here is (yet another) equivalent block diagram for this system, highlighting its decomposition into a sum:



We can understand this by looking at the responses of H_1 and of H_2 to the unit sample, and then summing them to recover the response of H to the unit sample. In the next figure, the blue stem plot is the overall signal, which is the sum of the green and red signals, which correspond to the top and bottom parts of the diagram, respectively.

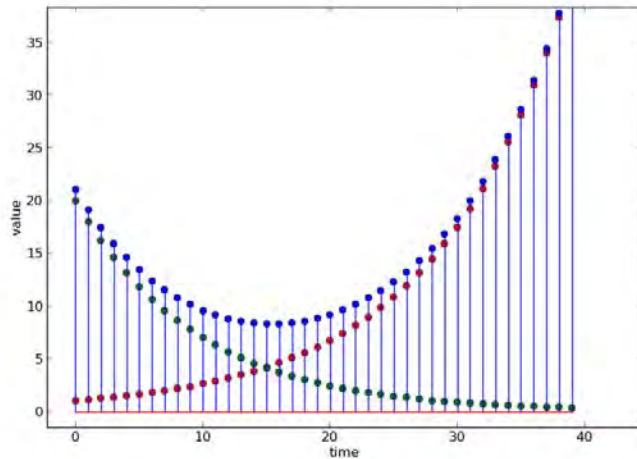


In this case, both of the poles (0.9 and 0.7) are less than 1, so the magnitude of the responses they generate decreases monotonically; their sum does not behave monotonically, but there is a time step at which the dominant pole completely dominates the other one, and the convergence is monotonic after that.

If, instead, we had a system with system function

$$H = \frac{Y}{X} = \frac{1}{1 - 1.1\mathcal{R}} + \frac{20}{1 - 0.9\mathcal{R}},$$

what would the unit-sample response be? The first mode (first term of the sum) has pole 1.1, which means it will generate monotonically increasing output values. The second mode has pole 0.9, and will decrease monotonically. The plot below illustrates the sum of these two components:

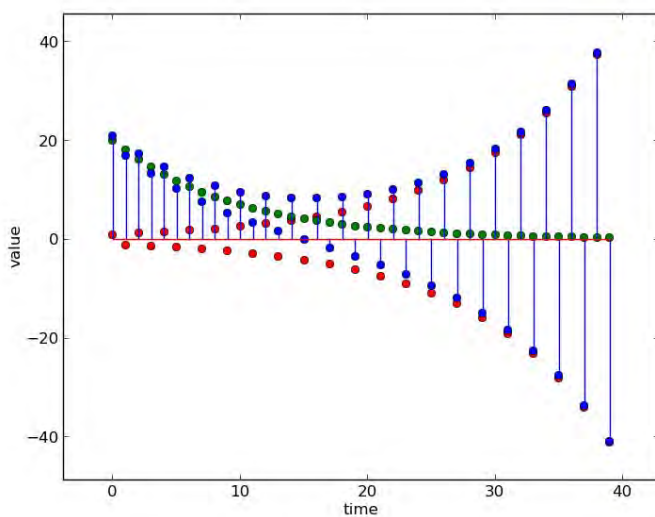


The red points correspond to the output from the mode with pole 1.1; the green points correspond to the output from the mode with pole 0.9; and the blue stem plot shows the sum.

Here is a plot of the output of the system described by

$$\frac{Y}{X} = \frac{21.1\mathcal{R} + 21}{-0.99\mathcal{R}^2 + 0.2\mathcal{R} + 1}.$$

It has poles 0.9 and -1.1 .



The green dots are generated by the component with pole 0.9, the red dots are generated by the component with pole -1.1 and the blue dots are generated by the sum.

In these examples, as in the general case, the long-term unit-sample response of the entire system is governed by the unit-sample response of the mode whose pole has the larger magnitude. In the long run, the rate of growth of the exponential with the largest exponent will always dominate.

5.5.2.2 Complex poles

Consider a system described by the operator equation:

$$Y = 2X + 2\mathcal{R}X - 2\mathcal{R}Y - 4\mathcal{R}^2Y .$$

It has system function

$$\frac{Y}{X} = \frac{2 + 2\mathcal{R}}{1 + 2\mathcal{R} + 4\mathcal{R}^2} . \quad (5.1)$$

But now, if we attempt to perform an additive decomposition on it, we find that the denominator cannot be factored to find real poles. Instead, we find that

$$\begin{aligned} \frac{Y}{X} &= \frac{2 + 2\mathcal{R}}{(1 - (-1 + \sqrt{-3})\mathcal{R})(1 - (-1 - \sqrt{-3})\mathcal{R})} \\ &\approx \frac{2 + 2\mathcal{R}}{(1 - (-1 + 1.732j)\mathcal{R})(1 - (-1 - 1.732j)\mathcal{R})} . \end{aligned}$$

So, the poles are $-1 + 1.732j$ and $-1 - 1.732j$. Note that we are using j to signal the imaginary part of a complex number.³⁶ What does that mean about the behavior of our system? Are the outputs real or complex? Do the output values grow or shrink with time?

Difference equations that represent physical systems have real-valued coefficients. For instance, a bank account with interest ρ might be described with difference equation

$$y[n] = (1 + \rho)y[n - 1] + x[n] .$$

The position of a robot moving toward a wall might be described with difference equation:

$$d_o[n] = d_o[n - 1] + KTd_o[n - 2] - KTd_i[n - 1] .$$

Difference equations with real-valued coefficients generate real-valued outputs from real-valued inputs. But, like the difference equation

$$y[n] = 2x[n] + 2x[n - 1] - 2y[n - 1] - 4y[n - 1] ,$$

corresponding to system function 5.1, they might still have complex poles.

Polar representation of complex numbers

Sometimes it's easier to think about a complex number $a + bj$ instead as $re^{j\Omega}$, where

³⁶ We use j instead of i because, to an electrical engineer, i stands for current, and can't be re-used!

$$a = r \cos \Omega$$

$$b = r \sin \Omega$$

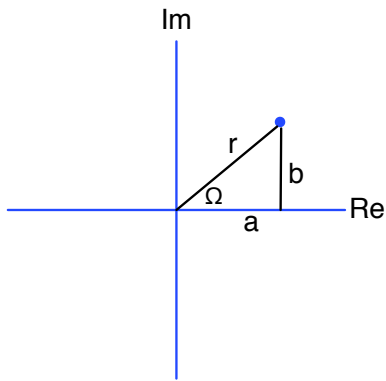
so that the *magnitude*, r , sometimes written as $|a + bj|$, is defined as

$$r = \sqrt{a^2 + b^2}$$

and the *angle*, Ω , is defined as

$$\Omega = \tan^{-1}(b, a) .$$

So, if we think of (a, b) as a point in the complex plane, then (r, Ω) is its representation in polar coordinates.



This representation is justified by Euler's equation

$$e^{xj} = \cos x + j \sin x ,$$

which can be directly derived from series expansions of e^z , $\sin z$ and $\cos z$. To see that this is reasonable, let's take our number, represent it as a complex exponential, and then apply Euler's equation:

$$\begin{aligned} a + bj &= re^{j\Omega} \\ &= r(\cos \Omega + j \sin \Omega) \\ &= \sqrt{a^2 + b^2}(\cos(\tan^{-1}(b, a)) + j \sin(\tan^{-1}(b, a))) \\ &= \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} + j \frac{b}{\sqrt{a^2 + b^2}}\right) \\ &= a + bj \end{aligned}$$

Why should we bother with this change of representation? There are some operations on complex numbers that are much more straightforwardly done in the exponential representation. In particular, let's consider raising a complex number to a power. In the Cartesian representation, we get complex trigonometric polynomials. In the exponential representation, we get, in the quadratic case,

$$\begin{aligned} (re^{j\Omega})^2 &= re^{j\Omega}re^{j\Omega} \\ &= r^2e^{j2\Omega} \end{aligned}$$

More generally, we have that

$$(re^{j\Omega})^n = r^n e^{jn\Omega} ,$$

which is much tidier. This is an instance of an important trick in math and engineering: changing representations. We will often find that representing something in a different way will allow us to do some kinds of manipulations more easily. This is why we use diagrams, difference equations, operator equations and system functions all to describe LTI systems. There is no one best representation; each has advantages under some circumstances (and disadvantages under others).

Complex modes

Now, we're equipped to understand how complex poles of a system generate behavior: they produce complex-valued modes. Remember that we can characterize the behavior of a mode as

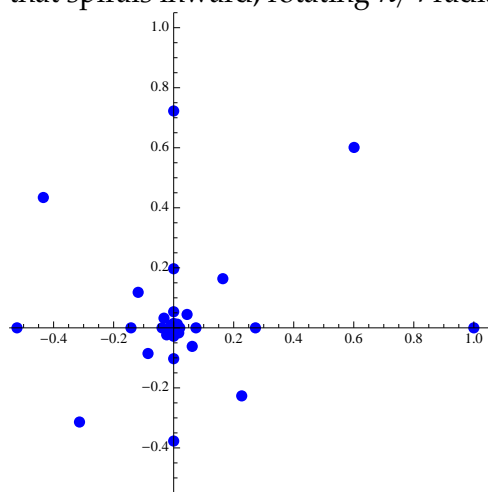
$$\frac{1}{1 - p\mathcal{R}} = 1 + p\mathcal{R} + p^2\mathcal{R}^2 + \dots + p^n\mathcal{R}^n + \dots .$$

For a complex pole $p = re^{j\Omega}$, $p^n = r^n e^{jn\Omega}$. So

$$\frac{1}{1 - re^{j\Omega}\mathcal{R}} = 1 + re^{j\Omega}\mathcal{R} + r^2 e^{j2\Omega}\mathcal{R}^2 + \dots$$

What happens as n tends to infinity when p is complex? Think of p^n as a point in the complex plane with coordinates $(r^n, \Omega n)$. The radius, r^n , will grow or shrink depending on the mode's magnitude, r . And the angle, Ωn , will simply rotate, but will not affect the magnitude of the resulting value. Note that each new point in the sequence p^n will be rotated by Ω from the previous one. We will say that the *period* of the output signal is $2\pi/\Omega$; that is the number of samples required to move through 2π radians (although this need not actually be an integer).

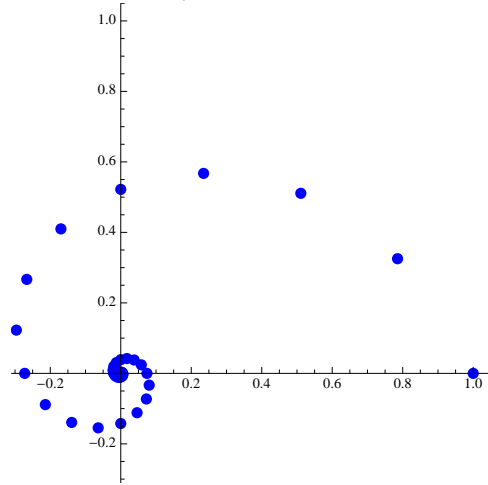
The sequence spirals around the origin of the complex plane. Here is a case for $r = 0.85$, $\Omega = \pi/4$, that spirals inward, rotating $\pi/4$ radians on each step:



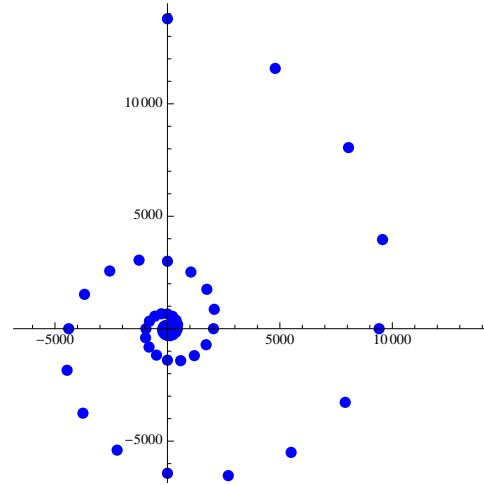
$$p = 0.85e^{j\pi/4} \approx 0.601 + 0.601j$$

Here, for example, are plots of two other complex modes, with different magnitudes and rates of oscillation. In all cases, we are plotting the unit-sample response, so the first element in the series

is the real value 1. The first signal clearly converges in magnitude to 0; the second one diverges (look carefully at the scales).



$$p = 0.85e^{j\pi/8} \approx 0.785 + 0.325j$$



$$p = 1.1e^{j\pi/8} \approx 1.016 + 0.421j$$

If complex modes occurred all by themselves, then our signals would have complex numbers in them. But isolated complex pole can result only from a difference equation with complex-valued coefficients. For example, to end up with this system function

$$\frac{Y}{X} = \frac{1}{1 - re^{j\Omega\mathcal{R}}}$$

we would need to have this difference equation, which has a complex coefficient.

$$y[n] - re^{j\Omega}y[n-1] = x[n]$$

But, we have difference equations with real parameters, so we can express the value of every sample of the output signal as a linear combination of inputs and previous values with real coefficients, so we know the output signal is real-valued at all samples. The reason this all works out is that, for polynomials with real coefficients, the complex poles always come in *conjugate pairs*: that is, pairs of complex numbers $p = a + bj$ and $p^* = a - bj$. (See section 5.5.4 for a proof of this fact.) In the polar representation, the conjugate pair becomes

$$\begin{aligned} a \pm bj &= \sqrt{a^2 + b^2} e^{j \tan^{-1}(\pm b, a)} \\ &= \sqrt{a^2 + b^2} e^{\pm j \tan^{-1}(b, a)} \end{aligned}$$

where the only difference is in the sign of the angular part.

If we look at a second-order system with complex-conjugate poles, the resulting polynomial has real-valued coefficients. To see this, consider a system with poles $re^{j\Omega}$ and $re^{-j\Omega}$, so that

$$\frac{Y}{X} = \frac{1}{(1 - re^{j\Omega\mathcal{R}})(1 - re^{-j\Omega\mathcal{R}})}$$

Let's slowly multiply out the denominator:

$$\begin{aligned}(1 - re^{j\Omega\mathcal{R}})(1 - re^{-j\Omega\mathcal{R}}) &= 1 - re^{j\Omega\mathcal{R}} - re^{-j\Omega\mathcal{R}} + r^2 e^{j\Omega} e^{-j\Omega} \mathcal{R}^2 \\ &= 1 - r(e^{j\Omega} + e^{-j\Omega})\mathcal{R} + r^2 e^{j\Omega-j\Omega}\mathcal{R}^2\end{aligned}$$

Using the definition of the complex exponential $e^{jx} = \cos x + j \sin x$,

$$= 1 - r(\cos \Omega + j \sin \Omega + \cos(-\Omega) + j \sin(-\Omega))\mathcal{R} + r^2 \mathcal{R}^2$$

Using trigonometric identities $\sin -x = -\sin x$ and $\cos -x = \cos x$,

$$\begin{aligned}&= 1 - r(\cos \Omega + j \sin \Omega + \cos \Omega - j \sin \Omega)\mathcal{R} + r^2 \mathcal{R}^2 \\ &= 1 - 2r \cos \Omega \mathcal{R} + r^2 \mathcal{R}^2\end{aligned}$$

So,

$$\frac{Y}{X} = \frac{1}{1 - 2r \cos \Omega \mathcal{R} + r^2 \mathcal{R}^2} .$$

This is pretty cool! All of the imaginary parts cancel, and we are left with a system function with only real coefficients, which corresponds to the difference equation

$$y[n] = x[n] + 2r \cos \Omega y[n - 1] - r^2 y[n - 2] .$$

Additive decomposition with complex poles

You can skip this section if you want to.

To really understand these complex modes and how they combine to generate the output signal, we need to do an additive decomposition. That means doing a partial fraction expansion of



$$\frac{Y}{X} = \frac{1}{(1 - re^{j\Omega\mathcal{R}})(1 - re^{-j\Omega\mathcal{R}})} .$$

It's a little trickier than before, because we have complex numbers, but the method is the same. The end result is that we can decompose this system additively to get

$$\frac{Y}{X} = \frac{\frac{1}{2}(1 - j \cot \Omega)}{1 - re^{j\Omega\mathcal{R}}} + \frac{\frac{1}{2}(1 + j \cot \Omega)}{1 - re^{-j\Omega\mathcal{R}}} .$$

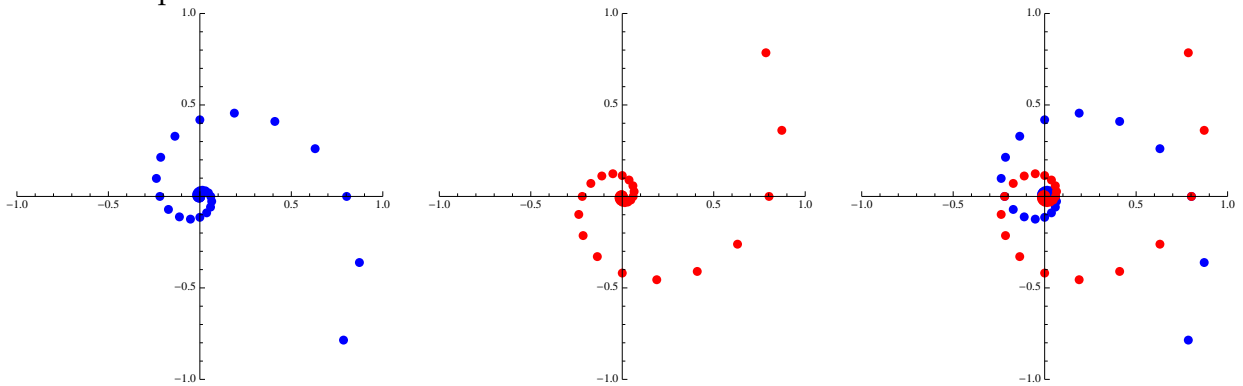
What is the unit-sample response of each of these modes? What is the unit-sample response of their sum? This might be making you nervous...it's hard to see how everything is going to come out to be real in the end.

But, let's examine the response of the additive decomposition; it's the sum of the outputs of the component systems. So, if $x[n] = \delta[n]$,

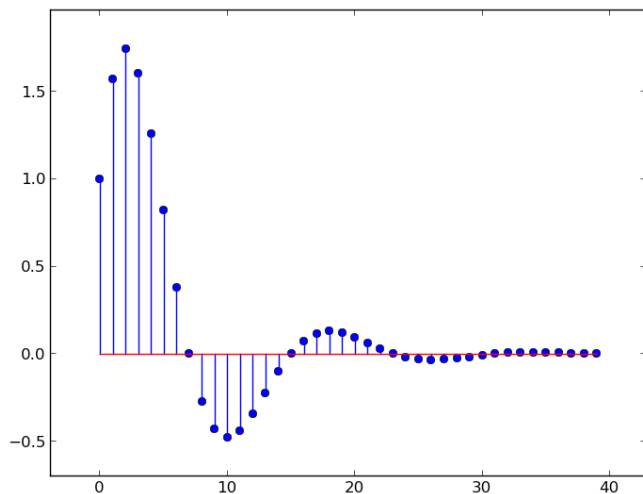
$$\begin{aligned}y[n] &= \frac{1}{2}(1 - j \cot \Omega)r^n e^{jn\Omega} + \frac{1}{2}(1 + j \cot \Omega)r^n e^{-jn\Omega} \\ &= r^n (\cos n\Omega + \cot \Omega \sin n\Omega) ,\end{aligned}$$

which is entirely real.

The figures below show the modes for a system with poles $0.85e^{j\pi/8}$ and $0.85e^{-j\pi/8}$: the blue series starts at $\frac{1}{2}(1 - j \cot(\pi/8))$ and traces out the unit sample response of the first component; the second red series starts at $\frac{1}{2}(1 + j \cot(\pi/8))$ and traces out the unit sample response of the second component.



Note, in the third figure, that the imaginary parts of the contributions of each of the modes cancel out, and that real parts are equal. Thus, the real part of the output is going to be twice the real part of these elements. The figure below shows the unit sample response of the entire system.



In the formula below,

$$y[n] = r^n (\cos n\Omega + \cot \Omega \sin n\Omega) ,$$

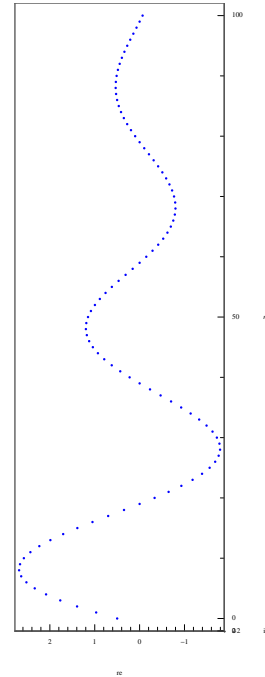
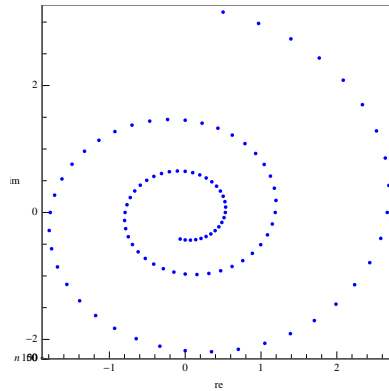
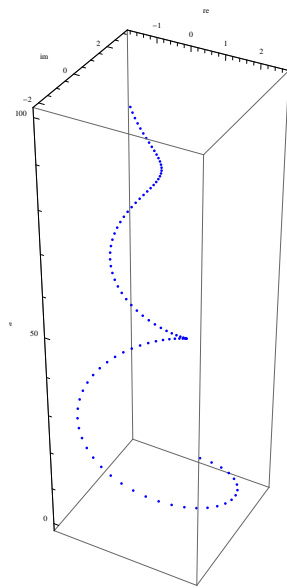
we know that

$$-\sqrt{1 + \cot^2 \Omega} \leq \cos n\Omega + \cot \Omega \sin n\Omega \leq \sqrt{1 + \cot^2 \Omega} ,$$

so

$$\begin{aligned} -\sqrt{1 + \cot^2 \Omega} r^n &\leq y[n] \leq \sqrt{1 + \cot^2 \Omega} r^n \\ -\frac{1}{\sin \Omega} r^n &\leq y[n] \leq \frac{1}{\sin \Omega} r^n . \end{aligned}$$

Just for fun, here is a three-dimensional plot of a single mode of a system with pole $0.98e^{j\pi/20}$. These values were chosen so that it would shrink slowly and also rotate slowly around the complex plane. The first figure shows n growing upward with the complex plane oriented horizontally; the second shows a view looking down onto the complex plane; the third shows a view that projects along the imaginary axis.



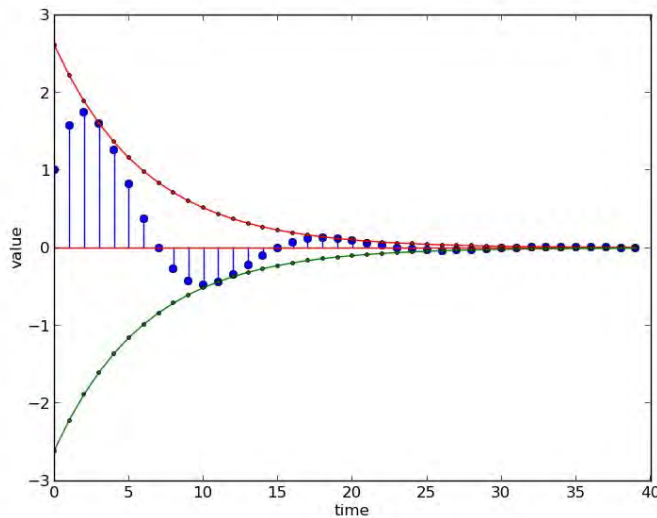
Importance of magnitude and period

Both r and Ω tells us something very useful about the way in which the system behaves. In the previous section, we derived an expression for the samples of the unit sample response for a system with a pair of complex poles. It has the form

$$y[n] = r^n (\cos n\Omega + \alpha \sin n\Omega) ,$$

where α is a constant. We know that $(\cos n\Omega + \alpha \sin n\Omega)$ cannot possibly be less than $-\sqrt{1 + \alpha^2}$ or greater than $\sqrt{1 + \alpha^2}$.

These bounds provide an *envelope* that constrains where the values of the signal can be as a function of time, and help us understand the rate of growth or decay of the signal, as shown below. The red and green curves are $\pm\sqrt{1 + \alpha^2}r^n$, for a system where $r = 0.85$, $\Omega = \pi/8$, and $\alpha = 2.414$.



The value of r governs the rate of exponential decrease. The value of Ω governs the rate of oscillation of the curve. It will take $2\pi/\Omega$ (the *period* of the oscillation) samples to go from peak to peak of the oscillation.³⁷ In our example, $\Omega = \pi/8$ so the period is 16; you should be able to count 16 samples from peak to peak.

5.5.2.3 Poles and behavior: summary

In a second-order system, if we let p_0 be the pole with *the largest magnitude*, then there is a time step at which the behavior of the dominant pole begins to dominate; *after that time step*

- If p_0 is real and
 - $p_0 < -1$, the magnitude increases to infinity and the sign alternates.
 - $-1 < p_0 < 0$, the magnitude decreases and the sign alternates.
 - $0 < p_0 < 1$, the magnitude decreases monotonically.
 - $p_0 > 1$, the magnitude increases monotonically to infinity.
- If p_0 is complex
 - and $|p_0| < 1$, the magnitude decreases monotonically.
 - and $|p_0| > 1$, the magnitude increases monotonically to infinity.
- If p_0 is complex and Ω is its angle, then the signal will be periodic, with period $2\pi/\Omega$.

As we have seen in our examples, when we add multiple modes, it is the mode with the largest pole that governs the long-term behavior.

5.5.3 Higher-order systems

Recall that we can describe any system in terms of a system function that is the ratio of two polynomials in \mathcal{R} (and assuming $a_0 = 1$):

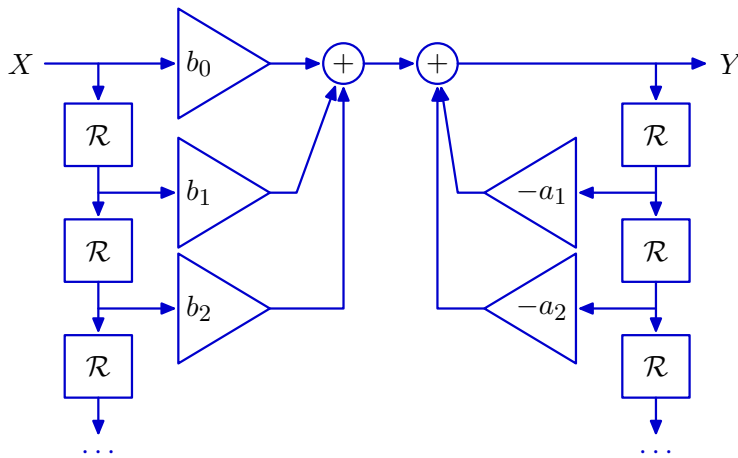
³⁷ We are being informal here, in two ways. First, the signal does not technically have a period, because unless $r = 1$, it doesn't return to the same point. Second, unless $r = 1$, then distance from peak to peak is not exactly $2\pi/\Omega$, however, for most signals, it will give a good basis for estimating Ω .

$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \dots}{1 + a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \dots}$$

Regrouping terms, we can write this as the operator equation:

$$Y = (b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \dots) X - (a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \dots) Y$$

and construct an equivalent block diagram:



Returning to the general polynomial ratio

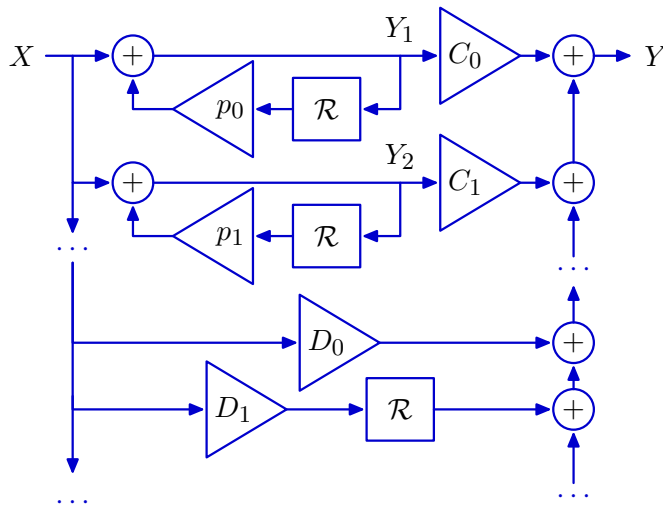
$$\frac{Y}{X} = \frac{b_0 + b_1\mathcal{R} + b_2\mathcal{R}^2 + b_3\mathcal{R}^3 + \dots}{1 + a_1\mathcal{R} + a_2\mathcal{R}^2 + a_3\mathcal{R}^3 + \dots}$$

We can factor the denominator of an n th-order polynomial into n factors, and then perform a partial fraction expansion, to turn it into the form of a sum of terms. We won't go over the details here (there are nice tutorials online), but it comes out in the form:

$$\frac{Y}{X} = \frac{C_0}{1 - p_0\mathcal{R}} + \frac{C_1}{1 - p_1\mathcal{R}} + \frac{C_2}{1 - p_2\mathcal{R}} + \dots + D_0 + D_1\mathcal{R} + D_2\mathcal{R}^2 + \dots$$

where the C_k and D_k are constants defined in terms of the a_i and b_j from the original polynomial ratio. It's actually a little bit trickier than this: if there are complex poles, then for each conjugate pair of complex poles, we would put in a second-order system with real coefficients that expresses the contribution of the sum of the complex modes.

The constant term D_0 and the terms with \mathcal{R}^k in the numerator occur if the numerator has equal or higher order than the denominator. They do not involve feedback and don't affect the long-term behavior of the system. One mode of the form p_i^n arises from each factor of the denominator. This modal decomposition leads us to an alternative block diagram:



We can fairly easily observe that the behavior is going to be the sum of the behaviors of the individual modes, and that, as in the second-order case, the mode whose pole has the largest magnitude will govern the qualitative long-term nature of the behavior of the system in response to a unit-sample input.

5.5.4 Finding poles

In general, we will find that if the denominator of the system function H is a k th order polynomial, then it can be factored into the form $(1 - p_0\mathcal{R})(1 - p_1\mathcal{R}) \dots (1 - p_{k-1}\mathcal{R})$. We will call the p_i values the *poles* of the system. The entire persistent output of the system can be expressed as a scaled sum of the signals arising from each of these individual poles.

We're doing something interesting here! We are using the PCAP system backwards for analysis. We have a complex thing that is hard to understand monolithically, so we are taking it apart into simpler pieces that we do understand.

It might seem like factoring polynomials in this way is tricky, but there is a straightforward way to find the poles of a system given its denominator polynomial in \mathcal{R} .

We'll start with an example. Assume the denominator is $12\mathcal{R}^2 - 7\mathcal{R} + 1$. If we play a quick trick, and introduce a new variable $z = 1/\mathcal{R}$, then our denominator becomes

$$\frac{12}{z^2} - \frac{7}{z} + 1 .$$

We'd like to find the roots of this polynomial, which, if you multiply through by z^2 , is equivalent to finding the roots of this polynomial:

$$12 - 7z + z^2 = 0 .$$

The roots are 3 and 4. If we go back to our original polynomial in \mathcal{R} , we can see that:

$$12\mathcal{R}^2 - 7\mathcal{R} + 1 = (1 - 4\mathcal{R})(1 - 3\mathcal{R}) .$$

so that our poles are 4 and 3. So, remember, *the poles are not the roots of the polynomial in \mathcal{R} , but are the roots of the polynomial in the reciprocal of \mathcal{R} .*

The roots of a polynomial can be a combination of real and complex numbers, with the requirement that if a complex number p is a root, then so is its complex conjugate, p^* .

Pole-Zero cancellation

For some systems, we may find that it is possible to cancel matching factors from the numerator and denominator of the system function. If the factors represent poles that are not at zero, then although it may be theoretically supportable to cancel them, it is unlikely that they would match *exactly* in a real system. If we were to cancel them in that case, we might end up with a system model that was a particularly poor match for reality.³⁸

So, we will only consider canceling poles and zeros at zero. In this example:

$$H = \frac{\mathcal{R}}{\mathcal{R} - 1.6\mathcal{R}^2 + 0.63\mathcal{R}^3}$$

we can cancel \mathcal{R} , getting

$$H = \frac{1}{1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2}$$

Now, we find the poles by finding the roots of the equation

$$z^2 - 1.6z + 0.63 = 0$$

Repeated roots

In some cases, the equation in z will have repeated roots. For example, $z^2 - z + 0.25$, which has two roots at 0.5. In this case, the system has a repeated pole; it is still possible to perform an additive decomposition, but it is somewhat trickier. Ultimately, however, it is still the magnitude of the largest root that governs the long-term convergence properties of the system.

5.5.5 Superposition

The principle of superposition states that the response of a LTI system to a sum of input signals is the sum of the responses of that system to the components of the input. So, given a system with system function H , and input $X = X_1 + X_2$,

$$Y = HX = H(X_1 + X_2) = HX_1 + HX_2$$

So, although we have been concentrating on the unit sample response of systems, we can see that, to find the response of a system to any finite signal, we must simply sum the responses to each of the components of that signal; and those responses will simply be scaled, delayed copies of the response to the unit sample.

If Φ is a polynomial in \mathcal{R} and $X = \Phi\Delta$, then we can use what we know about the algebra of polynomials in \mathcal{R} (remembering that H is a ratio of polynomials in \mathcal{R}) to determine that

$$Y = HX = H(\Phi\Delta) = \Phi(H\Delta)$$

³⁸ Don't worry too much if this doesn't make sense to you...take 6.003 to learn more.

So, for example, if $X = (-3\mathcal{R}^2 + 20\mathcal{R}^4)\Delta$, then $Y = HX = H(-3\mathcal{R}^2 + 20\mathcal{R}^4)\Delta = (-3\mathcal{R}^2 + 20\mathcal{R}^4)H\Delta$. From this equation, it is easy to see that once we understand the unit-sample response of a system, we can see how it will respond to any finite input.

We might be interested in understanding how a system H responds to a *step* input signal. Let's just consider the basic step signal, U , defined as

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} .$$

We can express U as an infinite sum of increasingly delayed unit-sample signals:

$$\begin{aligned} U &= \Delta + \mathcal{R}\Delta + \mathcal{R}^2\Delta + \mathcal{R}^3\Delta + \cdots \\ &= (1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots)\Delta . \end{aligned}$$

The response of a system to U will therefore be an infinite sum of unit-sample responses. Let $Z = H\Delta$ be the unit-sample response of H . Then

$$\begin{aligned} HU &= H(1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots)\Delta \\ &= (1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots)H\Delta \\ &= (1 + \mathcal{R} + \mathcal{R}^2 + \mathcal{R}^3 + \cdots)Z \end{aligned}$$

Let's consider the case where H is a first-order system with a pole at p . Then,

$$z[n] = p^n$$

If $Y = HU$, then

$$\begin{aligned} y[n] &= z[n] + z[n-1] + z[n-2] + \cdots + z[0] \\ &= \sum_{k=0}^n z[k] \\ &= \sum_{k=0}^n p^k \end{aligned}$$

It's clear that, if $|p| > 1$ then $y[n]$ will grow without bound; but if $0 < p < 1$ then, as n goes to infinity, $y[n]$ will converge to $1/(1-p)$.

We won't study this in any further detail, but it's useful to understand that the basis of our analysis of systems applies very broadly across LTI systems and inputs.

5.6 Designing systems

Will eventually include a discussion of root-locus plots.

For now, see [section 5.8.3](#) for a discussion of picking k for the wall finding robot.

5.7 Summary of system behavior

Here is some terminology that will help us characterize the long-term behavior of systems.

- A signal is *transient* if it has finitely many non-zero samples.
- Otherwise, it is *persistent*.
- A signal is *bounded* if there exist upper and lower bound values such that the samples of the signal never exceed those bounds; otherwise it is *unbounded*.

Now, using those terms, here is what we can say about system behavior.

- A transient input to an acyclic (feed-forward) system results in a transient output.
- A transient input to a cyclic (feed-back) system results in a persistent output.
- The poles of a system are the roots of the denominator polynomial of the system function in $1/\mathcal{R}$.
- The dominant pole is the pole with the largest magnitude.
- If the dominant pole has magnitude > 1 , then in response to a bounded input, the output signal will be unbounded.
- If the dominant pole has magnitude < 1 , then in response to a bounded input, the output signal will be bounded; in response to a transient input, the output signal will converge to 0.
- If the dominant pole has magnitude 1, then in response to a bounded input, the output signal will be bounded; in response to a transient input, it will converge to some constant value.
- If the dominant pole is real and positive, then in response to a transient input, the signal will, after finitely many steps, begin to increase or decrease monotonically.
- If the dominant pole is real and negative, then in response to a transient input, the signal will, after finitely many steps, begin to alternate signs.
- If the dominant pole is complex, then in response to a transient input, the signal will, after finitely many steps, begin to be periodic, with a period of $2\pi/\Omega$, where Ω is the 'angle' of the pole.

5.8 Worked Examples

5.8.1 Specifying difference equations

Here are some examples of LTI systems and the way they would be described as difference equations. It's useful to pay careful attention to the specification of the coefficients. As a reminder, here's the general form.

$$y[n] = c_0 y[n-1] + c_1 y[n-2] + \dots + c_{k-1} y[n-k] \\ + d_0 x[n] + d_1 x[n-1] + \dots + d_j x[n-j]$$

- Output at step n is 3 times the input at step n :

$$y[n] = 3x[n]$$

dCoeffs: 3, cCoeffs: none

- Output at step n is the input at step $n-1$:

$$y[n] = x[n-1]$$

dCoeffs: 0, 1, cCoeffs: none

- Output at step n is 2 times the input at step $n - 2$:

$$y[n] = 2x[n - 2]$$

dCoeffs: 0, 0, 2, cCoeffs: none

- Output at step n is 2 times the output at step $n - 1$:

$$y[n] = 2y[n - 1]$$

dCoeffs: none, cCoeffs: 2

- Output at step n is the input at step $n - 1$ plus the output at step $n - 2$:

$$y[n] = x[n - 1] + y[n - 2]$$

dCoeffs: 0, 1, cCoeffs: 0, 1

5.8.2 Difference equations and block diagrams

Let H represent a system whose input is a signal X and whose output is a signal Y . The system H is defined by the following difference equations:

$$y[n] = x[n] + z[n]$$

$$z[n] = y[n - 1] + z[n - 1]$$

Start by finding an operator equation.

$$Y = X + Z$$

$$Z = \mathcal{R}Y + \mathcal{R}Z$$

Now, we use the second equation to find an expression for Z :

$$Z = \frac{\mathcal{R}Y}{1 - \mathcal{R}}$$

and substitute that into the first equation, and solve for Y :

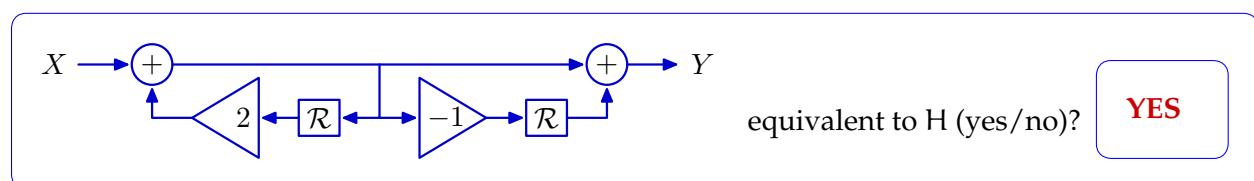
$$Y = X + \frac{\mathcal{R}Y}{1 - \mathcal{R}}$$

$$(1 - \mathcal{R})Y = (1 - \mathcal{R})X + \mathcal{R}Y$$

$$(1 - 2\mathcal{R})Y = (1 - \mathcal{R})X$$

$$Y = \frac{1 - \mathcal{R}}{1 - 2\mathcal{R}}X$$

Part a. Which of the following systems are valid representations of H ? (Remember that there can be multiple “equivalent” representations for a system.)



Find operator equations here; start by naming the signal that's flowing between the two adders. Let's call it W .

$$Y = W - \mathcal{R}W$$

$$W = X + 2\mathcal{R}W$$

Now, rewrite the first equation as

$$Y = (1 - \mathcal{R})W$$

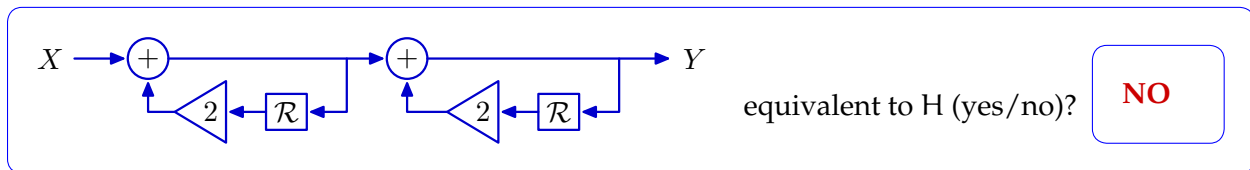
And the second one as

$$W = \frac{X}{1 - 2\mathcal{R}}$$

Then, combine them to get

$$\begin{aligned} Y &= (1 - \mathcal{R}) \frac{X}{1 - 2\mathcal{R}} \\ &= \frac{1 - \mathcal{R}}{1 - 2\mathcal{R}} X \end{aligned}$$

Showing that this system is the same as H.



This time, let's name the signal that's flowing between the two adders A . Now, we have equations

$$Y = A + 2\mathcal{R}Y$$

$$A = X + 2\mathcal{R}A$$

Rewrite the first equation as:

$$Y = \frac{A}{1 - 2\mathcal{R}}$$

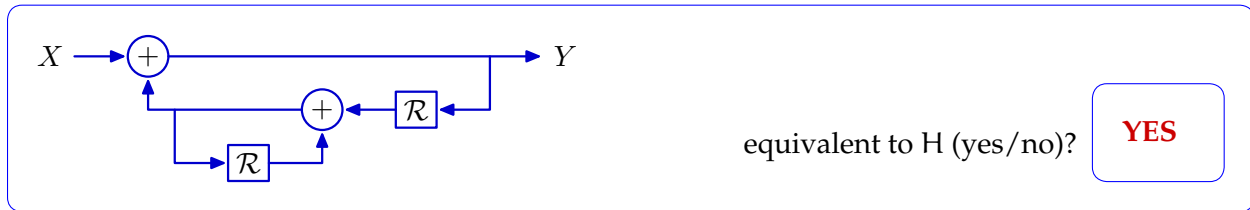
And the second as:

$$A = \frac{X}{1 - 2\mathcal{R}}$$

When we combine them, we get

$$Y = \frac{X}{(1 - 2\mathcal{R})(1 - 2\mathcal{R})}$$

which is not equivalent to the original system.



We'll name the signal flowing between the adders B. We get equations

$$Y = X + B$$

$$B = \mathcal{R}Y + \mathcal{R}B$$

Rewriting the second equation, we have

$$B = \frac{\mathcal{R}Y}{1 - \mathcal{R}}$$

Substituting into the first equation and solving, we get:

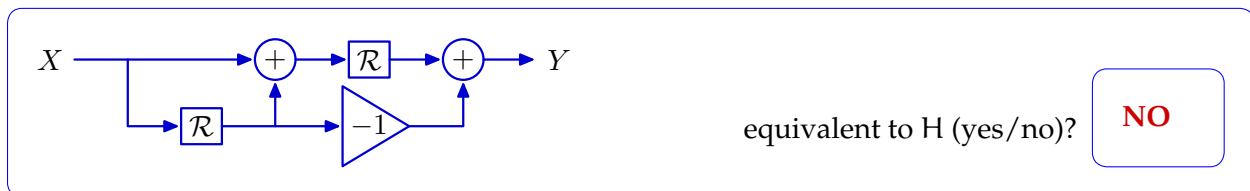
$$Y = X + \frac{\mathcal{R}Y}{1 - \mathcal{R}}$$

$$Y(1 - \mathcal{R}) = (1 - \mathcal{R})X + \mathcal{R}Y$$

$$Y(1 - 2\mathcal{R}) = (1 - \mathcal{R})X$$

$$Y = \frac{1 - \mathcal{R}}{1 - 2\mathcal{R}}X$$

So this system is equivalent to the original one.



Let's call the signal coming out of the first adder C. We get equations

$$Y = \mathcal{R}C - \mathcal{R}X$$

$$C = X + \mathcal{R}X$$

So

$$\begin{aligned} Y &= \mathcal{R}(X + \mathcal{R}X) - \mathcal{R}X \\ &= \mathcal{R}^2X \end{aligned}$$

which is not equivalent to the original system.

Part b. Assume that the system starts "at rest" and that the input signal X is the unit sample signal. Determine $y[3]$.

$$y[3] = \boxed{4}$$

n	$x[n]$	$z[n]$	$y[n]$
-1	0	0	0
0	1	0	1
1	0	1	1
2	0	2	2
2	0	4	4

Part c. Let p_o represent the dominant pole of H . Determine p_o .

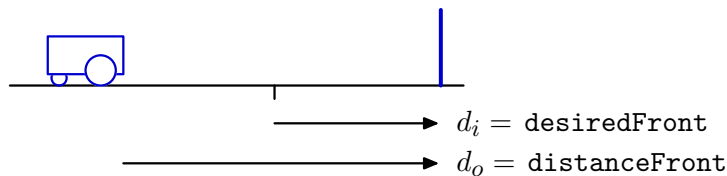
Enter p_o or **none** if there are no poles:

2

The denominator polynomial of the system function is $1 - 2\mathcal{R}$. This is directly in the form that exposes the pole as 2. But, we can also go step by step. We convert this into a polynomial in $z = 1/\mathcal{R}$ to get $1 - 2/z$. The roots of that equation are the same as the roots of $z - 2 = 0$; the single root is 2.

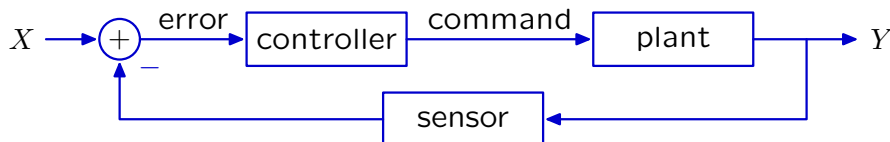
5.8.3 Wall finder

Let's think about a robot driving straight toward a wall. It has a distance sensor that allows it to observe the distance to the wall at time n , $D_s[n]$, and it desires to maintain a distance D_i from the wall.



The robot can execute velocity commands, and we program it to use this rule to set its forward velocity at time n , to be proportional to the error between D_i and its current sensor reading. In this section, we'll develop two different models of the system that includes the robot and the world around it, and try to see how we can use the models to understand how to select a controller.

In this block diagram, we will begin by considering the case in which the sensor box is just a wire; then consider what happens when it is a delay instead.



5.8.3.1 Version 1: No computational delay

In this first model, we'll assume that the sensor responds immediately, so that the robot's commanded velocity $v[n]$ depends *instantaneously* on its actual distance $d_o[n]$ from the wall. Of

course this dependence can't truly be instantaneous, but it might be quite small relative to the robot's 0.1-second cycle time and so might justifiably be ignored. So:

$$v[n] = k(d_i[n] - d_o[n]) ,$$

Although D_i will generally be a constant value, we'll allow for a more general case in which it may vary over time.

We can describe this system with the operator equation:

$$V = k(D_i - D_o) .$$

Now, we can think about the "plant"; that is, the relationship between the robot and the world. The distance of the robot to the wall changes at each time step depending on the robot's forward velocity and the length of the time steps. Let T be the length of time between velocity commands issued by the robot. Then we can describe the plant with the equation:

$$d_o[n] = d_o[n - 1] - Tv[n - 1] .$$

That is, the new distance from the wall is equal to the old distance from the wall, minus the robot's velocity towards the wall times the time interval of a step. In operator algebra terms, we have

$$D_o = \mathcal{R}D_o - T\mathcal{R}V$$

$$D_o - \mathcal{R}D_o = -T\mathcal{R}V$$

$$D_o(1 - \mathcal{R}) = -T\mathcal{R}V$$

Our overall system is a combination of the plant and the controller; we can combine the two equations to get

$$D_o(1 - \mathcal{R}) = -T\mathcal{R}V$$

$$D_o(1 - \mathcal{R}) = -T\mathcal{R}k(D_i - D_o)$$

$$D_o(1 - \mathcal{R} - T\mathcal{R}k) = T\mathcal{R}kD_i$$

$$D_o = \frac{-T\mathcal{R}k}{1 - (1 + T\mathcal{R}k)\mathcal{R}} D_i$$

We can solve for the poles analytically after substituting $1/z$ in for \mathcal{R} :

$$z - (1 + Tk) = 0 .$$

There is one pole at $1 + Tk$.

In order for the system to converge, we need

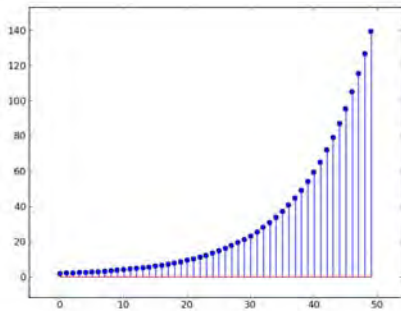
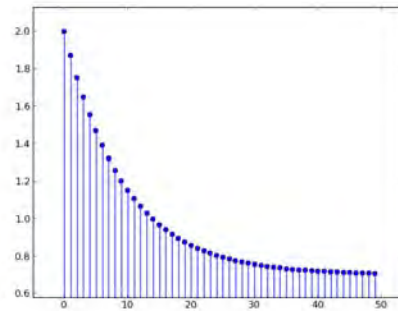
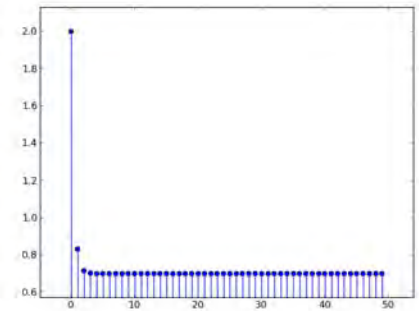
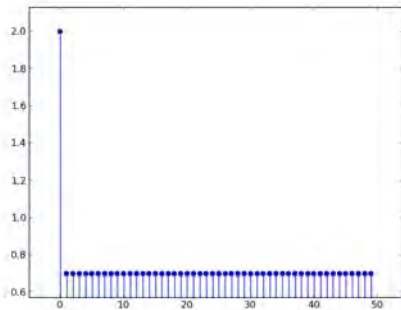
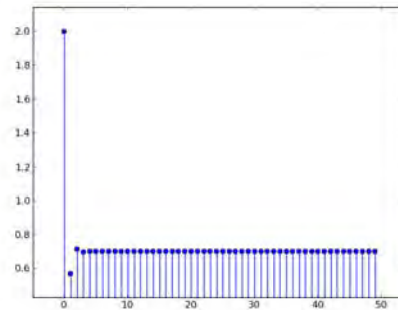
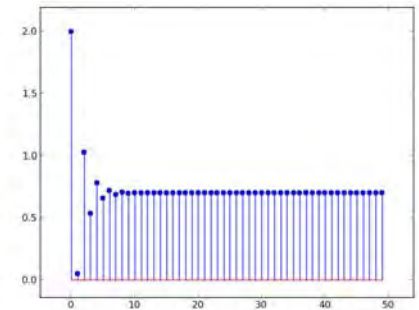
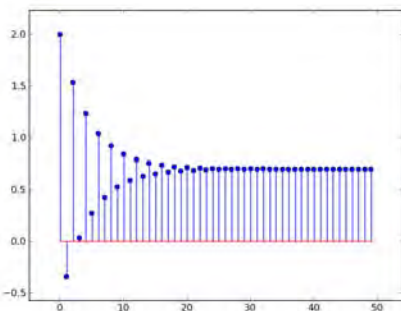
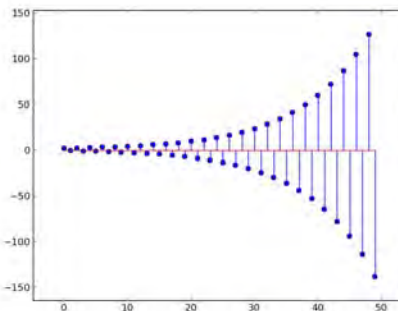
$$|1 + Tk| < 1$$

$$-1 < 1 + Tk < 1$$

$$-2 < Tk < 0$$

$$\frac{-2}{T} < k < 0$$

Assuming that $T = 0.1$ (which it is for our robots), then we can use this information to select k . Here are some plots of the evolution of the system, starting at distance 2.0, with an input $D_i = 0.7$.

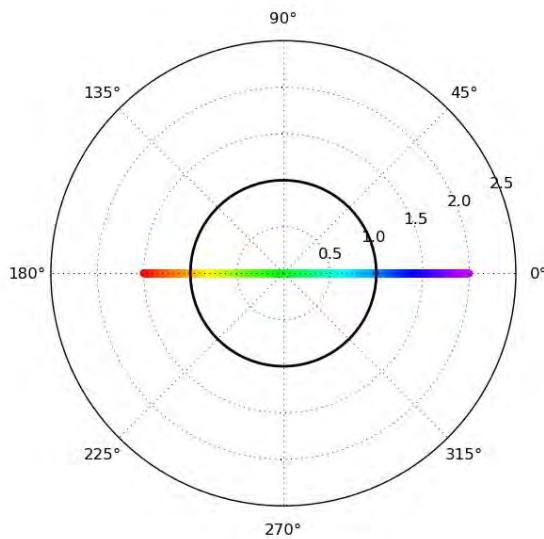
 $k = 1$  $k = -1$  $k = -9$  $k = -10$  $k = -11$  $k = -15$  $k = -18$  $k = -21$

Generally speaking, the closer the magnitude of the dominant pole is to 0, the faster the system will converge. For this system, $k = -10$ is the 'perfect' gain, which, in a perfect world, would make the robot jump, instantaneously, to the right place. This corresponds to having a pole of 0. (Note that, in this case, the system function degenerates into $\frac{D_o}{D_i} = \mathcal{R}$.)

Of course, in the real world, there will be error, which will cause the robot to overshoot or undershoot, and have to correct, etc. And, in the real world, we can't cause the robot to have an instantaneous increase (or decrease) in velocity, so we couldn't even approach the 'ideal' behavior of moving to the goal all in one step. Note that a positive gain causes the system to diverge as

does a gain less than -20 . And for gain values between -10 and -20 , it converges, but alternates sign.

Here is what is called a *root-locus* plot. It shows how the poles of the system (in this case just one pole) move in the complex plane as we vary parameter k . In this figure, we varied k from -25 to $+10$. The corresponding poles are plotted in different colors, starting with red corresponding to $k = -25$ through violet corresponding to $k = +10$. First, we can easily see that for any value of k , the pole is on the real line. Then, we observe that for the lowest values of k , the pole is outside the unit circle (drawn in dark black, any value inside it has magnitude less than 1), and will cause divergence, and for the highest values of k it is also outside the unit circle and will also diverge.



5.8.3.2 Model 2: A delay in the sensor

Now, we'll consider a version of the problem in which there is a delay in the sensor, so that the commanded velocity $v[n]$ depends on the distance at the *previous* time step, $d_o[n - 1]$, rather than on $d_o[n]$.

$$v[n] = k(d_i[n] - d_o[n - 1]) ,$$

We can describe this system with the operator equation:

$$V = k(D_i - \mathcal{R}D_o) .$$

We'll leave the model of the plant as it was above (but note that in a real situation, there might be additional delays in the plant, as well as, or instead of, the delay we're modeling in the controller).

$$D_o(1 - \mathcal{R}) = -\mathcal{R}V .$$

Our overall system is a combination of the plant and the controller; we can combine the two equations to get

$$D_o(1 - \mathcal{R}) = -\mathcal{T}\mathcal{R}V$$

$$D_o(1 - \mathcal{R}) = -\mathcal{T}\mathcal{R}k(D_i - \mathcal{R}D_o)$$

$$D_o(1 - \mathcal{R} - \mathcal{T}k\mathcal{R}^2) = -\mathcal{T}k\mathcal{R}D_i$$

$$D_o = \frac{-\mathcal{T}k\mathcal{R}}{1 - \mathcal{R} - \mathcal{T}k\mathcal{R}^2}D_i$$

We can solve for the poles analytically after substituting $1/z$ for \mathcal{R} :

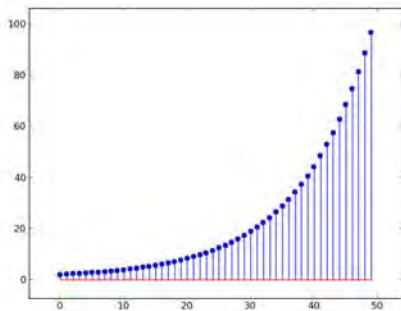
$$z^2 - z - \mathcal{T}k = 0 .$$

We find that the roots of this polynomial are

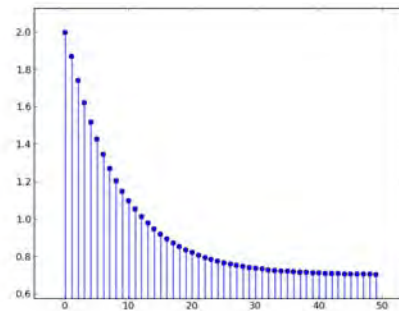
$$\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4k\mathcal{T}} .$$

To determine the behavior of the system for some value of k , you can plug it into this formula and see what the values are. Remember that the pole with the largest magnitude will govern the long-term behavior of the system.

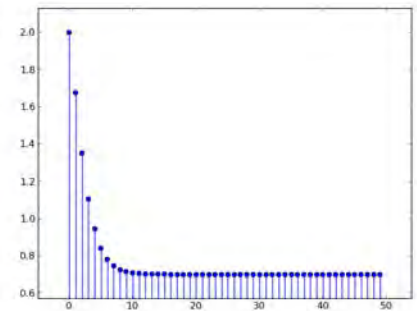
Here are some plots of the evolution of the system, starting at distance 2.0, with an input $D_i = 0.7$.



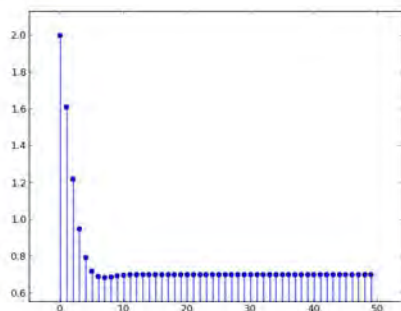
$k = 1$



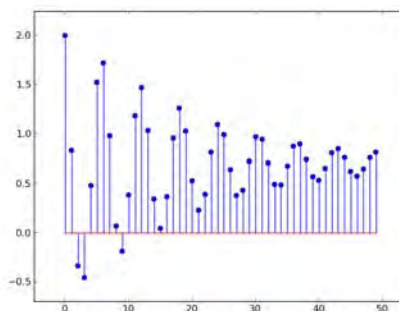
$k = -1$



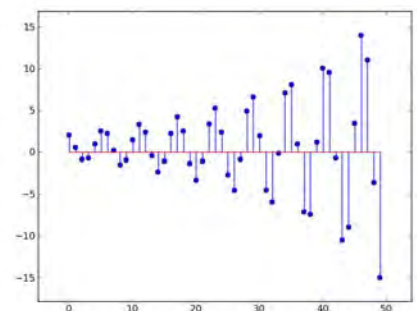
$k = -2.5$



$k = -3$



$k = -9$



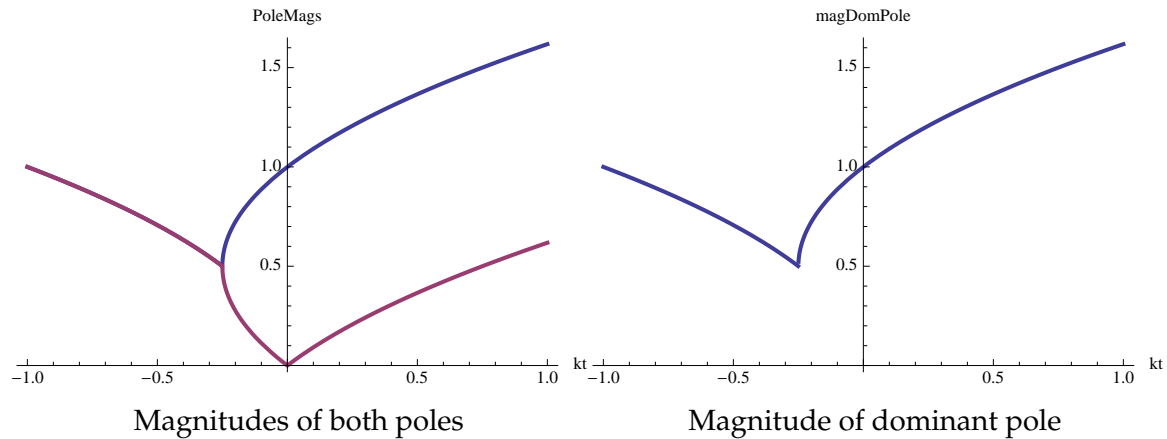
$k = -11$

$\mathcal{T} = 0.1$, the system is monotonically divergent for $k > 0$, monotonically convergent for $-2.5 < k < 0$, converges but oscillates for $-10 < k < -2.5$, and diverges while oscillating for $k < -10$.

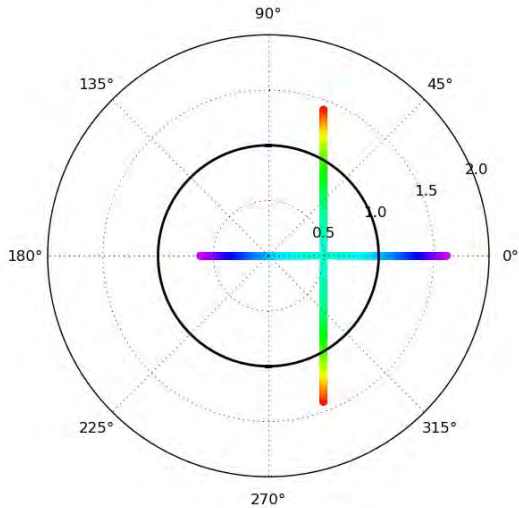
Below, on the left is

$$\left| \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4kT} \right|$$

that is, the magnitudes of the two poles as a function of kT . Note that, on the left branch, there are still two poles, but they are a complex conjugate pair with the same magnitude. On the right is the maximum of the pole magnitudes. We can see that it is minimized at $kT = -0.25$.



Here is a *root-locus* plot for this system. It shows how the poles of the system move in the complex plane as we vary parameter k . In this figure, we varied k from -20 to $+10$. The corresponding poles are plotted in different colors, starting with red corresponding to $k = -20$ through violet corresponding to $k = +10$. There are two red points, corresponding to the conjugate pair of complex poles arising when $k = -20$. This system is unstable, because the magnitudes of those complex poles are greater than 1 (outside the unit circle). As we increase k , these poles move down; that is, their real part stays constant and the imaginary part decreases, until we reach complex poles (greenish on the plot) that are stable. Finally, these two poles meet on the real line: one 'turns' right as k increases and one 'turns' left. Now, we have two real poles, one of which is closer to the unit circle than the other. As soon as the positive real pole is equal to 1 then the system will begin to turn unstable again (even though the other pole is still inside the unit circle). This, for $k = 10$, the system is unstable (eventually diverging monotonically) because the largest pole is positive and outside the unit circle.



5.8.4 Cool difference equations

Newton's law of cooling states that: the rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of its surroundings.

We can model this process in discrete time, by assuming that the change in an object's temperature from one time step to the next is proportional to the difference (on the earlier step) between the temperature of the object and the temperature of the environment, as well as to the length of the time step.

Let

- $o[n]$ be temperature of object
- $s[n]$ be temperature of environment
- T be the duration of a time step
- K be the constant of proportionality

Part a. Write a difference equation for Newton's law of cooling. Be sure the signs are such that the temperature of the object will eventually equilibrate with that of the environment.

$$o[n] = o[n - 1] + TK(s[n - 1] - o[n - 1])$$

Part b. Write the system function corresponding to this equation (show your work):

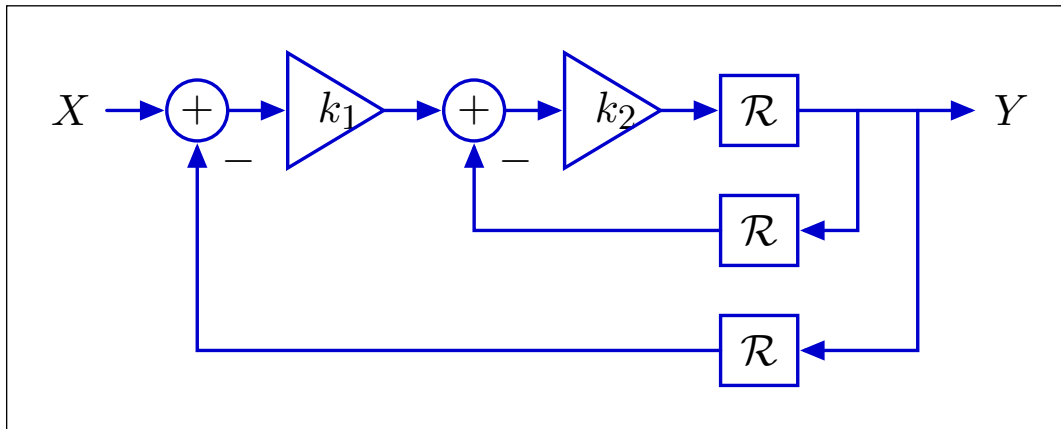
$$H = \frac{O}{S} = \frac{KT\mathcal{R}}{1 - (1 - KT)\mathcal{R}}$$

First, convert the difference equation to an operator equation, then solve for O in terms of S .

$$\begin{aligned}
 O &= \mathcal{R}O + KT(\mathcal{R}S - \mathcal{R}O) \\
 O - \mathcal{R}O + KT\mathcal{R}O &= KTR S \\
 O(1 - (1 - KT)\mathcal{R}) &= KTR S \\
 \frac{O}{S} &= \frac{KT\mathcal{R}}{1 - (1 - KT)\mathcal{R}}
 \end{aligned}$$

5.8.5 Modeling

Consider the following system:



Part a. Write the system function:

$$H = \frac{Y}{X} = \frac{k_1 k_2 \mathcal{R}}{1 + k_2 \mathcal{R}^2 (1 + k_1)}$$

We start by naming some of the internal signals. Let Z be the output of the gain of k_1 and W be the output of the gain of k_2 . Then we can write the following set of operator equations:

$$\begin{aligned}
 Z &= k_1(X - \mathcal{R}Y) \\
 W &= k_2(Z - \mathcal{R}Y) \\
 Y &= \mathcal{R}W
 \end{aligned}$$

Eliminating Z and W , we have:

$$\begin{aligned}
 Y &= \mathcal{R}k_2(Z - \mathcal{R}Y) \\
 &= \mathcal{R}k_2(k_1(X - \mathcal{R}Y) - \mathcal{R}Y) \\
 &= k_1 k_2 \mathcal{R}X - k_1 k_2 \mathcal{R}^2 Y - k_2 \mathcal{R}^2 Y
 \end{aligned}$$

Reorganizing terms, we have

$$Y + k_2(1 + k_1)\mathcal{R}^2 Y = k_1 k_2 \mathcal{R}X$$

which leads us to the answer.

Part b.

Let $k_1 = 1$ and $k_2 = -2$. Assume that the system starts “at rest” (all signals are zero) and that the input signal X is the unit sample signal. Determine $y[0]$ through $y[3]$.

$$y[0] = 0$$

$$y[1] = -2$$

$$y[2] = 0$$

$$y[3] = -8$$

First, we write the difference equation:

$$y[n] = -2x[n-1] + 4y[n-2]$$

Then we can calculate the values step by step.

- $y[0] = -2x[-1] + 4y[-1] = 0 + 0 = 0$
- $y[1] = -2x[0] + 4y[-1] = -2 + 0 = -2$
- $y[2] = -2x[1] + 4y[0] = 0 + 0 = 0$
- $y[3] = -2x[2] + 4y[1] = 0 + 4 \cdot (-2) = -8$

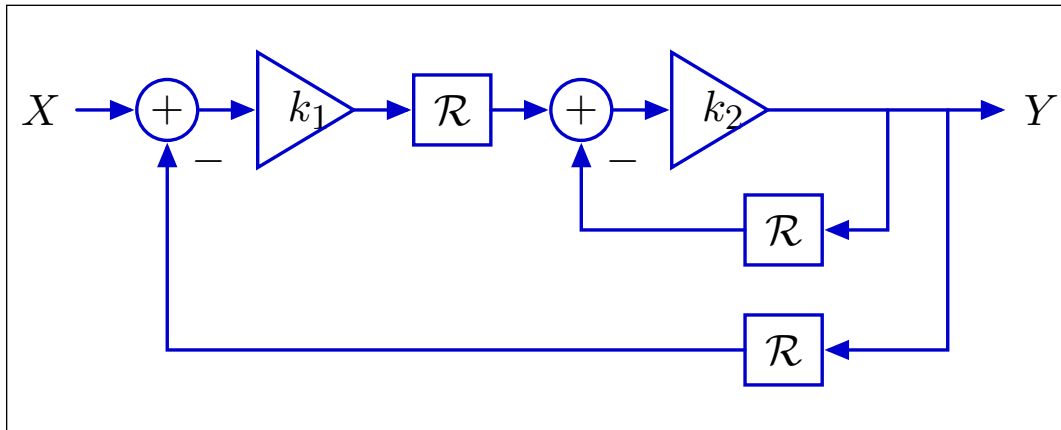
Part c. Let $k_1 = 1$ and $k_2 = -2$, determine the poles of H .

Enter poles or **none** if there are no poles:

2, -2

For these k values, the denominator polynomial is $1 - 4\mathcal{R}^2$. So, we need to find the roots of the polynomial $z^2 - 4$, which are ± 2 .

Part d. For each of the systems below indicate whether the system is equivalent to this one:



Equivalent to H (yes/no)?

No

Let the output of the gain of k_1 be W . Then we can write the following set of operator equations:

$$W = k_1(X - \mathcal{R}Y)$$

$$Y = k_2(\mathcal{R}W - \mathcal{R}Y)$$

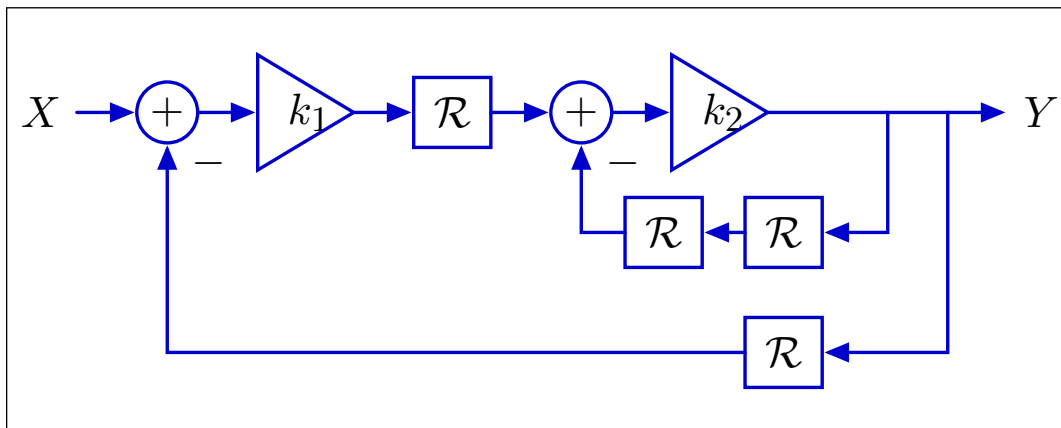
Eliminating W , we have:

$$Y = k_2(\mathcal{R}W - \mathcal{R}Y)$$

$$= k_2(\mathcal{R}k_1(X - \mathcal{R}Y) - \mathcal{R}Y)$$

$$= k_1k_2\mathcal{R}X - k_1k_2\mathcal{R}^2Y - k_2\mathcal{R}Y$$

which is not equal to the operator equation for the original system.



Equivalent to H (yes/no)?

Yes

Let the output of the gain of k_1 be W . Then we can write the following set of operator equations:

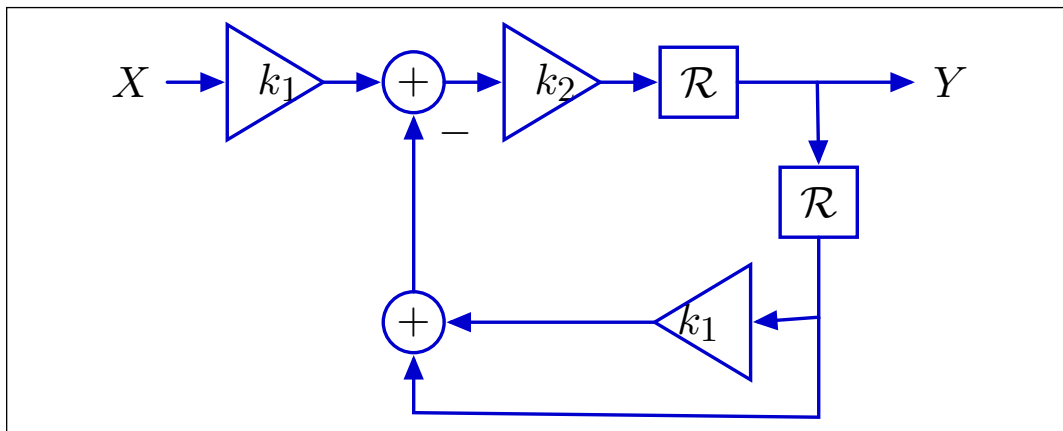
$$W = k_1(X - \mathcal{R}Y)$$

$$Y = k_2(\mathcal{R}W - \mathcal{R}^2Y)$$

Eliminating W , we have:

$$\begin{aligned} Y &= k_2(\mathcal{R}W - \mathcal{R}^2Y) \\ &= k_2(\mathcal{R}k_1(X - \mathcal{R}Y) - \mathcal{R}^2Y) \\ &= k_1k_2\mathcal{R}X - k_1k_2\mathcal{R}^2Y - k_2\mathcal{R}^2Y \end{aligned}$$

which is equal to the operator equation for the original system.



Equivalent to H (yes/no)?

Yes

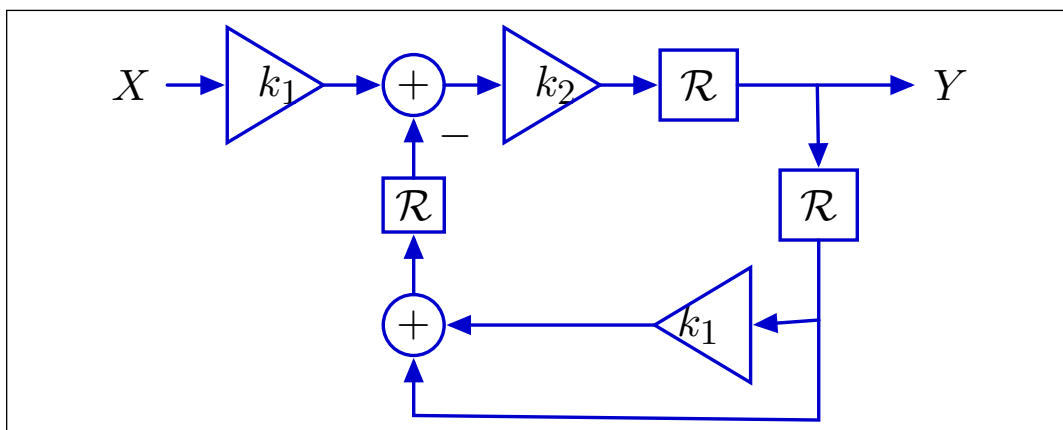
Let the output of the gain of k_2 be W . Then we can write the following set of operator equations:

$$\begin{aligned} W &= k_2(k_1X - (1 + k_1)\mathcal{R}Y) \\ Y &= \mathcal{R}W \end{aligned}$$

Eliminating W , we have:

$$\begin{aligned} Y &= \mathcal{R}k_2(k_1X - (1 + k_1)\mathcal{R}Y) \\ &= k_1k_2\mathcal{R}X - k_1k_2\mathcal{R}^2Y - k_2\mathcal{R}^2Y \end{aligned}$$

which is equal to the operator equation for the original system.



Equivalent to H (yes/no)?

No

This is like the previous system, but with an extra delay in the feedback path, so it cannot be equivalent to the original system.

5.8.6 SM to DE

Here is the definition of a class of state machines:

```
class Thing(SM):
    startState = [0, 0, 0, 0]
    def getNextValues(self, state, inp):
        result = state[0] * 2 + state[2] * 3
        newState = [state[1], result, state[3], inp]
        return (newState, result)
```

1. What is the result of evaluating

```
Thing().transduce([1, 2, 0, 1, 3])
```

```
[0, 0, 3, 6, 6]
```

2. The state machine above describes the behavior of an LTI system starting at rest. Write a difference equation that describes the same system as the state machine.

$$y[n] = 2y[n-2] + 3x[n-2]$$

The important thing to see here is that the values in the state are $(y[n-2], y[n-1], x[n-2], x[n-1])$, so that the output is $2y[n-2] + 3x[n-2]$.

5.8.7 On the Verge

For each difference equation below, say whether, for a unit sample input signal:

- the output of the system it describes will diverge or not,
- the output of the system it describes (a) will always be positive, (b) will alternate between positive and negative, or (c) will have a different pattern of oscillation

- 1.

$$10y[n] - y[n-1] = 8x[n-3]$$

diverge? Yes or No

No

positive/alternate/oscillate

Positive

We first write the operator equation:

$$10Y - \mathcal{R}Y = 8\mathcal{R}^3X$$

And the system function

$$\frac{Y}{X} = \frac{8\mathcal{R}^3}{10 - \mathcal{R}}$$

Find the root of the polynomial in $z = 1/\mathcal{R}$:

$$10z - 1 = 0$$

$$z = 0.1$$

The single pole is at 0.1. It is positive, so for a unit-sample input, the output will always be positive (assuming it starts at rest). It has magnitude less than 1, so it will converge.

2.

$$y[n] = -y[n-1] - 10y[n-2] + x[n]$$

diverge? Yes or No

Yes

positive/alternate/oscillate

Oscillates

We first write the operator equation:

$$Y + \mathcal{R}Y + 5\mathcal{R}^2Y = X$$

And the system function

$$\frac{Y}{X} = \frac{1}{1 + \mathcal{R} + 10\mathcal{R}^2}$$

Find the roots of the polynomial in $z = 1/\mathcal{R}$:

$$Z^2 + Z + 10 = 0$$

$$Z = \frac{-1 \pm \sqrt{1 - 100}}{2}$$

$$Z = 0.5 \pm 4.97j$$

The magnitude of the poles is 5, which is greater than 1, so it diverges. The poles are complex, so the output will oscillate.

3.

$$y[n] = -0.6y[n-1] + .16y[n-2] - 0.1x[n-1]$$

diverge? Yes or No

No

positive/alternate/oscillate

Oscillates

We first write the operator equation:

$$Y + 0.6\mathcal{R}Y - .16\mathcal{R}^2Y = -0.1\mathcal{R}X$$

And the system function

$$\frac{Y}{X} = \frac{-0.1\mathcal{R}}{1 + 0.6\mathcal{R} - 0.16\mathcal{R}^2}$$

Find the roots of the polynomial in $z = 1/\mathcal{R}$:

$$Z^2 + 0.6Z - 0.16 = 0$$

$$Z = \frac{-0.6 \pm \sqrt{.36 + .64}}{2}$$

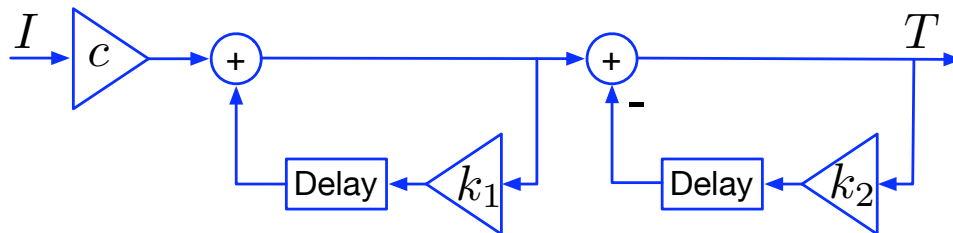
$$Z = (-0.8, 0.2)$$

The dominant pole is -0.8 , because it has the largest magnitude. Its magnitude is less than 1, so the system will converge. The pole is negative, so the system will alternate positive and negative signs.

5.8.8 What's Cooking?

Sous vide cooking involves cooking food at a very precise, fixed temperature T (typically, low enough to keep it moist, but high enough to kill any pathogens). In this problem, we model the behavior of the heater and water bath used for such cooking. Let I be the current going into the heater, and c be the proportionality constant such that Ic is the rate of heat input.

The system is thus described by the following diagram:



1. a. Give the system function:

$$\frac{c}{(1 - k_1\mathcal{R})(1 + k_2\mathcal{R})}$$

If we name the signal coming out of the first adder W , then we have operator equations

$$W = cI + k_1\mathcal{R}W$$

$$T = W - k_2\mathcal{R}T$$

Solving, we get

$$W - k_1\mathcal{R}W = cI$$

$$W(1 - k_1\mathcal{R}) = cI$$

$$W = \frac{cI}{1 - k_1\mathcal{R}}$$

$$T = \frac{cI}{1 - k_1\mathcal{R}} - k_2\mathcal{R}T$$

$$T(1 - k_1\mathcal{R}) = cI - k_2\mathcal{R}T(1 - k_1\mathcal{R})$$

$$T - k_1\mathcal{R}T + k_2\mathcal{R}T - k_1k_2\mathcal{R}^2T = cI$$

$$T(1 - k_1\mathcal{R})(1 + k_2\mathcal{R}) = cI$$

$$\frac{T}{I} = \frac{c}{(1 - k_1\mathcal{R})(1 + k_2\mathcal{R})}$$

b. Give a difference equation for the system: $t[n] =$

$$t[n] = (k_1 - k_2)t[n - 1] + k_1k_2t[n - 2] + ci[n]$$

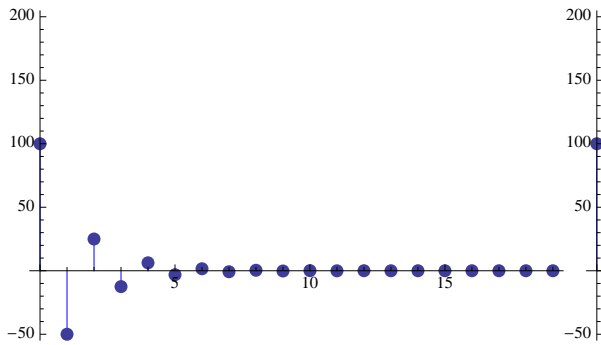
Starting with this form of the operator equation, taken from the derivation above, and then rearranging terms

$$T(1 - k_1\mathcal{R}) = cI - k_2\mathcal{R}T(1 - k_1\mathcal{R})$$

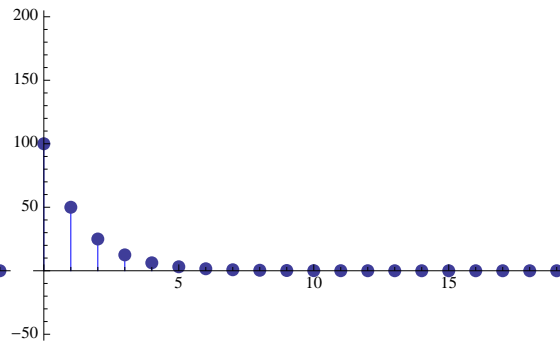
$$T = cI + k_1\mathcal{R}T - k_2\mathcal{R}T(1 - k_1\mathcal{R})$$

We get an equation that's easy to convert to the difference equation above.

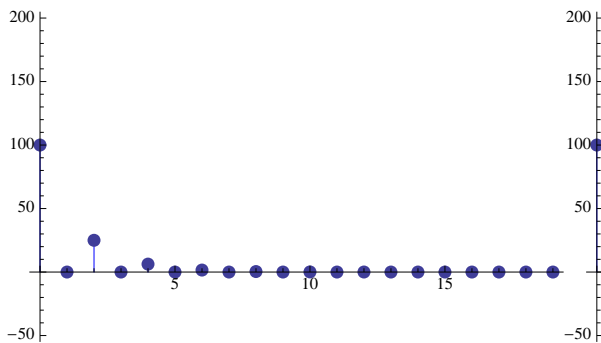
- Let the system start at rest (all signals are zero). Suppose $I[0] = 100$ and $I[n] = 0$ for $n > 0$. Here are plots of $T[n]$ as a function of n for this system for $c = 1$ and different values of k_1 and k_2 .



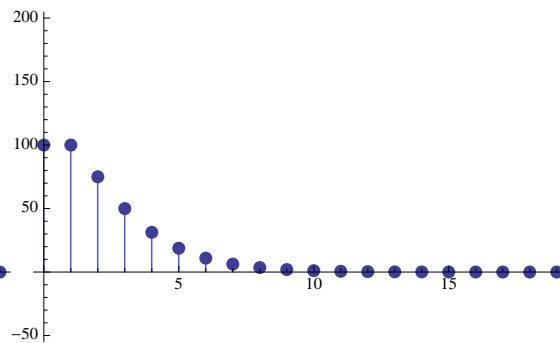
a



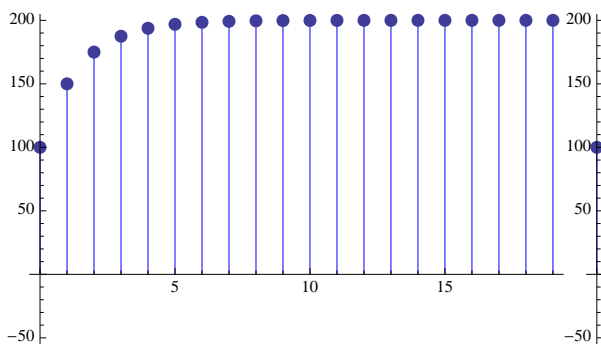
b



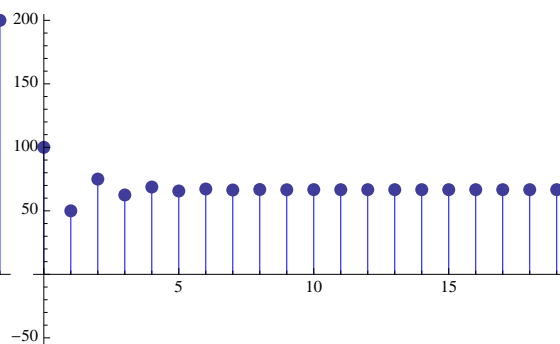
c



d



e



f

- a. Which of the plots above corresponds to $k_1 = 0.5$ and $k_2 = 0$?

Circle all correct answers: a b c d e f none

The denominator as written above is already factored, and so we know that the poles are k_1 and $-k_2$. So, with these values for k_1 and k_2 , there is a single pole at 0.5. So, we know that the system will converge monotonically, and that the each magnitude will be 0.5 of the magnitude on the previous step. The only plot that has this property is b.

- b. Which of the plots above corresponds to $k_1 = 1$ and $k_2 = 0.5$?

Circle all correct answers: a b c d e f none

In this case, the poles are 1 and -0.5 . With the dominant pole at 1, we expect the system to neither converge to 0, nor to diverge. The other pole at -0.5 will generate a component

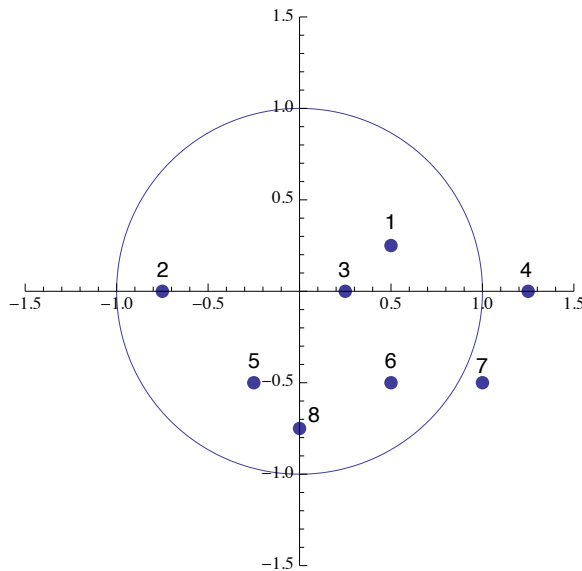
with alternating signs, but whose magnitude dies away over time. The graph in **f** shows a signal that is the sum of a long-term constant signal and a signal that is converging to zero with alternating signs.

3. Let $k_1 = 0.5$, $k_2 = 3$, and $c = 1$. Determine the poles of H , or **none** if there are no poles.

Looking at the factored form of the denominator, we can easily see that the poles are at k_1 and $-k_2$. If you didn't see that factored form, then you could explicitly see that the poles are roots of the equation $z^2 + (k_2 - k_1)z - k_1k_2 = 0$, which in this case is $z^2 + 2.5z - 1.5 = 0$. We can use the quadratic formula to find that the roots are at 0.5 and -3 .

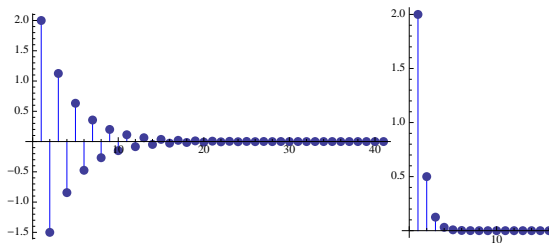
5.8.9 Pole Position

Consider eight poles located at the following locations in the z plane. The plots below show the unit-sample responses of eight linear, time-invariant systems. Match them with the dominant pole for each system (remember that the system may have more than one pole).

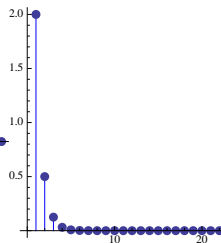


- A. This signal is alternating in sign and converging. Each magnitude is about 0.75 of the magnitude of the previous sample. So, we'd expect the dominant pole to be about -0.75 , which corresponds to pole 2 on the plot.
- B. This signal is converging monotonically; each sample is about 0.25 of the previous sample value. So, we expect a dominant pole of about 0.25. This corresponds to pole 3 on the plot.
- C. This signal is diverging monotonically; each sample is about 1.25 of the previous sample value. So, we expect a dominant pole of about 1.25. This corresponds to pole 4 on the plot.
- D. This signal is converging; it is neither monotonic, nor alternating in sign. It is oscillating with a period of 4, so we expect the dominant poles to be complex, with angle $\pm 2\pi/4 = \pm\pi/2$. We can see that the magnitude is about 0.3 of the previous magnitude after 4 steps, which means that the magnitude of the pole is about 0.75 (because $0.75^4 = 0.316$). Pole 8 has angle $-\pi/2$ and magnitude 0.75.

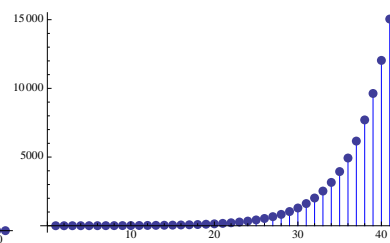
- E.** This signal is converging and oscillating. The period seems to be 8. So, we'd expect a pole at angle $\pm\pi/4$. The magnitude is a bit tricky to estimate. It seems to get from 2 to about 0.1 in 8 steps, so it's something like 0.7. That corresponds well to pole 6.
- F.** This signal is converging and oscillating. The period seems to be something like 3, and the magnitude even smaller than the previous two. Pole 5 has these characteristics.
- G.** This signal is diverging and oscillating. It seems to have a period of about 12, which would mean an angle of $\pm\pi/6$. The only pole on our picture with magnitude greater than 1 is at the correct angle, and so it must be 7.
- H.** Finally, we have a signal that converges and oscillates. The period seems to be 10 or 12 and the rate fairly fast; pole 1 has a smaller magnitude than pole 6, and this converges faster than E, so this must be pole 1.



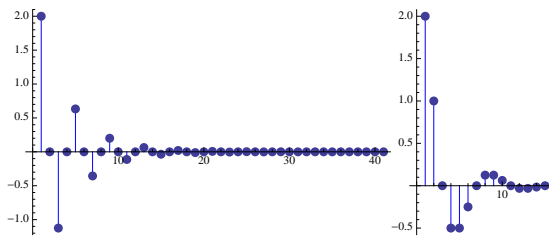
$$A = 2$$



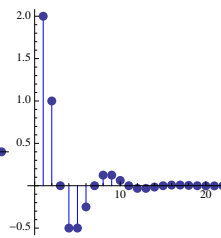
$$B = 3$$



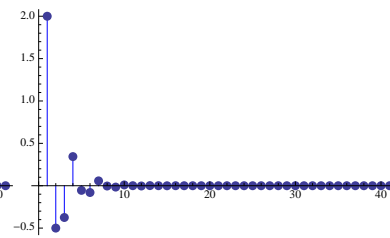
$$C = 4$$



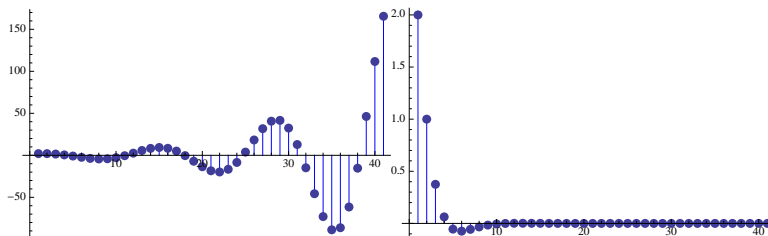
$$D = 8$$



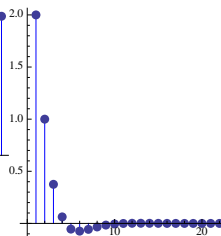
$$E = 6$$



$$F = 5$$



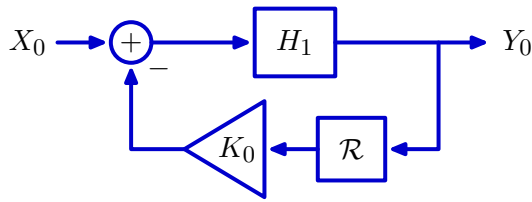
$$G = 7$$



$$H = 1$$

5.8.10 System functions

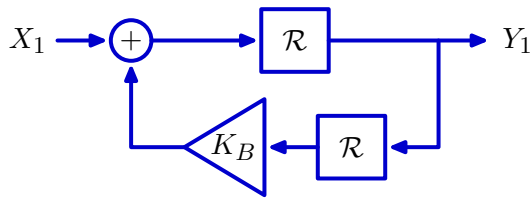
Let H_1 represent a subsystem that is part of the larger system shown below.



The system function for the larger system can be written as

$$H_0 = \frac{Y_0}{X_0} = \frac{H_1}{1 + K_0 R H_1}.$$

Assume that $H_1 = H_{1B} = \frac{Y_1}{X_1}$ as shown below.



What is the system function for H_{1B} ?

$$H_{1B} = \frac{\mathcal{R}}{1 - K_B \mathcal{R}^2}$$

Determine the system function H_0 for the larger system when $H_1 = H_{1B}$.

$$H_0 = \frac{\mathcal{R}}{1 + (K_0 - K_B) \mathcal{R}^2}$$

Under what conditions on K_0 and K_B is this system stable? Explain.

There are poles at $z = \pm\sqrt{K_B - K_0}$. To be stable, the poles should all have magnitudes less than 1. Thus the system is stable if $|K_B - K_0| < 1$.

Under what conditions on K_0 and K_B does the unit-sample response decay monotonically? Explain.

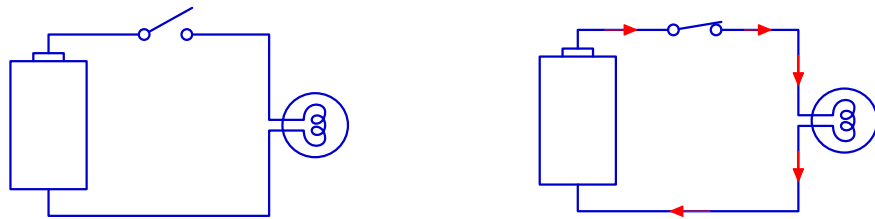
None. For monotonic convergence, **both** poles must have magnitudes between 0 and 1 (since there are two poles of equal magnitude). If $K_B < K_0$ then the poles have non-zero imaginary parts, and the response oscillates. If $K_B > K_0$ then one pole is on the positive real axis and one is on the negative real axis. The pole on the negative real axis causes the unit sample response to alternate. Thus there are no values of K_0 and K_B for which there is monotonic decay.

Chapter 6

Circuits

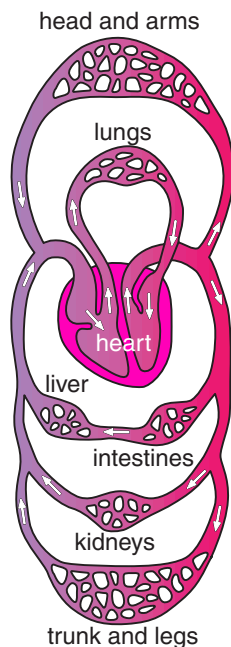
6.1 What is a circuit?

In conventional English, a circuit is a path or route that starts at one place and ultimately returns to that same place. The engineering sense of the word is similar. In electronics, a circuit is a closed path through which electrical currents can flow. The flow of electrical current through a flashlight illustrates the idea.



Three parts of the flashlight comprise the circuit: the battery, which supplies power, a bulb, which produces light when connected to the battery, and a switch, which can either be on (connected) or off (disconnected). The important feature of this circuit is that electrical currents flow through it in a loop, as illustrated in the right part of the figure above.

The rules that govern the flow of electrical current are similar to the rules that govern the flow of certain types of fluids; and it is often helpful to draw analogies between these flows. The following figure illustrates the flow of blood through the human circulatory system.



The right side (from the person’s viewpoint) of the heart pumps blood through the lungs and into the left side of the heart, where it is then pumped throughout the body. The heart and associated network of arteries, capillaries, and veins can be thought of and analyzed as a circuit. As with the flow of electrical current in the flashlight example, blood flows through the circulatory system in loops: it starts at one place and ultimately returns to that same place.

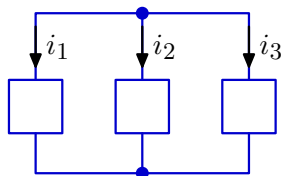
As we learn about circuits, we will point out analogies to fluid dynamics for two reasons. First, we all have developed some intuition for the flow of fluid as a result of our interactions with fluids as a part of everyday experience. We can leverage the similarity between fluid flow and the flow of electrical current to increase our intuition for electrical circuits. Second, the analogy between electrical circuits and fluid dynamics is a simple example of the use of circuits to make models of other phenomena. Such models are widely used in acoustics, hydrodynamics, cellular biophysics, systems biology, neurophysiology, and many other fields.

Electrical circuits are made up of components, such as resistors, capacitors, inductors, and transistors, connected together by wires. You can make arbitrarily amazing, complicated devices by hooking these things up in different ways, but in order to help with analysis and design of circuits, we need a systematic way of understanding how they work.

As usual, we can’t comprehend the whole thing at once: it’s too hard to analyze the system at the level of individual components, so, again, we’re going to build a model in terms of primitives, means of combination, and means of abstraction. The primitives will be the basic components, such as resistors and op-amps; the means of combination is wiring the primitives together into circuits. We’ll find that abstraction in circuits is different than in software or LTI systems. You can’t think of a circuit as “computing” the voltages on its wires: if you connect it to another circuit, then the voltages are very likely to be different. However, you *can* think of a circuit as enforcing a *constraint* on the voltages and currents that enter and exit it; this constraint will remain true, no matter what else you connect to the circuit.

6.1.1 Electrical circuits

Circuits are made up of *elements* connected by *nodes*. The following circuit has three elements, each represented with a box.



There are two nodes, each indicated by a dot. In an electrical circuit nodes can be thought of as wires³⁹ that connect a component.

Voltage is a difference in electrical potential between two different points in a circuit. We will often pick some point in a circuit and say that it is “ground” or has voltage 0. Now, every other

³⁹ Wire is typically made from metals through which electrons can flow to produce electrical currents. Generally, the strong forces of repulsion between electrons prevent their accumulation in wires. However, thinking about nodes as wires is only an approximation, because physical wires can have more complicated electrical properties.

point has a voltage defined with respect to ground. Because voltage is a relative concept, we could pick any point in the circuit and call it ground, and we would still get the same results.

Current is a flow of electrical charge through a path in the circuit. A positive current in a direction is generated by negative charges (electrons) moving in the opposite direction.

We'll start by considering a simple set of components that have two terminals (connection points into the circuit). Each component has a current flowing through it, and a voltage difference across its terminals. Each type of component has some special characteristics that govern the relationship between its voltage and current. We will restrict our attention, in this course, to components that exert a linear constraint on their current and voltage.

One way to model circuits is in terms of their *dynamics*. That is, to think of the currents and voltages in the system and how they change over time. Such systems are appropriately modeled, in fact, using differential or difference equations, connected together into complex systems, as we saw in the last couple of weeks. But for many purposes, the dynamic properties of a circuit converge quickly, and we can directly model the equilibrium state that they will converge to. The combination of the behavior of the components and the structure in which they're connected provides a set of constraints on the equilibrium state of the circuit. We'll work through this view by starting with the constraints that come from the structure, and then examining constraints for three simple types of components.

6.2 Conservation laws

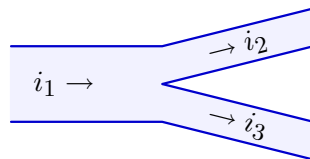
Conservation laws describe properties of a circuit that must hold, no matter what the particular elements are. There are two fundamental conservation laws, one that relates to currents and one that relates to voltages.

6.2.1 Kirchoff's current law

Fluids can be roughly classified as compressible or incompressible. Air is compressible. Doubling the pressure on a volume of air will decrease the volume by a factor of two. By contrast, water is nearly incompressible. Doubling the pressure on a volume of water (from 1 atmosphere to 2) will decrease the volume by less than one part in a billion.

The laws that govern incompressible fluid flows are relatively simple. The net flow of incompressible fluid into a region of fixed volume must be zero. It follows that if there is flow into one part of a fixed volume, then there must be an equal flow out of the other part.

As an example, consider the flow of water through a branching point, as shown below.

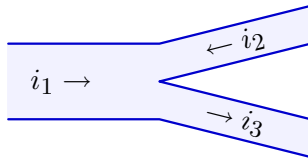


Let i_1 represent the flow into the branching point and i_2 and i_3 represent the flows out of it. Assume that i_1 , i_2 , and i_3 have the units of volume per unit time (e.g., m^3/s).

Example 1. What is the relation between i_1 , i_2 , and i_3 ?

In this example, $i_1 = i_2 + i_3$.

Example 2. What if i_2 represented flow in the opposite direction, as shown below?

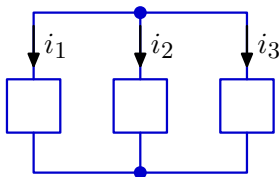


Now $i_1 + i_2 = i_3$ or equivalently $i_1 = -i_2 + i_3$.

Changing the direction of a flow variable is equivalent to changing its sign. For that reason, we think about flow variables as *signed* variables. The direction associated with the variable represents the direction that results when the variable is positive. The flow is in the opposite direction if the variable is negative. Thus, the direction that we associate with a flow is arbitrary. If we happen to associate the one direction, the flow variable will be positive. If we happen to associate the opposite direction, the flow variable will be negative.

The laws for the flow of electrical current are similar to those for the flow of an incompressible fluid. The net flow of electrical current into a **node** must be zero.

The following circuit has three elements, each represented with a box.

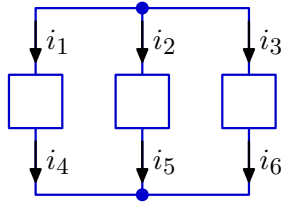


There are two nodes, each indicated by a dot. The net current into or out of each of these nodes is zero. Therefore $i_1 + i_2 + i_3 = 0$.

Similar current relations apply for all nodes in a circuit. Such relations are called **Kirchoff's Current Laws (KCL)** and represent the first of two fundamental conservation laws for circuits.

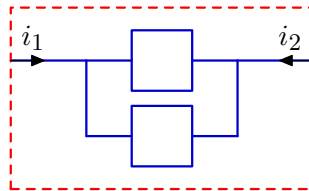
Kirchoff's Current Law: the sum of the currents that flow into a node is zero.

Electrical currents cannot accumulate in elements, so the current that flows into a circuit element must equal to the current that flows out.

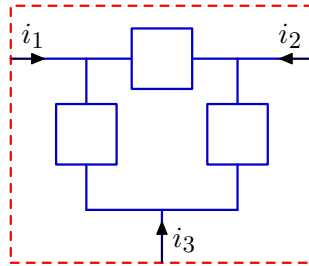


Therefore $i_1 = i_4$, $i_2 = i_5$, and $i_3 = i_6$.

Teach Yourself 1. Part a. A circuit is divided into the part that is inside the dashed red box (below) and the part that is outside the box (not shown). Find an equation that relates the currents i_1 and i_2 that flow **through** the dashed red box.

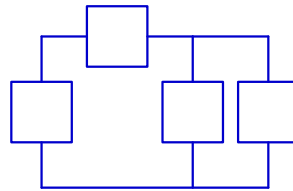


Part b. A more complicated circuit is shown inside the red box below. Find an equation that relates the currents i_1 , i_2 , and i_3 that flow **through** the dashed red box.



Part c. Generalize your results to an arbitrary closed surface. Explain how to prove the generalization.

Teach Yourself 2. Part a. How many linearly independent⁴⁰ KCL equations can be written for the following circuit?

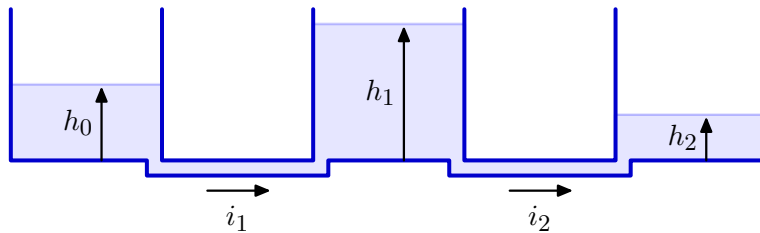


Part b. Generalize your result to determine the number of linearly independent KCL equations in an arbitrary circuit with n nodes. Prove your generalization.

6.2.2 Kirchoff's Voltage Law

What physical mechanisms cause flow? Blood circulates due to pressures generated in the heart. Hydraulic pressure also moves water through the pipes of your house. In similar fashion, voltages propel electrical currents. A voltage can be associated with every node in a circuit; and it is the difference between the voltages at two nodes that excites electrical currents to pass between those nodes.

The flow of water between water storage tanks provides a useful analogy for the flow of electrical currents. Consider three tanks connected as shown below.



⁴⁰ None of the equations in a linearly independent set can be written as a linear combination of the other equations. For example,

$$\begin{aligned}x + y &= 1 \\x + 2y &= 1\end{aligned}$$

are linearly independent equations. However, the equations

$$\begin{aligned}x + y + z &= 1 \\x + y &= 1 \\z &= 0\end{aligned}$$

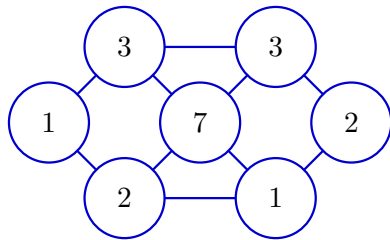
are not, since the last equation is the difference between the first two.

Example 3. What will be the signs of i_1 and i_2 ?
 Water will flow from the middle tank toward the left tank because $h_1 > h_0$. Thus $i_1 < 0$. Similarly, water will flow from the middle tank toward the right tank because $h_1 > h_2$. Thus $i_2 > 0$.

Voltages work similarly for driving electrical currents. It is the difference in voltage between two nodes that drives electrical current between the nodes. Absolute voltage is not important (or even well defined). Voltage differences are all that matter.

Teach Yourself 3. How would the flows in the tank system above change if each height of water in each tank were increased by 1 unit? ... by 1000 units?

Another similarity between the hydraulic pressures in the tank system and voltages in a circuit is that the voltage differences that accumulate along any closed path is zero. This is obvious for an array of tanks. Consider the array illustrated below in a top down view.

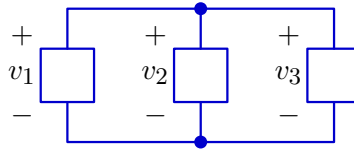


Assume that the lines between circles represent pipes and that the numbers indicate the height (in meters) of the water in each tank. Consider the clockwise path around the perimeter of the network starting and ending with the leftmost tank. The path visits heights of 1, 3, 3, 2, 1, 2, and 1. These heights increase by 2, 0, -1 , -1 , 1, and -1 , which sum to 0.

It's easy to see that the sum of differences in heights around any closed loop will be zero. If it were not, then the height of the first tank would not end up at the same value that it began. This same rule applies to voltage differences, where the rule is called *Kirchoff's Voltage Law (KVL)*.

Kirchoff's Voltage Law: the sum of voltage differences along a closed path is zero.

Example 4. How many KVL equations can be written for the following circuit.



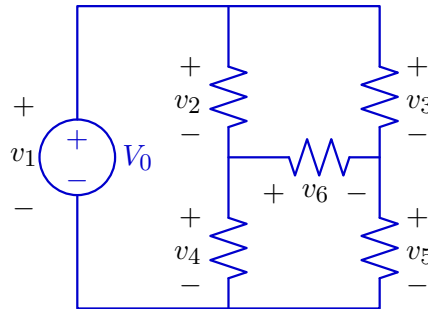
How many of the KVL equations are linearly independent?

There are three loops through this circuit. One through the left two elements yields the KVL equation $-v_1 + v_2 = 0$ or $v_1 = v_2$. One through the right two elements yields the KVL equation $-v_2 + v_3 = 0$ or $v_2 = v_3$. One through the outer two elements yields the KVL equation $-v_1 + v_3 = 0$ or $v_1 = v_3$.

Only two of the three KVL equations are linearly independent, since each equation can be derived from the other two.

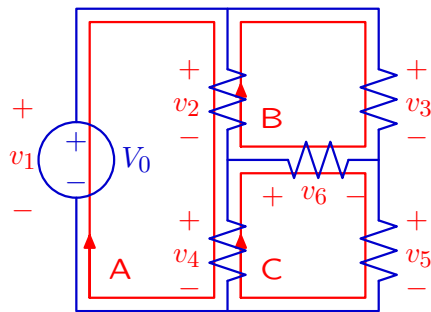
The particularly simple solution to this circuit is that $v_1 = v_2 = v_3$. This solution could also have been derived directly from the fact that there are just two nodes, and therefore only one possible potential difference.

Example 5. How many different KVL equations can be written for this circuit?



How many of the KCL equations are linearly independent?

One KVL equation can be written for every possible closed loop through the circuit. The most obvious loops are A, B, and C shown below.

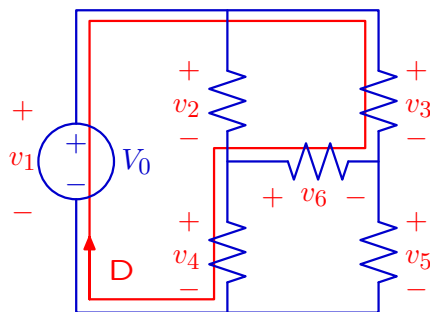


$$\text{A: } -v_1 + v_2 + v_4 = 0$$

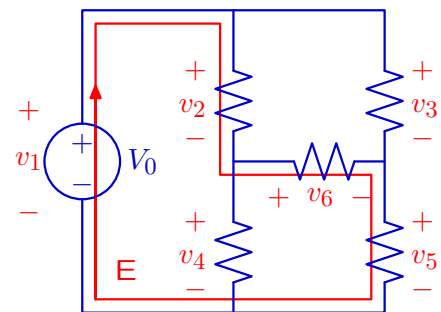
$$\text{B: } -v_2 + v_3 - v_6 = 0$$

$$\text{C: } -v_4 + v_6 + v_5 = 0$$

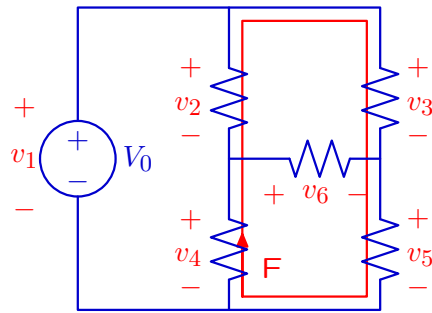
But there are several more, shown below.



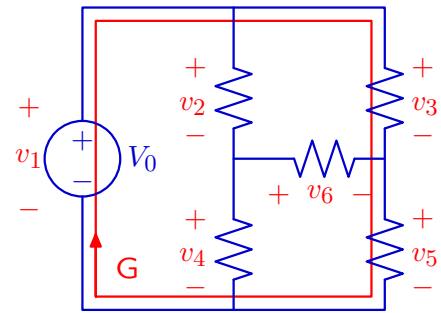
$$\text{D}=\text{A}+\text{B: } -v_1 + v_3 - v_6 + v_4 = 0$$



$$\text{E}=\text{A}+\text{C: } -v_1 + v_2 + v_6 + v_5 = 0$$



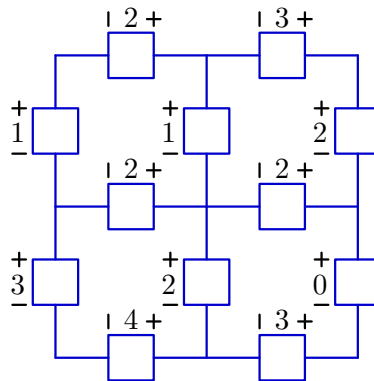
$$F=B+C: -v_4 - v_2 + v_3 + v_5 = 0$$



$$G=A+B+C: -v_1 + v_3 + v_5 = 0$$

These seven equations are not linearly independent. Equation D is the sum of Equations A and B (notice that the v_2 terms in Equations A and B cancel). Similarly, equation E is the sum of Equations A and C, equation F is the sum of Equations B and C, and equation G is the sum of Equations A, B, and C. Thus this circuit has three independent KVL equations and four additional dependent KVL equations.

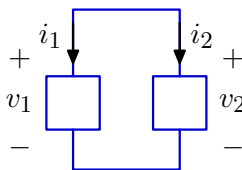
Teach Yourself 4. The following set of voltages are not consistent with Kirchoff's voltage laws but can be made consistent by changing just one. Which one?



Prove that the answer is unique.

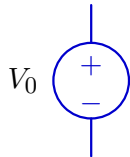
6.3 Circuit elements

Kirchoff's Current and Voltage Laws do not alone determine the behavior of a circuit. Consider the following circuit with just two elements. There are two elements, so there are two element currents i_1 and i_2 as well as two element voltages v_1 and v_2 , all of which are unknown.

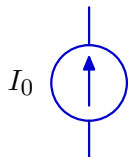


Example 6. List all of the KCL and KVL equations for this circuit. There are two nodes in this circuit. As we saw in [Teach Yourself 2](#), this implies that there is a single independent KCL equation, which is $i_1 + i_2 = 0$. There is a single loop and therefore a single KVL equation: $v_1 - v_2 = 0$. Thus we have four unknowns and just two equations. More information is needed to solve for the currents and voltages.

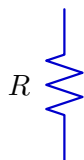
In general, every electrical element imposes a relation between the voltage across the element and the current through the element. This relation is called a **constitutive** or **element** relation. The simplest relations are those for **sources**. A voltage source is an element whose voltage is a constant (e.g., $v = V_0$) independent of the current through the element. We will denote a voltage source by a circle enclosing plus and minus symbols to denote voltage. A number beside the source will indicate the constant voltage generated by the source, as illustrated below. Voltages are specified with units called **volts** which are abbreviated as V.



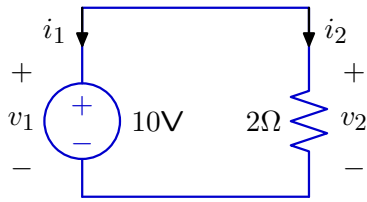
A current source is an element whose current is a constant (e.g., $i = I_0$) independent of the voltage across the element. We will denote a current source by a circle enclosing an arrow to denote current. A number beside the source will indicate the constant current generated by the source, as illustrated below. Currents are specified with units called **amperes** or simply **amps** which are abbreviated as A.



Our third simple element is a **resistor**, in which the voltage is proportional to the current and the proportionality constant is the resistance R , so that $v_R = Ri_R$. We will denote a resistor as follows. Resistances have the dimensions of **ohms** (abbreviated as Ω), which are equivalent to volts divided by amps.



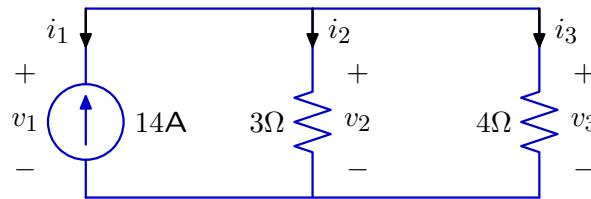
A circuit consisting of a 10V voltage source and 2Ω resistor can easily be solved.



Example 7. Find v_1 , v_2 , i_1 , and i_2 .

As before, $i_1 + i_2 = 0$ and $v_1 = v_2$. However, now we also know two constitutive relations: $v_1 = 10\text{V}$ and $v_2 = 2\Omega \times i_2$. Solving, we find that $v_1 = v_2 = 10\text{V}$, $i_2 = v_2/2\Omega = 5\text{A}$, and $i_1 = -i_2 = -5\text{A}$.

Example 8. Determine the currents and voltages in the following circuit.

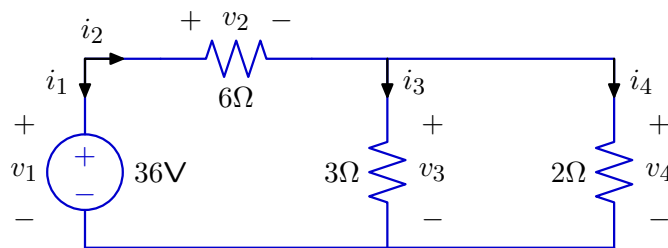


There is a single independent KCL equation: $i_1 + i_2 + i_3 = 0$. There are two independent KVL equations that together yield $v_1 = v_2 = v_3$. Thus, KCL and KVL together yield three equations in six unknowns. We need three more equations — one from each element. From the current source constitutive relation we have $i_1 = -14\text{A}$. From the resistor constitutive relations we have $v_2 = 3\Omega \times i_2$ and $v_3 = 4\Omega \times i_3$. Solving

$$v_1 = v_2 = 3\Omega \times i_2 = v_3 = 4\Omega \times i_3 = 4\Omega \times (-i_1 - i_2) = 4\Omega \times (14 - i_2)$$

so that $3\Omega \times i_2 = 4\Omega \times (14 - i_2)$ so that $i_2 = 8\text{A}$. Therefore $i_1 = -14\text{A}$, $i_2 = 8\text{A}$, and $i_3 = 6\text{A}$. Also, $v_1 = v_2 = 3\Omega \times 8\text{A} = v_3 = 4\Omega \times 6\text{A} = 24\text{V}$.

Example 9. Solve the following circuit.



Answer:

$$i_1 = -5\text{A}; \quad i_2 = 5\text{A}; \quad i_3 = 2\text{A}; \quad i_4 = 3\text{A};$$

$$v_1 = 36\text{V}; \quad v_2 = 30\text{V}; \quad v_3 = 6\text{V}; \quad v_4 = 6\text{V}$$

6.4 Solving circuits

Given a description of the circuit's wiring and a specification of the individual components, we can *solve* the circuit to answer questions about particular voltages or currents in the circuit. There are several ways to approach this problem: we will start with a completely systematic method that is particularly useful if we want to solve circuits using a computer; then, we'll see some simpler method and patterns that are easier for humans to use.

The most obvious strategy would be to write down all the KVL and KCL equations, add the constituent equations for each element, and then solve for the voltage and current at each component. But we have a problem with redundant KVL equations. The voltages across circuit elements are constrained by KVL so that the sum of the voltages around any closed loop is zero. However, there are many possible closed loops through even simple circuits (see [section 6.2.2](#)). If we are dealing with a planar circuit,⁴¹ then we know that all loops can be written as combinations of primitive loops with non-overlapping interiors (see [example 5](#)).

Teach Yourself 5. Each of four nodes are connected to each of the others by a 1Ω resistor. How many resistors are there? Is the network planar?

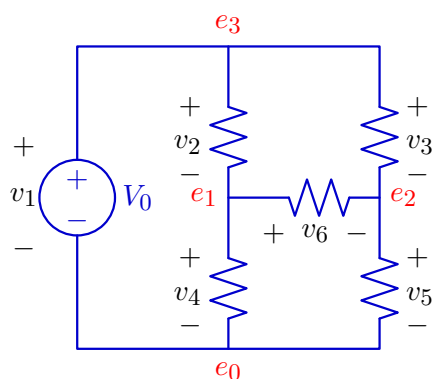
Teach Yourself 6. Each of five nodes are connected to each of the others by a 1Ω resistor. How many resistors are there? Is the network planar?

So far, the only general way that we have seen to deal with this redundancy is to (1) find **all** of the possible KVL equations, and then (2) eliminate the redundant ones. This process seems needlessly complicated.

⁴¹ A circuit is "planar" if it is possible to draw all of the elements and all of the lines that connect the elements without any of the lines or elements crossing over one another.

One good way to eliminate KVL redundancy is to use **node voltages**. Node voltages are voltages associated with circuit nodes rather than with circuit elements. Given a complete set of node voltages, it is easy to determine the element voltages: each element voltage is the difference between the voltage of the node at the positive end of the element minus that at its negative end.

Example 10. Determine expressions for each of the element voltages from the node voltages $e_0, e_1, e_2,$ and e_3 .



Answers: $v_1 = e_3 - e_0$; $v_2 = e_3 - e_1$; $v_3 = e_3 - e_2$; $v_4 = e_1 - e_0$; $v_5 = e_2 - e_0$; $v_6 = e_1 - e_2$;

Example 11. How would the element voltages in the previous example change if 1V were added to each of the node voltages?

Adding a constant voltage to each of the node voltages of a circuit has no effect on any of the element voltages. Thus, substituting node voltages for element voltages removes redundancies that are implicit in KCL equations, but also introduces a new redundancy, which we can think of as a global offset voltage. If we are only interested in element voltages (which is typically the case), we can exclude effects of a global offset voltage by arbitrarily assigning the potential of one node to zero. We refer to that special node with zero potential as the “ground.”

We will explore two methods that use node voltages: the node-voltage-and-component-current (NVCC) method, which has variables for voltages at all of the nodes and currents through all of the components, and the node method, which only has variables for voltages at the nodes. The NVCC method is completely general, but rather long-winded (and is perhaps better for computers to use than humans). The node method is terser, with potentially many fewer equations, but can become complicated when there are multiple voltage sources.

6.4.1 NVCC Method

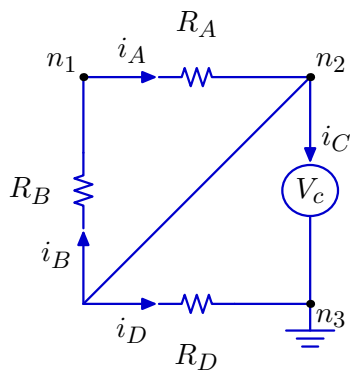
In the NVCC method, we take the following steps:

1. Label all of the *nodes* (places where there is a wire between two or more components) with names n_1, \dots, n_n , and make variables v_1, \dots, v_n , one for the voltage at each node.
2. Declare one of them, n_g , to be the *ground node* (this can be any node; the voltages will be computed relative to the voltage at that node) and set $v_g = 0$.
3. Make current variables i_1, \dots, i_m for each component (resistor or source) in the network. Label a direction for each current in your network (it doesn't matter what directions you pick as long as you handle them consistently from here on out).
4. Write down $n - 1$ KCL equations, one for each node except for n_g . These equations assert that the sum of currents entering each node is 0.
5. Write down m constitutive equations, one for each component, describing that component's linear relationship between its current i_k and the voltage difference across the component. The voltage across the component is $v_{k+} - v_{k-}$, where v_{k+} is the node voltage at the positive terminal of the component and v_{k-} is the node voltage at its negative terminal voltage. The direction of the current defines what constitutes the 'positive' and 'negative' terminals of the component: the current runs from positive to negative.

For a resistor with resistance R , the equation is $v_{k+} - v_{k-} = i_k R$; for a voltage source with voltage V_s , the equation is $v_{k+} - v_{k-} = V_s$; for a current source with current C_s , the equation is $i_k = C_s$.

6. Solve these equations to determine the node voltage and component currents.

So, for this circuit,



using the current directions for each component as drawn, we have the following equations:

- **Ground**

$$v_3 = 0 .$$

- **KCL**

$$i_A - i_B - i_D - i_C = 0$$

$$i_B - i_A = 0 .$$

- **Constitutive equations**

$$(v_1 - v_2) = i_A \cdot R_A$$

$$(v_2 - v_1) = i_B \cdot R_B$$

$$(v_2 - v_3) = i_D \cdot R_D$$

$$v_2 - v_3 = V_c .$$

So, now, if we know R_A , R_B , R_D , and V_c , which are the specifications of our components, we have 7 linear equations in 7 unknowns (v_1 , v_2 , v_3 , i_A , i_B , i_C , and i_D). Just a small (though possibly tedious) matter of algebra, and we're done.

As an example, let $R_A = 100\Omega$, $R_B = 200\Omega$, $R_D = 100\Omega$, and $V_c = 10V$. Then, we get $v_2 = 10V$; $i_A = i_B = 0A$ (that's reasonable: why would any current bother going that way, when it can just run through the diagonal wire?); and $i_D = 0.1A$, which is pretty straightforward.

6.4.2 Solution strategy

You can see that the KCL equations are all in terms of currents; and the constitutive equations give us direct expressions for almost all of the currents. So, a good solution strategy is to work primarily with the KCL equations, trying to eliminate as many current variables as possible, by substituting in expressions derived from the constitutive equations. In the example above, here are the KCL equations:

$$i_A - i_B - i_D - i_C = 0$$

$$i_B - i_A = 0 .$$

Now, substituting in expressions from the constitutive equations, we can rewrite them as:

$$\frac{v_1 - v_2}{R_A} - \frac{v_2 - v_1}{R_B} - \frac{V_c}{R_D} - i_C = 0$$

$$\frac{v_2 - v_1}{R_B} - \frac{v_1 - v_2}{R_A} = 0 .$$

We also know, from the ground equation $v_3 = 0$, that $v_2 = V_c$, so we can rewrite our equations as:

$$\frac{v_1 - V_c}{R_A} - \frac{V_c - v_1}{R_B} - \frac{V_c}{R_D} - i_C = 0$$

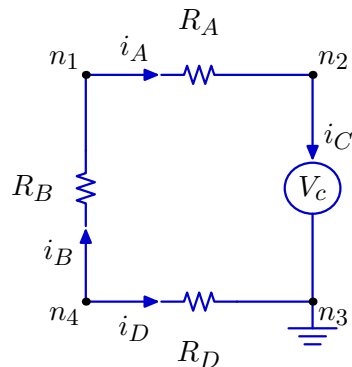
$$\frac{V_c - v_1}{R_B} - \frac{v_1 - V_c}{R_A} = 0 .$$

In most cases, you can write equations of this form down directly, without bothering to write the KCL and the constituent equations separately, and then substituting in for the currents.

Now we only have two unknowns: v_1 and i_C , and two equations. The second equation tells us that $V_c = V_1$, and, substituting that into the first equation, we have

$$i_C = \frac{V_c}{R_D}$$

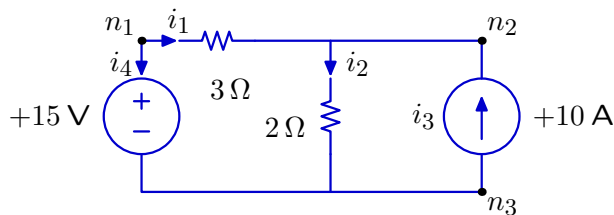
Exercise 6.1. What happens when we take out the diagonal wire? We have to add a new node, v_4 , and then proceed as we did above.



Ultimately, we can find that $i_A = i_B = i_C = -0.025\text{A}$, $i_D = 0.025\text{A}$, $v_1 = 7.5\text{V}$, $v_2 = 10\text{V}$, and $v_4 = 2.5\text{V}$. Verify this solution by writing out the equations and solving them using a process similar to the one outlined above.

6.4.3 An example

Let's try to find the node voltages and constituent currents for this circuit:



NVCC

Here are the steps of the NVCC method:

1. We have labeled the nodes n_1 , n_2 , and n_3 . Remember that we don't need a node for every corner or join in the circuit: a single 'node' can cover a stretch of wire with several connections, because it will all have the same voltage as long as no components intervene.
2. Declare one of them to be the *ground node*. When there's a single voltage source, it is conventional (but not necessary!) to set its negative terminal to be the ground node; so we'll set $v_3 = 0$.
3. Make current variables for each component in the network and label them with arrows. We did this already in the figure.
4. Write down $n - 1$ KCL equations, one for each node except for the ground node.

$$-i_4 - i_1 = 0$$

$$i_1 - i_2 + i_3 = 0$$

5. Write down m constitutive equations, one for each component, describing that component's linear relationship between its current i_k and the voltage difference across the component.

$$v_1 = 15$$

$$v_1 - v_2 = 3i_1$$

$$v_2 - v_3 = 2i_2$$

$$i_3 = 10$$

6. Solve these equations to determine the node voltage and component currents. There are lots of ways to work through this algebra. Here's one meandering path to the right answer.

We can do some plugging in of v_1 , v_3 , and i_3 to simplify matters.

$$15 - v_2 = 3i_1$$

$$v_2 = 2i_2$$

$$-i_4 - i_1 = 0$$

$$i_1 - i_2 + 10 = 0$$

Plugging expressions for i_1 and i_2 derived from the first two equations into the last one, we get

$$\frac{15 - v_2}{3} - \frac{v_2}{2} = -10$$

so

$$v_2 = 18$$

And now it's easy to see that $i_1 = -1$, $i_2 = 9$, and $i_4 = 1$.

NVCC Shortcut

If we want to shortcut the NVCC method, we can follow the first three steps above, but then try to go straight to versions of the KCL equations that substitute in expressions for current that are derived from the constituent constraints. So, we could observe that if $v_3 = 0$ then $v_1 = 15$. And then we could write the KCL equations as:

$$-i_4 - \frac{15 - v_2}{3} = 0$$

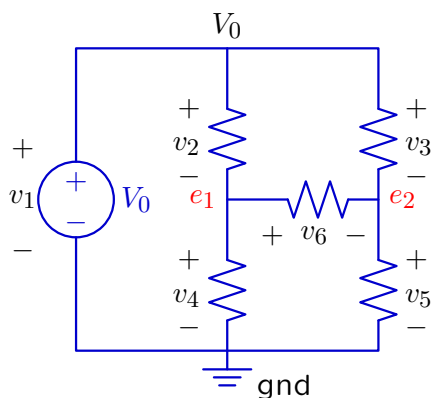
$$\frac{15 - v_2}{3} + 10 - \frac{v_2}{2} = 0$$

Solving the second equation we find that $v_2 = 18$; from this we can easily determine the other relevant quantities.

6.4.4 Node method (optional)

Node analysis is similar to NVCC, but with the goal of writing down one equation per node.

We can further simplify the node representation of circuits that contain voltage sources. When a voltage source connects two nodes, it constrains the two node voltages so that one can be determined from the other, as shown in the following figure, where e_0 in the previous figure is taken as ground.

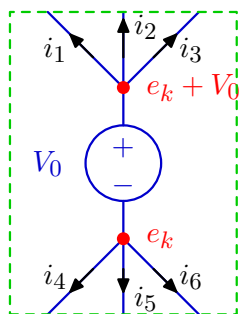


So far, we have seen that node voltages are a concise representation of all of the information that is necessary to compute the element voltages of a circuit. Furthermore, node voltages automatically guarantee that KVL is satisfied by all of the element voltages. To complete the analysis of a circuit using node voltages, we must next consider the element currents and KCL.

In general, one can write KCL at each of the nodes of a circuit. However, as we saw in [TeachYourself 2](#), one of those equations will always be linearly dependent on the others. Thus, if the circuit has n nodes, we can write $n - 1$ independent KCL equations.

If there are no voltage sources in the circuit, then we can express each KCL equation as a sum of element currents that can each be determined from the node voltages by using the element's constitutive equation. The result is a system of $n - 1$ KCL equations that can be expressed in terms of $n - 1$ node voltage variables.

If there are voltage sources in the circuit, then two additional issues arise. First, each voltage source connects two nodes, but is represented by just one node variable (since the voltage of the source determines the voltage at the other node). Second, the current through a voltage source **cannot** be determined from its constitutive equation. The constitutive equation for a voltage source is $v_s = V_0$, which must be true for all currents. Thus, knowing that $v_s = V_0$ provides no information about the current through the source. These constraints are illustrated below.



A consistent set of equations can be derived by treating the entire voltage source as a single node, represented by the dashed green box above. The KCL equation for that node,

$$i_1 + i_2 + i_3 + i_4 + i_5 + i_6 = 0$$

involves element currents that can each be written (using their constitutive equations) in terms of node voltages outside the box plus e_k and $e_k + V_0$. Like the KCL equations outside the box, this KCL equation adds a single new unknown, which is e_k .

Example 12. Does the above analysis imply that the current through the voltage source is zero?

No. Let i_v represent the current through the voltage source. KCL at the top node requires that

$$i_1 + i_2 + i_3 + i_v = 0$$

KCL at the bottom node requires that

$$i_4 + i_5 + i_6 - i_v = 0$$

The sum of these equations is the KCL equation given above for the dashed green box. It holds regardless of the value of the current through the voltage source. To find the value of the current through the current source, first solve for i_1 , i_2 , i_3 , i_4 , i_5 , i_6 , and e_k , ignoring i_v . Then $i_v = -(i_1 + i_2 + i_3)$, which will be the same as $i_4 + i_5 + i_6$.

Teach Yourself 7. How would you handle a circuit in which i_1 (above) connects directly to another voltage source outside the dashed green box?

Node Analysis: Summary

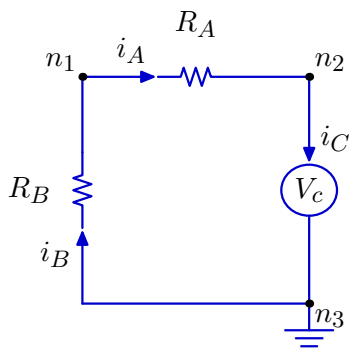
- Assign a voltage expression to each node in the circuit, as follows:
 - Assign a voltage of zero to one node, which is called the “ground.”
 - Assign voltages to nodes that are connected to voltage sources so that the voltage source relations are satisfied:
 - ★ If the voltage on one terminal of a voltage source is known (e.g., ground), then use the voltage source relation to determine the node voltage on the other terminal.
 - ★ If neither terminal of the voltage source connects to a known voltage, then assign a variable to represent the node voltage at one terminal and use the voltage source relation to determine the node voltage at the other terminal.
 - Assign variables to represent the voltages at all remaining nodes.
- For each element in the circuit that is not a voltage source, use the associated constitutive law to express the current through the element in terms of the node voltages at its terminals.
- Write one KCL equation for each unknown node voltage, as follows:
 - Express KCL for each node that is not connected to a voltage source as a sum of element currents.
 - For each voltage source, write a single KCL to express the sum of currents entering the nodes connected to both its positive **and** negative terminals.

6.4.5 Common Patterns

There are some common patterns of resistors that are important to understand and that can be used over and over again as design elements. In this section, we will lead into a systematic approach to solving circuits. The first step in that approach is to associate voltages with nodes (points at which components connect) in the circuit rather than with the components themselves. This only makes sense if we establish a fixed reference voltage. We pick a particular node in the circuit (it won't matter which one, but it is, for example, conventional to pick the negative terminal of a voltage source) and call it *ground* and assert that it has voltage 0. Now, when we speak of a node in the circuit having voltage v , what we mean is that the voltage difference between that node and the ground node is v . Ground is indicated in a circuit diagram by a special single-terminal component made of three parallel lines.

6.4.5.1 Resistors in series

The figure below shows two resistors connected together in a circuit with a voltage source, the negative terminal of which is connected to ground.



It induces a simple set of constraints (remember that we don't need a KVL equation for the ground node):

$$i_A - i_C = 0$$

$$i_B - i_A = 0$$

$$v_3 = 0$$

$$v_1 - v_2 = i_A \cdot R_A$$

$$v_3 - v_1 = i_B \cdot R_B$$

$$v_2 - v_3 = V_c$$

What happens when we solve? First, it's easy to see that because there's a single loop, KCL implies that the current across each of the nodes is the same. Let's call it i . Now, we can add together the fourth and fifth equations, and then use the last equation to get

$$v_3 - v_2 = i_A R_A + i_B R_B$$

$$v_3 - v_2 = i(R_A + R_B)$$

$$-V_c = i(R_A + R_B)$$

$$-i = \frac{V_c}{R_A + R_B}$$

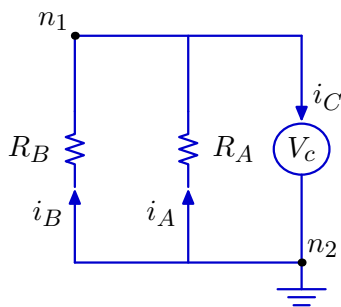
The interesting thing to see here is that we get exactly the same result as we would have had if there were a single resistor R , with resistance $R_A + R_B$. So, if you ever see two or more resistors in series in a circuit, with no other connections from the point between them to other components, you can treat them as if it were one resistor with the sum of the resistance values. This is a nice small piece of abstraction.

It might bother you that we got something that looks like $v = -iR$ instead of $v = iR$. Did we do something wrong? Not really. The reason that it seems funny is that the directions we picked for the currents i_A and i_B turn out to be “backwards”, in the sense that, in fact, the current is running in the other direction, given the way we hooked them up to the voltage source. But the answer is still correct.

Exercise 6.2. Consider the circuit we just analyzed. You should be able to construct an equivalent circuit with only one resistor. What is its resistance value?

6.4.5.2 Resistors in parallel

Now, in this figure,



we have a simple circuit with two resistors in parallel. Even though there are a lot of wires being connected together, there are really only two nodes: places where multiple components are connected. Let's write down the equations governing this system.

First, applying KCL to n_1 ,

$$i_A + i_B - i_C = 0 .$$

Now, setting v_2 to ground, and describing the individual components, we have:

$$\begin{aligned}v_2 &= 0 \\v_2 - v_1 &= i_A \cdot R_A \\v_2 - v_1 &= i_B \cdot R_B \\v_1 - v_2 &= V_c\end{aligned}$$

We can simplify this last set of constraints to

$$\begin{aligned}-V_C &= i_A \cdot R_A \\-V_C &= i_B \cdot R_B\end{aligned}$$

so

$$\begin{aligned}i_A &= -\frac{V_c}{R_A} \\i_B &= -\frac{V_c}{R_B}\end{aligned}$$

Plugging these into the KCL equation, we get:

$$\begin{aligned}i_A + i_B - i_C &= 0 \\-\frac{V_c}{R_A} - \frac{V_c}{R_B} &= i_C \\-V_c \frac{R_A + R_B}{R_A R_B} &= i_C \\-V_c &= i_C \frac{R_A R_B}{R_A + R_B}\end{aligned}$$

What we can see from this is that two resistances, R_A and R_B , wired up in parallel, act like a single resistor with resistance

$$\frac{R_A R_B}{R_A + R_B}.$$

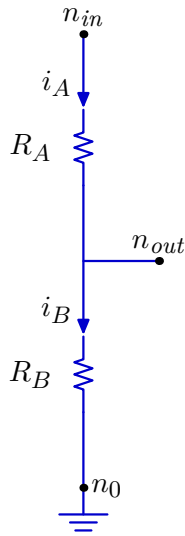
This is another common pattern for both analysis and design. If you see a circuit with parallel resistors connected at nodes n_1 and n_2 , you can simplify it to a circuit that replaces those two paths between n_1 and n_2 with a single path with a single resistor.

To get some intuition, think about what happens when R_A is much bigger than R_B . Because the voltage across both resistors is the same, it must be that the current through R_A is much smaller than the current through R_B . The appropriate intuition here is that current goes through the resistor that's easier to pass through.

Exercise 6.3. If $R_A = 10000\Omega$ and $R_B = 10\Omega$, what is the effective resistance of R_A and R_B in parallel?

6.4.5.3 Voltage divider

The figure below shows part of a circuit, in a configuration known as a *voltage divider*.



Using what we know about circuit constraints, we can determine the following relationship between V_{out} and V_{in} :

$$V_{out} = \frac{R_B}{R_A + R_B} V_{in} .$$

Let's go step by step. Here are the basic equations:

$$v_0 = 0$$

$$i_A - i_B = 0$$

$$V_{in} - V_{out} = i_A R_A$$

$$V_{out} - v_0 = i_B R_B$$

We can start by seeing that $i_A = i_B$; let's just call it i . Now, we add the last two equations to each other, and do some algebra:

$$V_{in} - v_0 = iR_A + iR_B$$

$$V_{in} = i(R_A + R_B)$$

$$i = \frac{V_{in}}{R_A + R_B}$$

$$V_{in} - V_{out} = iR_A$$

$$V_{in} - V_{out} = V_{in} \frac{R_A}{R_A + R_B}$$

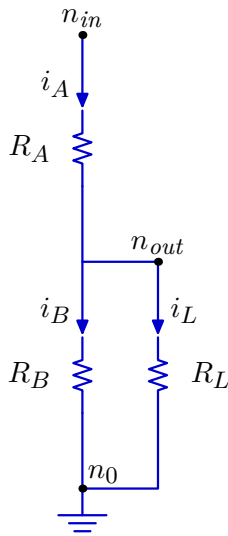
$$V_{in}(R_A + R_B) - V_{out}(R_A + R_B) = V_{in} R_A$$

$$V_{in} R_B = V_{out}(R_A + R_B)$$

$$V_{out} = V_{in} \frac{R_B}{R_A + R_B}$$

So, for example, if $R_A = R_B$, then $V_{out} = V_{in}/2$. This is a very handy thing: if you need a voltage in your circuit that is between two values that you already have available, you can choose an appropriate R_A and R_B to create that voltage.

Well, almost. When we wrote $i_A - i_B = 0$, we were assuming that there was no current flowing out V_{out} . But, of course, in general, that won't be true. Consider the following figure:



We've shown an additional "load" on the circuit at V_{out} with a resistor R_L standing for whatever resistance that additional load might offer to the ground node n_0 . This changes matters considerably.

To see what is going to happen, we could solve the whole circuit again. Or, we could observe that, between the node labeled V_{out} and n_0 , we have two resistances, R_B and R_L in parallel. And we've already seen that resistances in parallel behave as if they are a single resistor with value $R_B R_L / (R_B + R_L)$. So, (you do the algebra), our result will be that

$$V_{out} = V_{in} \frac{R_B}{R_A + R_B + \frac{R_A R_B}{R_L}} .$$

The lesson here is that the modularity in circuits is not as strong as that in programs or our difference equation models of linear systems. How a circuit will behave can be highly dependent on how it is connected to other components. Still, the constraints that it exerts on the overall system remain the same.

6.5 Circuit Equivalents

We just saw that pieces of circuits cannot be abstracted as input/output elements; the actual voltages and currents in them will depend on how they are connected to the rest of a larger circuit. However, we can still abstract them as sets of constraints on the values involved.

In fact, when a circuit includes only resistors, current sources, and voltage sources, we can derive a much simpler circuit that induces the same constraints on currents and voltages as the original one. This is a kind of abstraction that's similar to the abstraction that we saw in linear systems: we can take a complex circuit and treat it as if it were a much simpler circuit.

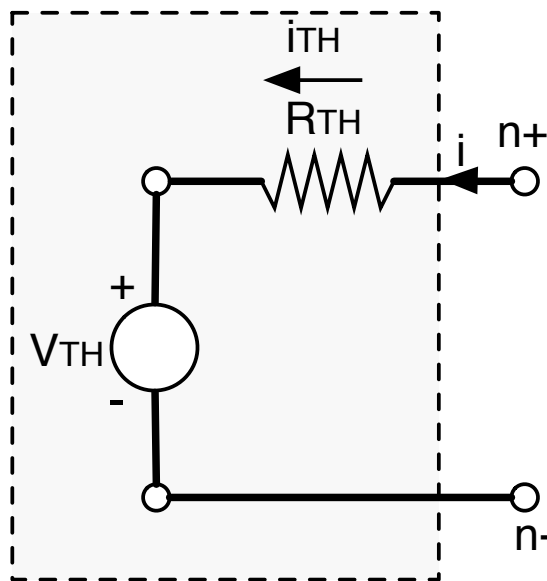
If somebody gave you a circuit and put it in a black box with two wires coming out, labeled + and -, what could you do with it? You could try to figure out what constraints that box puts on the voltage between and current through the wires coming out of the box.

We can start by figuring out the *open-circuit voltage* across the two terminals. That is the voltage drop we'd see across the two wires if nothing were connected to them. We'll call that V_{oc} . Another thing we could do is connect the two wires together, and see how much current runs through them; this is called the *short-circuit current*. We'll call that i_{sc} .

Because all of the relationships in our circuits are linear, these two values are sufficient to characterize the constraint that this whole box will exert on a circuit connected to it. The constraint will be a relationship between the voltage across its terminals and the current flowing through the box. We can derive it by using Thévenin's theorem:

Any combination of current sources, voltage sources, and resistances with two terminals can be replaced by a single voltage source V_{th} and a single series resistor R_{th} . The value of V_{th} is the open circuit voltage at the terminals V_{oc} , and the value of R_{th} is V_{th} divided by the current with the terminals short circuited ($-i_{sc}$).

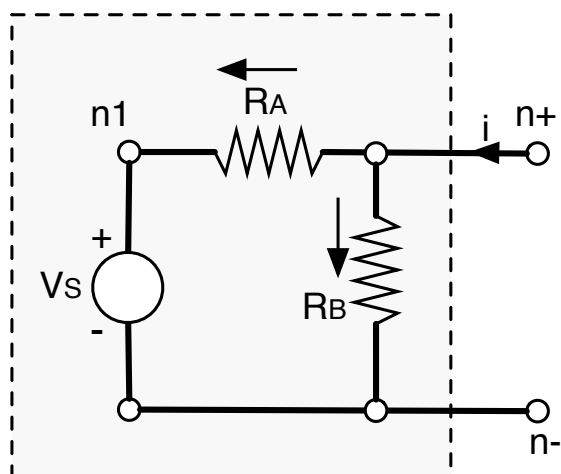
Let's look at a picture, then an example.



Here, we show a picture of a black (well, gray) box, abstracted as being made up of a circuit with a single voltage source V_{th} and a single resistor R_{th} in series. The open-circuit voltage from n_+ to n_- is clearly V_{th} . The short-circuit current i_{sc} (in the direction of the arrow) is $-V_{th}/R_{th}$. So, this circuit would have the desired measured properties.⁴²

Now, here is an actual circuit. We'll compute its associated open-circuit voltage and short-circuit current, construct the associated *Thévenin equivalent* circuit, and be sure it has the same properties.

⁴² The minus sign here can be kind of confusing. The issue is this: when we are treating this circuit as a black box with terminals n_+ and n_- , we think of the current flowing *out* of n_+ and *in* to n_- , which is consistent with the voltage difference $V_{th} = V_+ - V_-$. But when we compute the short-circuit current by wiring n_+ and n_- together, we are continuing to think of i_{sc} as flowing *out* of n_+ , but now it is coming *out* of n_- and *in* to n_+ , which is the opposite direction. So, we have to change its sign to compute R_{th} .



The first step is to compute the open-circuit voltage. This just means figuring out the difference between the voltage at nodes n_+ and n_- , under the assumption that the current $i = 0$. An easy way to do this is to set n_- as ground and then find the node voltage at n_+ . Let's write down the equations:

$$\begin{aligned} v_+ - v_1 &= i_A R_A \\ v_1 - v_- &= V_s \\ v_+ - v_- &= i_B R_B \\ -i_A - i_B &= 0 \\ i_A - i_s &= 0 \\ v_- &= 0 \end{aligned}$$

We can solve these pretty straightforwardly to find that

$$v_+ = V_s \frac{R_B}{R_A + R_B} .$$

So, we know that, for this circuit, $V_{th} = V_s \frac{R_B}{R_A + R_B}$.

Now, we need the short-circuit current, i_{sc} . To find this, imagine a wire connecting n_+ to n_- ; we want to solve for the current passing through this wire. We can use the equations we had before, but adding equation 6.1 wiring n_+ to n_- , and adding the current i_{sc} to the KCL equation 6.2.

$$\begin{aligned} v_+ - v_1 &= i_A R_A \\ v_1 - v_- &= V_s \\ v_+ - v_- &= i_B R_B \\ v_+ &= v_- \end{aligned} \tag{6.1}$$

$$\begin{aligned} i_{sc} - i_A - i_B &= 0 \\ i_A - i_s &= 0 \\ v_- &= 0 \end{aligned} \tag{6.2}$$

We can solve this system to find that

$$i_{sc} = -\frac{V_s}{R_A},$$

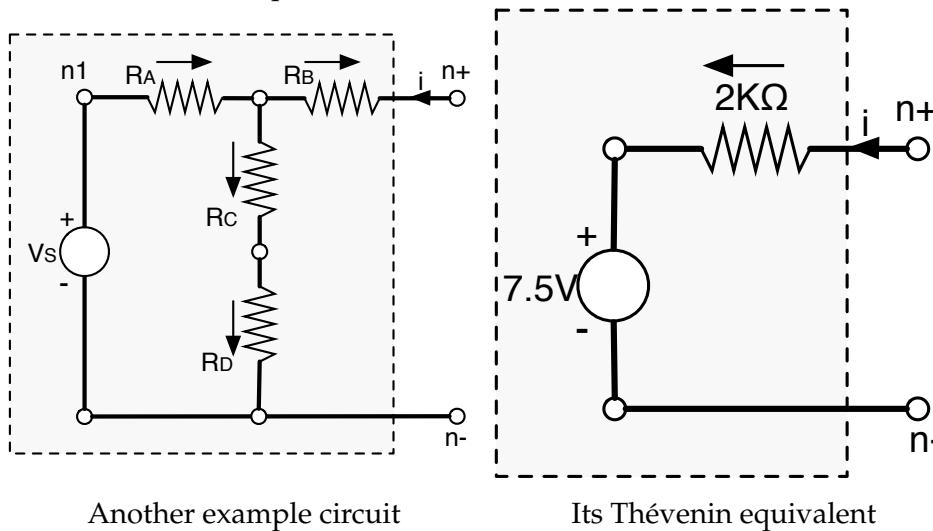
and therefore that

$$\begin{aligned} R_{th} &= -\frac{V_{th}}{i_{sc}} \\ &= V_s \frac{R_B}{R_A + R_B} \frac{V_s}{R_A} \\ &= \frac{R_A R_B}{R_A + R_B} \end{aligned}$$

What can we do with this information? We could use it during circuit analysis to simplify parts of a circuit model, individually, making it easier to solve the whole system. We could also use it in design, to construct a simpler implementation of a more complex network design. One important point is that the Thévenin equivalent circuit is not exactly the same as the original one. It will exert the same constraints on the voltages and currents of a circuit that it is connected to, but will, for example, have different heat dissipation properties.

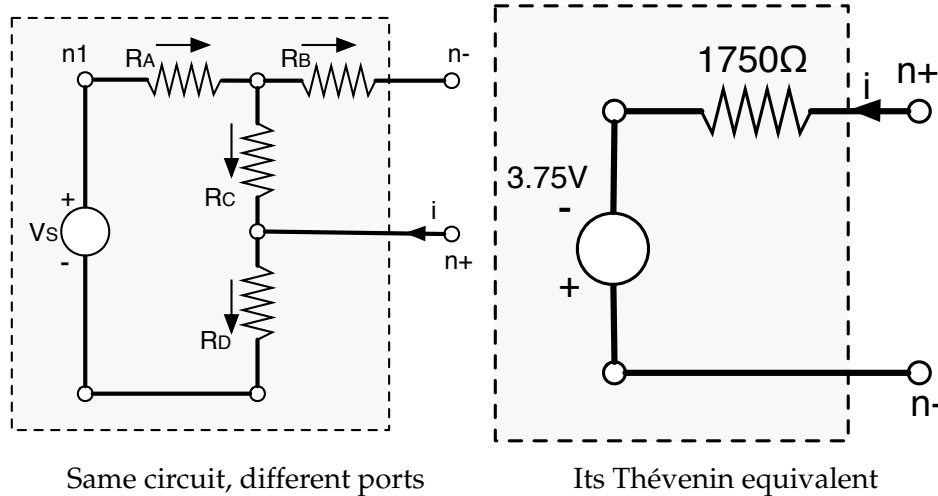
6.5.1 Example

Here's another example.



It's a bit more hassle than the previous one, but you can write down the equations to describe the constituents and KCL constraints, as before. If we let $R_A = 2K\Omega$, $R_B = R_C = R_D = 1K\Omega$, and $V_s = 15V$, then we can solve for $V_{th} = 7.5V$ and $R_{th} = 2K\Omega$. So, it is indistinguishable by current and voltage from the circuit shown in the equivalent circuit.

Here is the same circuit, but with the connections that run outside the box made to different nodes in the circuit.



Note also that the top lead is marked n_- and the bottom one n_+ . If we solve, using the same values for the resistors and voltage source as before, we find that $V_{th} = -3.75V$ and $R_{th} = 1750\Omega$. We show the Thévenin equivalent circuit next to the original. We've changed the polarity of the voltage source and made it 3.75V (instead of having the + terminal at the top and a voltage of -3.75), but that's just a matter of drawing.

These results are quite different: so, the moral is, it matters which wires you connect up to what!

6.5.2 Norton Equivalents

It is also possible to derive an equivalent circuit consisting of a single resistor in parallel with a single current source. Such a circuit is called a *Norton equivalent* circuit. The process for deriving Norton equivalents is similar in spirit to the process outlined above.

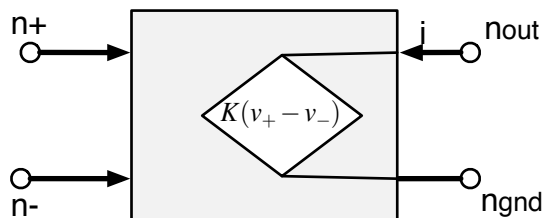
6.6 Op Amps

So far, we have considered circuits with resistors, voltage sources, and current sources. Now we are going to introduce a new component, called an *operational amplifier* or op-amp, for short. Op-amps are a crucial tool for decoupling some aspects of complex circuits, so that they can be designed independently and then connected. They are also important for amplifying small voltage differences into larger ones, and can be used to scale, invert, and add voltages.

6.6.1 Voltage-controlled voltage sources

In fact, op-amps are complex assemblages of transistors and other circuit components. We'll need a simpler model than that of what they do, so that we can analyze circuits that contain them, and design new circuits that use them.

We will think about an op-amp as a *voltage-controlled voltage source* (VCVS). Op-amps have four terminals, two of which are often called the input terminals and two, the output terminals. We can think of the op-amp as a voltage source, in which the voltage difference across the output terminals is controlled by the voltage difference between the input terminals. The crucial thing about a voltage source is that it maintains the voltage difference that it is trying to source, no matter what it is connected to.



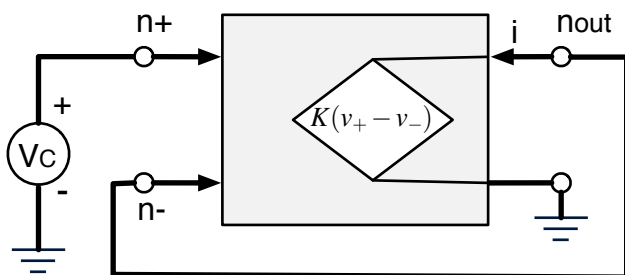
A model of the constraint that an op-amp exerts on the circuit it is connected to relates the voltages at its four terminals in this way:

$$v_{out} - v_{gnd} = K(v_+ - v_-) ,$$

where K is a very large gain, on the order of 10,000, and asserts that no current flows into the op amp.

$$i_+ = i_- = 0 .$$

So, we can think of n_{out} and n_{gnd} as constituting a voltage source, whose voltage is defined to be $K(v_+ - v_-)$. We can see it as *amplifying* the voltage difference $v_+ - v_-$. It's hard to really understand how this model works without seeing it in context. So, we'll study an op-amp that is wired up in a configuration called a *voltage follower*:



Note that we have connected a wire from the output, n_{out} , to the negative input, n_- , and so those constitute one node with a single voltage, which we call v_{out} . We can write down the equations:

$$v_+ = V_c$$

$$v_{out} - v_{gnd} = K(v_+ - v_{out})$$

$$v_{gnd} = 0$$

Solving this system, we find that

$$v_{out} = V_c \frac{K}{K+1} .$$

So, for large K , the voltage difference between the output terminals is very close to the voltage difference between the input terminals.

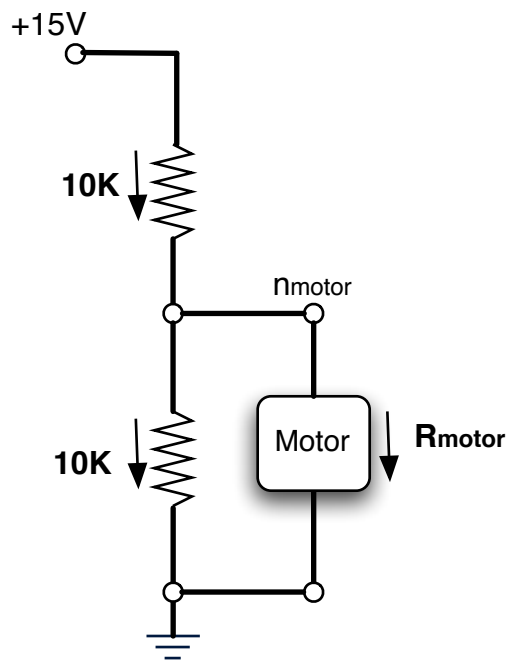
In this circuit we have set $v_{gnd} = 0$; in practice this doesn't matter. If we had left v_{gnd} in our equations, we would have found that:

$$v_{out} = V_c \frac{K}{K+1} + v_{gnd} \frac{1}{K+1} .$$

Since K is very large, that second term can be ignored. In fact, if you look at an actual op-amp, you won't even find an output corresponding to n_{gnd} . Therefore, we will always treat it as being zero.

The crucial additional fact about an op-amp is that no current flows between the input and output terminals, so the op-amp acts as a *buffer* between the input and output signal. It provides a disconnect between the currents on either side of it. This is a big deal: it gives us a kind of modularity in our circuits that we haven't had before, by limiting the kinds of influence of one part of the circuit on the other. The ability to partially disconnect sub-parts of our circuits will make it easier to do complex designs.

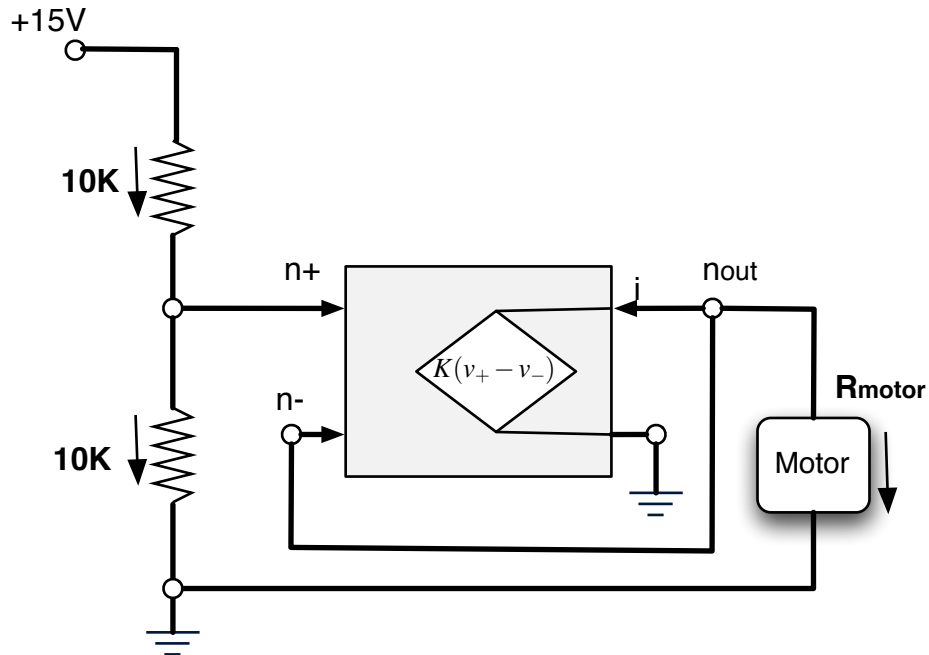
We can appreciate the value of a buffer in the context of using a variable voltage to control a motor. If we have a 15V supply, but only want to put 7.5V across the motor terminals, what should we do? A voltage divider seems like a good strategy: we can use one with two equal resistances, to make 7.5V, and then connect it to the motor as shown here



But what will the voltage v_{motor} end up being? It all depends on the resistance of the motor. If the motor is offering little resistance, say 100Ω , then the voltage v_{motor} will be very close to 0.⁴³ So, this is not an effective solution to the problem of supplying 7.5V to the motor.

So, now, back to the motor. In this figure,

⁴³ Go back and review the discussion of adding a load to a voltage divider, if this doesn't seem clear.



we have used a voltage follower to connect the voltage divider to the motor. Based on our previous analysis of the follower, we expect the voltage at n_{out} to be 7.5V, at least before we connect it up to the motor. Let's see how this all works out by writing the equations out for the whole circuit: Letting the current through the top resistor be i_1 , the current through the second resistor be i_2 , the current through the motor be i_m , and recalling that $v_{out} = v_-$ (since they are the same node), we have

$$\begin{aligned} i_1 - i_2 - i_+ &= 0 \\ -i_{out} - i_- - i_m &= 0 \\ i_+ &= 0 \\ i_- &= 0 \\ (15 - v_+) &= 10000i_1 \\ (v_+ - 0) &= 10000i_2 \\ (v_- - 0) &= K(v_+ - v_-) \\ (v_- - 0) &= i_m R_m \end{aligned}$$

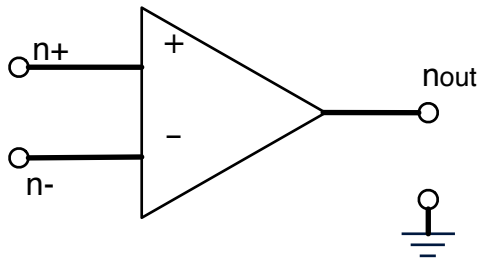
Solving these equations, we find that

$$\begin{aligned} v_+ &= 7.5 \\ v_- &= \frac{K}{K+1} 7.5 \\ i_m &= \frac{K}{K+1} \frac{7.5}{R_m} \end{aligned}$$

So, now, the voltage across the motor stays near 7.5V, and because of the isolation provided by the op-amp, it will remain that way, no matter what the resistance of the motor. Further, if in fact the motor has resistance of 5Ω , then we can find that the (maximum) current through the motor, i_m , is approximately 1.5A.

6.6.2 Simplified model

The voltage-controlled voltage-source model allows us to use our standard linear-equation form of reasoning about circuit values, but it can become cumbersome to deal with the K's all the time. In most of the usual applications of an op-amp, we will find that we can use a simpler model to reason about what is happening. That simpler model also comes with a simpler picture:

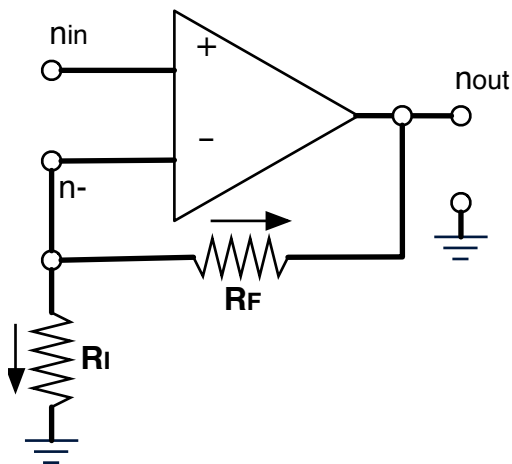


The simplified behavioral model is that the op-amp adjusts v_{out} in order to try to maintain the constraint that $v_+ \approx v_-$ and that no current flows in to n_+ or n_- . In what follows we will assume that the op-amp enforces $v_+ = v_-$.

In this section, we will explore several standard applications of op-amps, using this simplified model. If it becomes confusing, you can always go back and apply the VCVS model from the previous section.

Non-inverting amplifier

Not surprisingly, a primary use of an op-amp is as an amplifier. Here is an amplifier configuration, like this.



Let's see if we can figure out the relationship between v_{in} and v_{out} . The circuit constraints tell us that

$$\begin{aligned} v_- &= i_I R_I \\ v_- - v_{out} &= i_F R_F \\ -i_I - i_F &= 0 \end{aligned} \tag{6.3}$$

$$v_{in} = v_- \tag{6.4}$$

The KCL equation (6.3) has no term for the current into the op-amp, because we assume it is zero. Equation 6.4 is the op-amp constraint. So, we find that

$$v_{out} = v_{in} \frac{R_F + R_I}{R_I} .$$

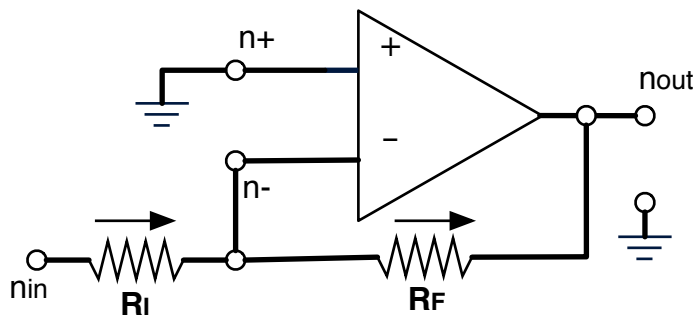
This is cool. We've arranged for the output voltage to be greater than the input voltage. We can get any gain (greater than one) by choosing the values of R_F and R_I .

We can think intuitively about how it works by examining some cases. First, if $R_F = 0$, then we'll have $v_{out} = v_{in}$, so there's not a particularly interesting change in the voltages. This is still a useful device, essentially the voltage follower, which we have already seen, with an additional resistor. This extra resistor does not change the voltage relationships but does increase the current flowing from the op-amp. Note also that choosing R_I very much larger than R_F also gives us a follower.

Now let's think about a more interesting case, but simplify matters by setting $R_F = R_I$. We can look at the part of the circuit running from V_{out} through R_F and R_I to ground. This looks a lot like a voltage divider, with v_- coming out of the middle of it. Because v_- needs to be the same as v_{in} , and it is v_{out} being divided in half, then v_{out} clearly has to be $2v_{in}$.

Inverting amplifier

Here is a very similar configuration, called an *inverting amplifier*.



The difference is that the + terminal of the op-amp is connected to ground, and we're thinking of the path through the resistors as the terminal of the resulting circuit. Let's figure out the relationship between v_{in} and v_{out} for this one. The circuit constraints tell us that

$$\begin{aligned}v_{in} - v_- &= i_I R_I \\v_- - v_{out} &= i_F R_F \\i_I - i_F &= 0 \\v_+ &= v_- \\v_+ &= 0\end{aligned}$$

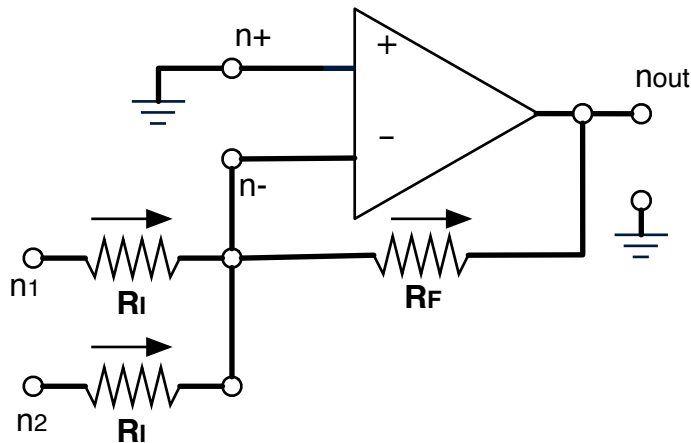
Solving, we discover that

$$v_{out} = -v_{in} \frac{R_F}{R_I} .$$

If $R_F = R_I$, then this circuit simply inverts the incoming voltage. So, for example, if v_{in} is +10V with respect to ground, then v_{out} will be -10V. Again, we can see the path from n_{in} through the resistors, to n_{out} , as a voltage divider. Knowing that v_- has to be 0, we can see that v_{out} has to be equal to $-v_{in}$. If we want to scale the voltage, as well as invert it, we can do that by selecting appropriate values of R_F and R_I .

Voltage summer

A *voltage summer*⁴⁴ circuit, shown here, can be thought of as having three terminals, with the voltage at n_{out} constrained to be a scaled, inverted, sum of the voltages at n_1 and n_2 .



You should be able to write down the equations for this circuit, which is very similar to the inverting amplifier, and derive the relationship:

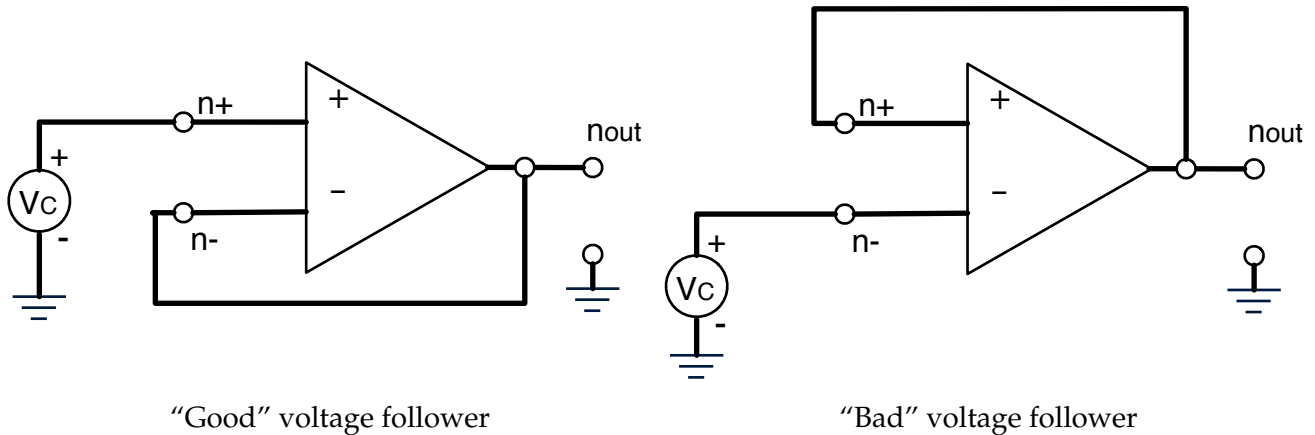
$$v_{out} = -\frac{R_F}{R_I}(v_1 + v_2) .$$

6.6.3 Op-amps and feedback

We have now seen two different models of op-amps. Each of those models can be useful in some circumstances, but now we'll explore some situations that require us to use an even more detailed and faithful model of how an op-amp really behaves.

⁴⁴ As in thing that sums, not as in endless summer.

Here are two voltage followers, a “good” one and a “bad” one.



If we use our simplest model to predict their behavior, in both cases, we’ll predict that $v_{out} = V_c$. If we use the VCVS model, we’ll predict that, for the good follower $v_{out} = V_c \frac{K}{K+1}$. For the bad follower, things are connected a little bit differently:

$$\begin{aligned} v_- &= V_c \\ v_{out} - v_{gnd} &= K(v_+ - v_-) \\ v_+ &= v_{out} \\ v_{gnd} &= 0 \end{aligned}$$

Solving this system, we find that

$$v_{out} = V_c \frac{K}{K-1} .$$

Those two predictions are basically the same for large K . But, in fact, the prediction about the behavior of the bad follower is **completely bogus!** In fact, if you connect an op-amp this way, it is likely to burn out, possibly generating smoke and noise in the process.

To see why this happens, we’ll actually need to make a more detailed model.

Dynamics model

Our new model is going to take the actual dynamics of the circuit into account. We can model what is going on in an op-amp by using a difference equation to describe the value of the output at a time n as a linear function of its values at previous times and of the values of v_+ and v_- .

Note that we’re using $v_o = v_{out}$ in the following.

You might imagine that the way the op-amp ‘tries’ to make V_o equal $K(V_+ - V_-)$ is by gradually adjusting V_o to decrease the error, E , making adjustments that are proportional, with a positive ‘gain’ g , to the error. That is, that

$$\begin{aligned} v_o[t] &= v_o[t-1] - g \cdot e[t-1] \\ &= v_o[t-1] - g \cdot (v_o[t-1] - K(v_+[t-1] - v_-[t-1])) \end{aligned} \tag{6.5}$$

We would expect this system to adjust v_o to be closer to $K(v_+ - v_-)$ on every step. The gain g is a property of the design of the op-amp.

Now, what happens when we use this model to predict the temporal behavior of the voltage followers? Let's start with the good one.

Good follower

In this case, we have $v_-[t] = v_o[t]$ for all t . So, [equation 6.5](#) becomes

$$\begin{aligned} v_o[t] &= v_o[t-1] - g \cdot (v_o[t-1] - K(v_+[t-1] - v_o[t-1])) \\ &= (1 - g - gK)v_o[t-1] + gKv_+[t-1] \end{aligned}$$

Writing this as an operator equation, and then a system function, we get

$$\begin{aligned} V_o &= (1 - g - gK)\mathcal{R}V_o + gK\mathcal{R}V_+ \\ V_o(1 - \mathcal{R}(1 - g - gK)) &= gK\mathcal{R}V_+ \\ \frac{V_o}{V_+} &= \frac{gK\mathcal{R}}{1 - \mathcal{R}(1 - g - gK)} \end{aligned}$$

This system has a pole at $1 - g - gK$, so if we let $g = 1/(1 + K)$, then we have a pole at 0, which means the system will, ideally, converge in a single step, with

$$V_o = \frac{K}{K+1}\mathcal{R}V_+ ,$$

which agrees with our previous model.

Bad follower

In this case, we have $v_+[t] = v_o[t]$ for all t . This time, [equation 6.5](#) becomes

$$\begin{aligned} v_o[t] &= v_o[t-1] - g \cdot (v_o[t-1] - K(v_o[t-1] - v_-[t-1])) \\ &= (1 - g + gK)v_o[t-1] - gKv_-[t-1] \end{aligned}$$

Writing this as an operator equation, and finding the system function, we get

$$\begin{aligned} V_o &= (1 - g + gK)\mathcal{R}V_o - gK\mathcal{R}V_- \\ V_o(1 - \mathcal{R}(1 - g + gK)) &= -gK\mathcal{R}V_- \\ \frac{V_o}{V_-} &= \frac{-gK\mathcal{R}}{1 - \mathcal{R}(1 - g + gK)} \end{aligned}$$

This system has a pole at $1 - g + gK$, so with $g = 1/(1 + K)$ as before, we have a pole at $2K/(1 + K)$, which means the system will diverge for large positive values of K . There is no positive value of g that will make this system stable (and, if we were to pick a negative value, it would make the good follower become a bad follower).

The prediction we get for $g = 1/(1 + K)$ agrees with the empirical behavior of op-amps, at least in so far as op-amps wired up in feedback to the positive input terminal do not converge to a stable output voltage, but typically burn themselves out, or behave otherwise unpredictably.

6.6.4 Where does the current come from?

In all of the models we have seen so far, we've left out an important point. As you know from lab, op-amps, in fact, need two more connections, which is to V_{CC} , the main positive voltage supply, and to $V_{EE/GND}$, which is the negative power supply (which can be ground). Why? If no current flows "through" the op-amp, from n_+ or n_- to the other side, then how can it maintain the necessary voltage difference on the output side, and supply current to, e.g., drive the motor? The answer is that it uses current from the power supplies to regulate the output voltage and to supply the necessary output current.

One metaphorical way of thinking about this is that V_{CC} is a big main water-supply pipe and that $v_+ - v_-$ is the controlling the handle of a faucet. As we change the size of the difference $v_+ - v_-$, we open or close the faucet; but the water is being supplied from somewhere else entirely.

This metaphor is appropriate in another way: the total voltage on the output of the op-amp is limited by V_{CC} . So, if the $V_{CC} = 10$, then the output of the op-amp is limited to be between 0 and 10. If we connect the ground connection of the op-amp to a negative supply voltage, then the output of the op-amp can be negative as well as positive; this has many applications.

Chapter 7

Probabilistic State Estimation

Consider trying to drive through an unfamiliar city. Even if we are able to plan a route from our starting location to our destination, navigation can fail on two counts: sometimes we don't know where we are on a map, and sometimes, due to traffic or road work or bad driving, we fail to execute a turn we had intended to take.

In such situations, we have some information about where we are: we can make observations of our local surroundings, which give us useful information; and we know what actions we have taken and the consequences those are likely to have on our location. So, the question is: how can we take information from a sequence of actions and local observations and integrate it into some sort of estimate of where we are? What form should that estimate take?

We'll consider a probabilistic approach to answering this question. We'll assume that, as you navigate, you maintain a *belief state* which contains your best information about what state you're in, which is represented as a probability distribution over all possible states. So, it might say that you're sure you're somewhere in Boston, and you're pretty sure it's Storrow drive, but you don't know whether you're past the Mass Ave bridge or not (of course, it will specify this all much more precisely).

We'll start by defining probability distributions on simple and complex spaces, and develop a set of methods for defining and manipulating them. Then, we'll formulate problems like the navigation problem discussed above as *state estimation* problems in *stochastic state machines*. Finally, we'll develop an algorithm for estimating the unobservable state of system, based on observations of its outputs.

7.1 State spaces

We have been using state machines to model systems and how they change over time. The *state* of a system is a description of the aspects of the system that allow us to predict its behavior over time. We have seen systems with finite and infinite state spaces:

- A basic enumeration of states, such as ('closed', 'closing', 'open', 'opening') for an elevator controller, we will call an *atomic* finite state space. The states are atomic because they don't have any further detailed structure.
- A state space that is described using more than one variable, such as a counter for seconds that goes from 0 to 59 and a counter for minutes that goes from 0 to 59 can be described as having two *state variables*: seconds and minutes. We would say that this state space is *factored*: a state is actually described by a value of each of the two variables.

- A state space described by a single integer is a countably infinite atomic state space.
- A state space described by a real number is uncountably infinite, but still atomic.
- A state space described by more than one integer or real number (or a combination of continuous and discrete state variables) is a factored state space.

In this chapter, we will concentrate on factored, finite state spaces, and see how to represent and manipulate probability distributions on them.

7.2 Probability distributions on atomic state spaces

Probability theory is a calculus that allows us to assign numerical assessments of uncertainty to possible events, and then do calculations with them in a way that preserves their meaning. (A similar system that you might be more familiar with is algebra: you start with some facts that you know, and the axioms of algebra allow you to make certain manipulations of your equations that you know will preserve their truth).

The typical informal interpretation of probability statements is that they are long-term frequencies: to say “the probability that this coin will come up heads when flipped is 0.5” is to say that, in the long run, the proportion of flips that come up heads will be 0.5. This is known as the *frequentist interpretation* of probability. But then, what does it mean to say “there is a 0.7 probability that it will rain somewhere in Boston sometime on April 29, 2017”? How can we repeat that process a lot of times, when there will only be one April 29, 2017? Another way to interpret probabilities is that they are measures of a person’s (or robot’s) *degree of belief* in the statement. This is sometimes referred to as the *Bayesian interpretation*. In either interpretation, the formal calculus is the same.

So, studying and applying the axioms of probability will help us make true statements about long-run frequencies and make consistent statements about our beliefs by deriving sensible consequences from initial assumptions.

We will restrict our attention to discrete sample spaces⁴⁵, so we’ll let U be the *universe* or sample space, which is a set of *atomic events*. An atomic event is just a state: an outcome or a way the world could be. It might be a die roll, or whether the robot is in a particular room, for example. Exactly one (no more, no less) event in the sample space is guaranteed to occur; we’ll say that the atomic events are “mutually exclusive” (no two can happen at once) and “collectively exhaustive” (one of them is guaranteed to happen).

⁴⁵ In probability, it is typical to talk about a ‘sample space’ rather than a ‘state space’, but they both come to the same thing: a space of possible situations.

Example 13.

- The sample space for a coin flip might be {H, T}, standing for heads and tails.
- The sample space for a sequence of three coin flips might be {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}: all possible sequences of three heads or tails.
- The sample space for a robot navigating in a city might be the set of intersections in the city.
- The sample space for a randomly generated document might be all strings of fewer than 1000 words drawn from a particular dictionary.

An event is a subset of \mathcal{U} ; it will contain zero or more atomic events.

Example 14.

- An event in the three-coin-flip space might be that there are at least two heads: {HHH, HHT, HTH, THH}.
- An event in the robot navigation problem might be that the robot is within one mile of MIT.
- An event in the document domain might be that the document contains, in sequence, the words '6.01' and 'rules'.

A probability measure \Pr is a mapping from events to numbers that satisfy the following axioms:

$$\Pr(\mathcal{U}) = 1$$

$$\Pr(\{\}) = 0$$

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2)$$

Or, in English:

- The probability that something will happen is 1.
- The probability that nothing will happen is 0.
- The probability that an atomic event in the set E_1 or an atomic event in the set E_2 will happen is the probability that an atomic event of E_1 will happen plus the probability that an atomic event of E_2 will happen, minus the probability that an atomic event that is in both E_1 and E_2 will happen (because those events effectively got counted twice in the sum of $\Pr(E_1)$ and $\Pr(E_2)$).

Armed with these axioms, we are prepared to do anything that can be done with discrete probability!

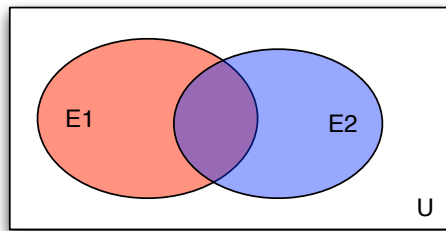
Conditional probability

One of the most important operations in probabilistic reasoning is incorporating evidence into our models. So, we might wonder what the probability of an event is *given* or *conditioned on* some other relevant information we have received. For example, we might want to know the probability of getting a die roll greater than 3, if we already know that the die roll will be odd.

We will write conditional probabilities as $\Pr(E_1 | E_2)$, pronounced “the probability of E_1 given E_2 ”, where E_1 and E_2 are events (subset of the atomic sample space). The formal definition of conditional probability is this:

$$\Pr(E_1 | E_2) = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_2)} .$$

In the figure below, E_1 is the red ellipse, E_2 is the blue ellipse, and $E_1 \cap E_2$ is the purple intersection. When we condition on E_2 , we restrict our attention entirely to the blue ellipse, and ask what percentage of the blue ellipse is also in the red ellipse. This is the same as asking for the ratio of the purple intersection to the whole blue ellipse.



Example 15. What is the conditional probability of getting a die roll greater than 3, given that it will be odd? The probability of a die roll greater than 3 (in a fair six-sided die) is $1/2$. But if we know the roll will be odd, then we have to consider ratio of the probabilities of two events: that the die roll will be odd and greater than three, and that the die roll will be odd. This is $1/6$ divided by $1/2$, which is $1/3$. So, learning that the die roll will be odd decreases our belief that it will be greater than three.

7.3 Random variables

Just as some state spaces are naturally described in their factored form (it’s easier to say that there are 10 coins, each of which can be heads or tails, than to enumerate all 2^{10} possible sequences), we often want to describe probability distributions over one or more variables. We will call a probability distribution that is described over one dimension of a state space a *random variable*.

A discrete random variable is a discrete set of values, $v_1 \dots v_n$, and a mapping of those values to probabilities $p_1 \dots p_n$ such that $p_i \in [0, 1]$ and $\sum_i p_i = 1$. So, for instance, the random variable associated with flipping a somewhat biased coin might be $\{heads : 0.6, tails : 0.4\}$. We will speak of random variables having distributions: so, for example, two flips of the same biased coin are actually two different random variables, but they have the same distribution.

In a world that is appropriately described with multiple random variables, the atomic event space is the *Cartesian product* of the value spaces of the variables. So, for example, consider two random variables, C for *cavity* and A for *toothache*. If they can each take on the values T or F (for *true* and *false*), then the universe is pairs of values, one for each variable:

$$C \times A = \{(T, T), (T, F), (F, T), (F, F)\} .$$

We will systematically use the following notation when working with random variables:

- A : capital letters stand for random variables;
- a : small letters stand for possible values of random variables; so a is an element of the domain (possible set of values) of random variable A ;
- $A = a$: an equality statement with a random variable in it is an event: in this case, the event that random variable A has value a

Some non-atomic events in a universe described by random variables A and C might be $C = c$ (which includes atomic events with all possible values of A), or $C = A$ (which includes all atomic events consisting of pairs of the same value).

Joint distribution

The *joint distribution* of a set of random variables is a function from elements of the product space to probability values that sum to 1 over the whole space. So, we would say that $(C = c, A = a)$ (with a comma connecting the equalities), which is short for $(C = c \text{ and } A = a)$ is an atomic event in the joint distribution of C, A .

In most of our examples, we'll consider joint distributions of two random variables, but all of the ideas extend to joint distributions over any finite number of variables: if there are n random variables, then the domain of the joint distribution is all n -tuples of values, one drawn from the domain of each of the component random variables. We will write $\Pr(A, B, \dots, N)$ to stand for an entire joint distribution over two or more variables.

Conditional distribution

In [section 7.2](#) we gave the basic definition of conditional probabilities, in terms of events on atomic subspaces. Sometimes, it will be useful to define conditional probabilities directly. A *conditional probability distribution*, written $\Pr(A | B)$, where A and B are random variables (we can generalize this so that they are groups of random variables), is a function from values, b , of B , to probability distributions on A . We can think of it this way:

$$\Pr(A | B) = \lambda b. \Pr(A | B = b)$$

Here, we are using λ in the same way that it is used in Python: to say that this is a function that takes a value b as an argument, and returns a distribution over A .

Example 16. Conditional distributions are often used to model the efficacy of medical tests. Consider two random variables: D , which has value *disease* if someone has a disease and value *nodisease* otherwise; and T , which has value *positive* if the test comes out positive and value *negative* otherwise. We can characterize the efficacy of the test by specifying $\Pr(T | D)$, that is, a conditional distribution on the test results given whether a person has the disease. We might specify it as:

$$\Pr(T | D) = \begin{cases} \{positive : 0.99, negative : 0.01\} & \text{if } D = \textit{disease} \\ \{positive : 0.001, negative : 0.999\} & \text{if } D = \textit{nodisease} \end{cases}$$

7.3.1 Python representations of distributions

We can represent distributions in Python in a number of ways. We'll use a simple discrete distribution class, called `DDist`, which stores its entries in a dictionary, where the elements of the sample space are the keys and their probabilities are the values.

```
class DDist:
    def __init__(self, dictionary):
        self.d = dictionary
```

The primary method of the `DDist` class is `prob`, which takes as an argument an element of the domain of this distribution and returns the probability associated with it. If the element is not present in the dictionary, we return 0. This feature allows us to represent distributions over large sets efficiently, as long as they are sparse, in the sense of not having too many non-zero entries.

```
def prob(self, elt):
    if elt in self.d:
        return self.d[elt]
    else:
        return 0
```

(The expression `elt in self.d` is a nicer way to say `self.d.has_key(elt)`. That is a call to a built-in method of the Python dictionary class, which returns `True` if the dictionary contains the key `elt` and `False` otherwise.)

It is useful to know the support of the distribution, which is a list of elements that have non-zero probability. Just in case there are some zero-probability elements stored explicitly in the dictionary, we filter to be sure they do not get returned.

```
def support(self):
    return [k for k in self.d.keys() if self.prob(k) > 0]
```

If we want to use the probability distribution to make a simulation, or to do something like shuffle or deal cards, it's useful to be able to draw from it. This method returns an element from the sample space of the distribution, selected at random according to the specified distribution.

```
def draw(self):
    r = random.random()
    sum = 0.0
    for val in self.support():
        sum += self.prob(val)
        if r < sum:
            return val
```

We can represent a joint distribution on two random variables simply as a `DDist` on pairs of their values. So, for example, this distribution

```
dist.DDist({(0, 0) : 0.5, (0, 1): 0.2, (1, 0): 0.1, (1, 1): 0.2})
```

can be seen as the joint distribution on two random variables, each of which can take on values 0 or 1. (Remember that expressions like `key1 : v1, key2 : v2` create a new dictionary).

Finally, we will represent conditional distributions as Python procedures, from values of the conditioning variable to distributions on the conditioned variable. This distribution

$$\Pr(T | D) = \begin{cases} \{positive : 0.99, negative : 0.01\} & \text{if } D = \textit{disease} \\ \{positive : 0.001, negative : 0.999\} & \text{if } D = \textit{nodisease} \end{cases}$$

would be represented in Python as

```
def TgivenD(D):
    if D == 'disease':
        return dist.DDist({'positive' : 0.99, 'negative' : 0.01})
    elif D == 'nodisease':
        return dist.DDist({'positive' : 0.001, 'negative' : 0.999})
    else:
        raise Exception, 'invalid value for D'
```

To find a value for $\Pr(T = \textit{negative} | D = \textit{disease})$, we would evaluate this Python expression:

```
>>> TgivenD('disease').prob('negative')
```

7.4 Operations on random variables

Now that we can talk about random variables, that is, distributions over their sets of values, we can follow the PCAP principle to define a systematic way of combining them to make new distributions. In this section, we will define important basic operations.

7.4.1 Constructing a joint distribution

A convenient way to construct a joint distribution is as a product of factors. We can specify a joint distribution on C and A by the product of a distribution $\Pr(A)$ and a conditional distribution $\Pr(C | A)$ by computing the individual elements of the joint, for every pair of values a in the domain of A and c in the domain of C :

$$\Pr(C = c, A = a) = \Pr(C = c) \Pr(A = a | C = c)$$

It is also true that

$$\Pr(C = c, A = a) = \Pr(A = a) \Pr(C = c | A = a)$$

Exercise 7.1. Use the definition of conditional probability to verify that the above formulas are correct.

Example 17.

In the domain where the random variable C stands for whether a person has a cavity and A for whether they have a toothache, we might know:

- The probability of a randomly chosen patient having a cavity:

$$\Pr(C) = \{T : 0.15, F : 0.85\}$$

- The conditional probability of someone having a toothache given that they have a cavity:

$$\Pr(A | C) = \begin{cases} \{T : 0.333, F : 0.667\} & \text{if } C = T \\ \{T : 0.0588, F : 0.9412\} & \text{if } C = F \end{cases}$$

Then we could construct the following table representing the joint distribution:

		C	
		T	F
A	T	0.05	0.05
	F	0.1	0.8

The numbers in the table make up the joint probability distribution. They are assignments of probability values to atomic events, which are complete specifications of the values of all of the random variables. For example, $\Pr(C = T, A = F) = 0.1$; that is, the probability of the atomic event that random variable C has value T and random variable A has value F is 0.1. Other events can be made up of the union of these primitive events, and specified by the assignments of values to only some of the variables. So, for instance, the event $A = T$ is really a set of primitive events: $\{(A = T, C = F), (A = T, C = T)\}$, which means that

$$\Pr(A = T) = \Pr(A = T, C = T) + \Pr(A = T, C = F) ,$$

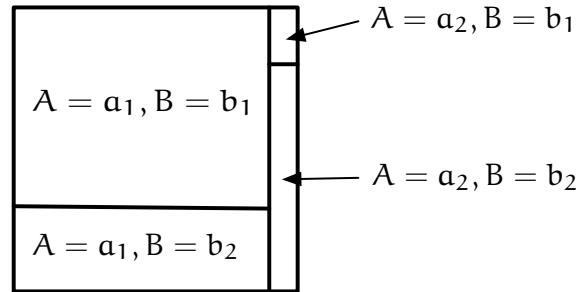
which is just the sum of the row in the table.

Here is another example of forming a joint distribution. Imagine that we are given the following distributions:

$$\Pr(A) = \{a_1 : 0.9, a_2 : 0.1\}$$

$$\Pr(B | A) = \begin{cases} \{b_1 : 0.7, b_2 : 0.3\} & \text{if } A = a_1 \\ \{b_1 : 0.2, b_2 : 0.8\} & \text{if } A = a_2 \end{cases}$$

You can visualize the joint distribution spatially, like this:



The sample space is divided vertically according to the distribution $\Pr(A)$, and then, for each value of A , it is divided horizontally according to the distribution $\Pr(B \mid A = a)$. This joint distribution is represented numerically by the table:

		A	
		a_1	a_2
B	b_1	0.63	0.02
	b_2	0.27	0.08

We can also think of this joint distribution as just another regular distribution on a larger state space:

$$\Pr(A, B) = \{(a_1, b_1) : 0.63, (a_1, b_2) : 0.27, (a_2, b_1) : 0.02, (a_2, b_2) : 0.08\}$$

More generally, we can construct a joint distribution on an arbitrary number of random variables, $\Pr(V_1, \dots, V_n)$, as follows:

$$\begin{aligned} \Pr(V_1 = v_1, \dots, V_n = v_n) &= \Pr(V_1 = v_1 \mid V_2 = v_2, \dots, V_n = v_n) \\ &\quad \cdot \Pr(V_2 = v_2 \mid V_3 = v_3, \dots, V_n = v_n) \\ &\quad \dots \\ &\quad \cdot \Pr(V_{n-1} = v_{n-1} \mid V_n = v_n) \\ &\quad \cdot \Pr(V_n = v_n) \end{aligned}$$

This can be done with the variables in any order.

7.4.2 Marginalization

A *marginal distribution* over any individual random variable can be obtained from the joint distribution by summing over all assignments to the other random variables in the joint distribution. In two dimensional tables, this means summing the rows or the columns:

$$\Pr(A = a) = \sum_b \Pr(A = a, B = b)$$

Example 18. In our example with toothaches and cavities, we can compute the marginal distributions:

$$\Pr(A) = \{T : 0.1, F : 0.9\}$$

$$\Pr(C) = \{T : 0.15, F : 0.85\}$$

Although you can compute the marginal distributions from the joint distribution, **you cannot in general compute the joint distribution from the marginal distributions!!** In the **very special case** when two random variables A and B do not influence one another, we say that they are *independent*, which is mathematically defined as

$$\Pr(A = a, B = b) = \Pr(A = a) \Pr(B = b) .$$

If we only knew the marginals of toothaches and cavities, and assumed they were independent, we would find that $\Pr(C = T, A = T) = 0.015$, which is much less than the value in our joint distribution. This is because, although cavity and toothache are relatively rare events, they are highly dependent.

7.4.3 Conditioning

One more important operation on a joint distribution is *conditioning*. It is fundamentally the same operation as computing a conditional probability, but when the conditioning event (on the right hand side of the bar) is that a random variable has a particular value, then we get a nice simplification. So, for example, if we wanted to condition our joint toothache/cavity distribution on the event $A = T$, we could write $\Pr(C, A | A = T)$. But since the value for A is already made explicit in the conditioning event, we typically write this as $\Pr(C | A = T)$. It is obtained by:

- Finding the row (or column) of the joint distribution corresponding to the conditioning event. In our example, it would be the row for $A = T$, which consists of $\Pr(A = T, C = T)$ and $\Pr(A = T, C = F)$.
- Dividing each of the numbers in that row (or column) by the probability of the conditioning event, which is the marginal probability of that row (or column). In our example, we would divide 0.05 by 0.1 to get

$$\Pr(C | A = T) = \{T : 0.5, F : 0.5\}$$

So, in this example, although a cavity is relatively unlikely, it becomes much more likely conditioned on knowing that the person has a toothache.

We have described conditioning in the case that our distribution is only over two variables, but it applies to joint distributions over any number of variables. You just have to think of selecting the entries whose value for the specified random variable equals the specified value, and then renormalizing their probabilities so they sum to 1.

7.4.4 Bayesian reasoning

Frequently, for medical diagnosis or characterizing the quality of a sensor, it's easiest to measure conditional probabilities of the form $\Pr(\textit{Symptom} | \textit{Disease})$, indicating what proportion of diseased patients have a particular symptom. (These numbers are often more useful, because they

tend to be the same everywhere, even though the proportion of the population that has disease may differ.) But in these cases, when a patient comes in and demonstrates some actual symptom s , we really want to know $\Pr(\text{Disease} \mid \text{Symptom} = s)$. We can compute that if we also know a *prior* or *base rate* distribution, $\Pr(\text{Disease})$. The computation can be done in two steps, using operations we already know:

1. Form the joint distribution: $\Pr(\text{Disease}, \text{Symptom})$
2. Condition on the event $\text{Symptom} = s$, which means selecting the row (or column) of the joint distribution corresponding to $\text{Symptom} = s$ and dividing through by $\Pr(\text{Symptom} = s)$.

So, given an actual observation of symptoms s , we can determine a conditional distribution over Disease , by computing for every value of d ,

$$\Pr(\text{Disease} = d \mid \text{Symptom} = s) = \frac{\Pr(\text{Symptom} = s \mid \text{Disease} = d) \Pr(\text{Disease} = d)}{\Pr(\text{Symptom} = s)} .$$

The formula, when written this way, is called *Bayes' Rule*, after Rev. Thomas Bayes who first formulated this solution to the 'inverse probability problem,' in the 18th century.

7.4.5 Total probability

Another common pattern of reasoning is sometimes known as *the law of total probability*. What if we have our basic distributions specified in an inconvenient form: we know, for example, $\Pr(A)$ and $\Pr(B \mid A)$, but what we really care about is $\Pr(B)$? We can form the joint distribution over A and B , and then marginalize it by summing over values of A . To compute one entry of the resulting distribution on B , we would do:

$$\Pr(B = b) = \sum_a \Pr(B = b \mid A = a) \Pr(A = a)$$

but it's easier to think about as an operation on the whole distributions.

7.4.6 Python operations on distributions

We can implement the operations described in this section as operations on `DDist` instances.

Constructing a joint distribution

We start by defining a procedure that takes a distribution $\Pr(A)$, named `PA`, and a conditional distribution $\Pr(B \mid A)$, named `PBgA`, and returns a joint distribution $\Pr(A, B)$, represented as a Python `dist.DDist` instance. It must be the case that the domain of A in `PA` is the same as in `PBgA`. It creates a new instance of `dist.DDist` with entries (a, b) for all a with support in `PA` and b with support in `PB`. The Python expression `PA.prob(a)` corresponds to $\Pr(A = a)$; `PBgA` is a conditional probability distribution, so `PBgA(a)` is the distribution $\Pr(B \mid A = a)$ on B , and `PBgA(a).prob(b)` is $\Pr(B = b \mid A = a)$.

So, for example, we can re-do our joint distribution on A and B as:

```

PA = dist.DDist({'a1' : 0.9, 'a2' : 0.1})
def PBgA(a):
    if a == 'a1':
        return dist.DDist({'b1' : 0.7, 'b2' : 0.3})
    else:
        return dist.DDist({'b1' : 0.2, 'b2' : 0.8})

>>> PAB = JDist(PA, PBgA)
>>> PAB
DDist((a1, b2): 0.270000, (a1, b1): 0.630000, (a2, b2): 0.080000, (a2, b1): 0.020000)

```

We have constructed a new joint distribution. We leave the implementation as an exercise.

Marginalization

Now, we can add a method to the `DDist` class to marginalize out a variable. It is only appropriately applied to instances of `DDist` whose domain is pairs or tuples of values (corresponding to a joint distribution). It takes, as input, the index of the variable that we want to *marginalize out*.

It can be implemented using two utility procedures: `removeElt` takes a list and an index and returns a new list that is a *copy* of the first list with the element at the specified index removed; `incrDictEntry` takes a dictionary, a key, and a numeric increment and adds the increment to the value of the key, adding the key if it was not previously in the dictionary.

```

def removeElt(items, i):
    result = items[:i] + items[i+1:]
    if len(result) == 1:
        return result[0]
    else:
        return result

def incrDictEntry(d, k, v):
    if d.has_key(k):
        d[k] += v
    else:
        d[k] = v

```

Now, we can understand `marginalizeOut` as making a new dictionary, with entries that have the variable at the specified index removed; the probability associated with each of these entries is the sum of the old entries that agree on the remaining indices. So, for example, we could take the joint distribution, `PAB`, that we defined above, and marginalize out the variable `A` (by specifying index 0) or `B` (by specifying index 1):

```

>>> PAB.marginalizeOut(0)
DDist(b1: 0.650000, b2: 0.350000)
>>> PAB.marginalizeOut(1)
DDist(a1: 0.900000, a2: 0.100000)

```

Conditioning

We can also add a `conditionOnVar` method to `DDist` which, like `marginalizeOut`, should only be applied to joint distributions. It takes as input an index of the value to be conditioned on, and a value for that variable, and returns a `DDist` on the remaining variables. It operates in three steps:

- Collect all of the value-tuples in the joint distribution that have the specified value at the specified index. This is the new universe of values, over which we will construct a distribution.
- Compute the sum of the probabilities of those elements.
- Create a new distribution by removing the elements at the specified index (they are redundant at this point, since they are all equal) and dividing the probability values by the sum of the probability mass in this set. The result is guaranteed to be a distribution in the sense that the probability values properly sum to 1.

Now, we can compute, for example, the distribution on A , given that $B = b_1$:

```
>>> PAB.conditionOnVar(1, 'b1')
DDist(a1: 0.969231, a2: 0.030769)
```

Note that this is *not* a conditional distribution, because it is not a function from values of B to distributions on A . At this point, it is simply a distribution on A .

Exercise 7.2. Define a method `condDist(self, index)` of the `DDist` class that makes a new conditional probability distribution. Remember that a conditional distribution is *not a distribution*. It is a function that takes as input a value of the random variable we are conditioning on, and returns, as a result a probability distribution over the other variable(s). So, this method takes an index (of the variable we are conditioning on) and returns a conditional probability distribution of the other variables in the joint distribution, given the variable at the specified index.

Answer:

```
def condDist(self, index):
    return lambda val: self.conditionOnVar(index, val)
```

Bayesian Evidence

The operation of updating a distribution on a random variable A , given evidence in the form of the value b of a random variable B , can be implemented as a procedure that takes as arguments the prior distribution $\Pr(A)$, named `PA`, a conditional distribution $\Pr(B | A)$, named `PBgA`, and the actual evidence b , named `b`. It starts by constructing the joint distribution $\Pr(A, B)$ with `JDist`. Then, remembering that the order of the variables in the joint distribution is (A, B) , it conditions on the variable with index 1 (that is, B) having value b , and returns the resulting distribution over A .

So, for example, given a prior distribution on the prevalence of disease in the population

```
pDis = dist.DDist({True: 0.001, False: 0.999})
```

and the conditional distribution of test results given disease:

```
def pTestGivenDis(disease):
    if disease:
        return dist.DDist({True: 0.99, False: 0.01})
    else:
        return dist.DDist({True: 0.001, False: 0.999})
```

we can determine the probability that someone has the disease if the test is positive:

```
>>> dist.bayesEvidence(pDis, pTestGivenDis, True)
DDist(False: 0.502262, True: 0.497738)
```

Exercise 7.3. Does the result above surprise you? What happens if the prevalence of disease in the population is one in a million? One in ten?

Total Probability

Finally, we can implement the law of total probability straightforwardly in Python. Given a distribution $\Pr(A)$, called `PA`, and $\Pr(B | A)$, called `PBgA`, we compute $\Pr(B)$. We do this by constructing the joint distribution and then marginalizing out A (which is the variable with index 0).

To compute the probability distribution of test results in the example above, we can do:

```
>>> dist.totalProbability(pDis, pTestGivenDis)
DDist(False: 0.998011, True: 0.001989)
```

7.5 Modeling with distributions

When we have a small number of discrete states, it is relatively easy to specify probability distributions. But, as domains become more complex, we will need to develop another PCAP system, just for constructing distributions. In this section, we'll describe a collection of relatively standard primitive distributions, and a method, called a *mixture distribution* for combining them, and show how they can be implemented in Python.

7.5.1 Primitives

Delta

Sometimes we'd like to construct a distribution with all of the probability mass on a single element. Here's a handy way to create *delta* distributions, with a probability spike on a single element:

```
def DeltaDist(v):
    return DDist({v:1.0})
```

Uniform

Another common distribution is the *uniform* distribution. On a discrete set of size n , it assigns probability $1/n$ to each of the elements:

```
def UniformDist(elts):
    p = 1.0 / len(elts)
    return DDist(dict([(e, p) for e in elts]))
```

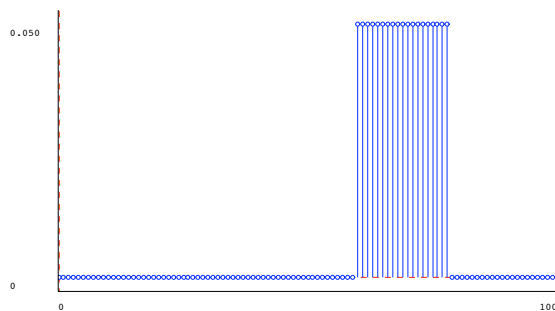
Square

There are some distributions that are particularly valuable in numeric spaces. Since we are only dealing with discrete distributions, we will consider distributions on the integers.

One useful distribution on the integers is a *square distribution*. It is defined by parameters lo and hi , and assigns probability

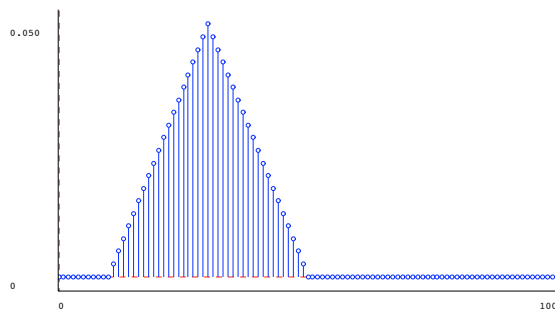
$$p = \frac{1}{hi - lo}$$

to all integers from lo to $hi - 1$. Here is a square distribution (from 60 to 80):



Because it is non-zero on 20 values, those values have probability 0.05.

Another useful distribution is a *triangle distribution*. It is defined by parameters $peak$ and $halfWidth$. It defines a shape that has its maximum value at index $peak$, and has linearly decreasing values at each of $halfWidth - 1$ points on either side of the peak. The values at the indices are scaled so that they sum to 1. Here is a triangular distribution with peak 30 and half-width 20.



7.5.2 Mixture distribution

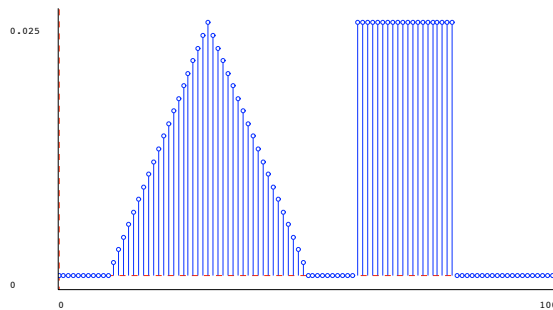
We can combine distributions by *mixing* them. To create a mixture distribution, we specify two distributions, d_1 and d_2 and a *mixing parameter* p , with $0 \leq p \leq 1$. The intuition is that, to draw an element from a mixture distribution, first we first flip a coin that comes up heads with probability p . If it is heads, then we make a random draw from d_1 and return that; otherwise we make a random draw from d_2 and return it. Another way to see it is that, if we think of a random variable D_1 having distribution d_1 and another random variable D_2 having distribution d_2 , then

$$\Pr_{\text{mix}}(x) = p \Pr(D_1 = x) + (1 - p) \Pr(D_2 = x)$$

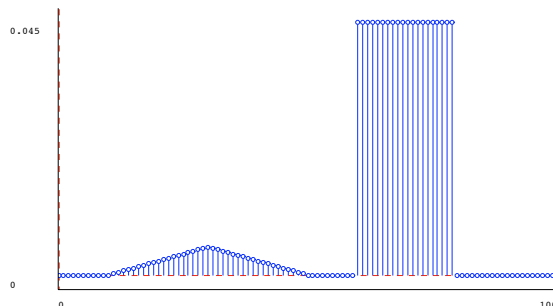
that is, the probability of an element x under the mixture distribution is p times its probability under distribution d_1 plus $1 - p$ times its probability under distribution d_2 .

Exercise 7.4. Convince yourself that if both d_1 and d_2 are proper probability distributions, in that they sum to 1 over their domains, then any mixture of them will also be a proper probability distribution.

We can make a mixture of the square and triangle distributions shown above, with mixture parameter 0.5:



Here it is, with mixture parameter 0.9, where the square is d_1 and the triangle is d_2 :

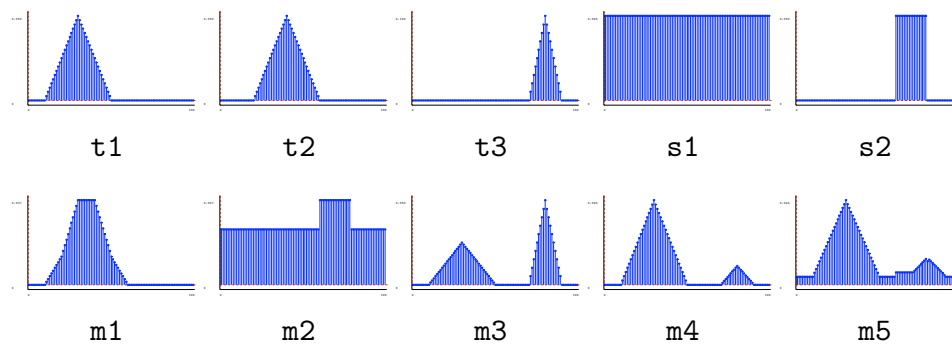


To implement a mixture distributions, we have two choices. One would be to go through the supports of both component distributions and create a new explicit `DDist`. Below, we have taken the 'lazy' approach, similar to the way we handled signal composition. We define a new class that

stores the two component distributions and the mixture parameter, and computes the appropriate probabilities as necessary.

Here are some example mixture distributions, created in Python and plotted below.

```
t1 = dist.triangleDist(30, 20)
t2 = dist.triangleDist(40, 20)
t3 = dist.triangleDist(80, 10)
s1 = dist.squareDist(0, 100)
s2 = dist.squareDist(60, 80)
m1 = dist.MixtureDist(t1, t2, 0.5)
m2 = dist.MixtureDist(s1, s2, 0.95)
m3 = dist.MixtureDist(t1, t3, 0.5)
m4 = dist.MixtureDist(t1, t3, 0.9)
m5 = dist.MixtureDist(m2, m4, 0.5)
```



7.6 Stochastic state machines

Now we can return to the application of primary interest: there is a system moving through some sequence of states over time, but instead of getting to see the states, we only get to make a sequence of observations of the system, where the observations give partial, noisy information about the actual underlying state. The question is: what can we infer about the current state of the system given the history of observations we have made?

As a very simple example, let's consider a copy machine: we'll model it in terms of two possible internal states: *good* and *bad*. But since we don't get to see inside the machine, we can only make observations of the copies it generates; they can either be *perfect*, *smudged*, or *all black*. There is only one input we can generate to the system, which is to ask it to make a copy.

We have to start by building a model of the system of interest. We can model this problem using a probabilistic generalization of state machines, called *stochastic*⁴⁶ *state machines* (SSMs).

In an SSM, we model time as a discrete sequence of steps, and we can think about the state, input, and the observation at each step. So, we'll use random variables S_0, S_1, S_2, \dots to model the state at each time step, the random variables O_0, O_1, \dots to model the observation at each time step, and the random variables I_0, I_1, \dots to model the input at each time step.

Our problem will be to compute a probability distribution over the state at some time $t + 1$ given the past history of inputs and observations $o_0, i_0, \dots, o_t, i_t$; that is, to compute

⁴⁶ 'Stochastic' is a synonym for probabilistic.

$$\Pr(S_{t+1} \mid O_0 = o_0, I_0 = i_0, \dots, O_t = o_t, I_t = i_t) .$$

A stochastic state-machine model makes a strong assumption about the system: that the state at time t is sufficient to determine the probability distribution over the observation at time t and the state at time $t + 1$. This is the essence of what it means to be a *state* of the system; that is, that it summarizes all past history. Furthermore, we assume that the system is time-invariant, so that way the state at time 3 depends on the state and input at time 2 is the same as the way that the state at time 2 depends on the state and input at time 1, and so on; similarly for the observations.

So, in order to specify our model of how this system works, we need to provide three probability distributions:

- **Initial state distribution:** We need to have some idea of the state that the machine will be in at the very first step of time that we're modeling. This is often also called the *prior* distribution on the state. We'll write it as

$$\Pr(S_0) .$$

It will be a distribution over possible states of the system.

- **State transition model:** Next, we need to specify how the state of the system will change over time. It is described by the conditional probability distribution

$$\Pr(S_{t+1} \mid S_t, I_t) ,$$

which specifies for each possible old state r and input i , a probability distribution over the state the system will be in at time $t + 1$ if it had been in state r and received input i at time t . We will assume that this same probabilistic relationship holds for all times t .

- **Observation model:** Finally, we need to specify how the observations we make of the system depend on its underlying state. This is often also called the *sensor model*. It is described by the conditional probability distribution

$$\Pr(O_t \mid S_t) ,$$

which specifies for each possible state of the system, s , a distribution over the observations that will be obtained when the system is in that state.

7.6.1 Copy machine example

We'll specify a stochastic state-machine model for our copy machine. Because there is only a single possible input, to keep notation simple, we'll omit the input throughout the example.

- **Initial state distribution:**

$$\Pr(S_0) = \{good : 0.9, bad : 0.1\}$$

- **State transition model:**

$$\Pr(S_{t+1} \mid S_t) = \begin{cases} \{good : 0.7, bad : 0.3\} & \text{if } S_t = good \\ \{good : 0.1, bad : 0.9\} & \text{if } S_t = bad \end{cases}$$

- **Observation model:**

$$\Pr(O_t \mid S_t) = \begin{cases} \{perfect : 0.8, smudged : 0.1, black : 0.1\} & \text{if } S_t = good \\ \{perfect : 0.1, smudged : 0.7, black : 0.2\} & \text{if } S_t = bad \end{cases}$$

7.6.2 Representation in Python

We can represent these basic distributions in Python using the `DDist` class that we have already developed. First, we have the initial state distribution, which is quite straightforward:

```
initialStateDistribution = dist.DDist({'good': 0.9, 'bad': 0.1})
```

The observation model is simply a conditional probability distribution; that is, a procedure that takes a state as input and returns the appropriate distribution over observations.

```
def observationModel(s):
    if s == 'good':
        return dist.DDist({'perfect' : 0.8, 'smudged' : 0.1, 'black' : 0.1})
    else:
        return dist.DDist({'perfect' : 0.1, 'smudged' : 0.7, 'black' : 0.2})
```

The transition model is just a little bit trickier. For reasons that we will see later, it is most convenient to organize the transition model as a procedure that takes an input i as input, and then returns a conditional probability distribution of $\Pr(S_{t+1} \mid S_t, I_t = i)$. Remember that the conditional probability distribution is, itself, a procedure that takes an old state and returns a distribution over new states. This seems particularly silly for the copy machine example, since we are ignoring the input, but we still have to make the definitions this way so that the types will match those of more complex models we will build later.

```
def transitionModel(i):
    def transitionGivenI(oldState):
        if oldState == 'good':
            return dist.DDist({'good' : 0.7, 'bad' : 0.3})
        else:
            return dist.DDist({'good' : 0.1, 'bad' : 0.9})
    return transitionGivenI
```

Finally, we can define a new class, called `StochasticSM`, which is a subclass of `SM`. We have had to generalize the `SM` class slightly, so that there is an option of supplying a `startState` method instead of a fixed start state, so that the actual starting state can be the result of calling a procedure. To get a starting state, it makes a random draw from the initial state distribution; to get the next values, it makes random draws from the transition and observation distributions:

```
class StochasticSM(sm.SM):
    def __init__(self, startDistribution, transitionDistribution,
                 observationDistribution):
        self.startDistribution = startDistribution
        self.transitionDistribution = transitionDistribution
        self.observationDistribution = observationDistribution

    def startState(self):
        return self.startDistribution.draw()

    def getNextValues(self, state, inp):
        return (self.transitionDistribution(inp)(state).draw(),
                self.observationDistribution(state).draw())
```

Now, we can define our copy machine:

```
copyMachine = ssm.StochasticSM(initialStateDistribution,
                               transitionModel, observationModel)
```

Because `StochasticSM` is a subclass of `SM`, we can use the standard methods of `SM`. Here, we ask the copy machine to make 20 copies, and observe the sequence of outputs it generates:

```
>>> copyMachine.transduce(['copy']* 20)
['perfect', 'smudged', 'perfect', 'perfect', 'perfect', 'perfect',
 'perfect', 'smudged', 'smudged', 'black', 'smudged', 'black',
 'perfect', 'perfect', 'black', 'perfect', 'smudged', 'smudged',
 'black', 'smudged']
```

7.7 State estimation

Now, given the model of the way a system changes over time, and how its outputs reflect its internal state, we can do *state estimation*. The problem of state estimation is to take a sequence of inputs and observations, and determine the sequence of hidden states of the system. Of course, we won't be able to determine that sequence exactly, but we can derive some useful probability distributions.

We will concentrate on the problem of *filtering* or *sequential state estimation*, in which we imagine sitting and watching the stream of inputs and observations go by, and we are required, on each step, to produce a state estimate, in the form

$$\Pr(S_{t+1} \mid O_0, \dots, O_t, I_0, \dots, I_t)$$

We will develop a procedure for doing this by working through a few steps of the example with the copy machine, and then present it more generally.

7.7.1 Our first copy

Let's assume we get a brand new copy machine in the mail, and we think it is probably (0.9) good, but we're not entirely sure. We print out a page, and it looks perfect. Yay! Now, what do we believe about the state of the machine? We'd like to compute

$$\Pr(S_1 \mid O_0 = \textit{perfect}) .$$

We'll do this in two steps. First, we'll consider what information we have gained about the machine's state at time 0 from the observation, and then we'll consider what state it might have transitioned to on step 1.

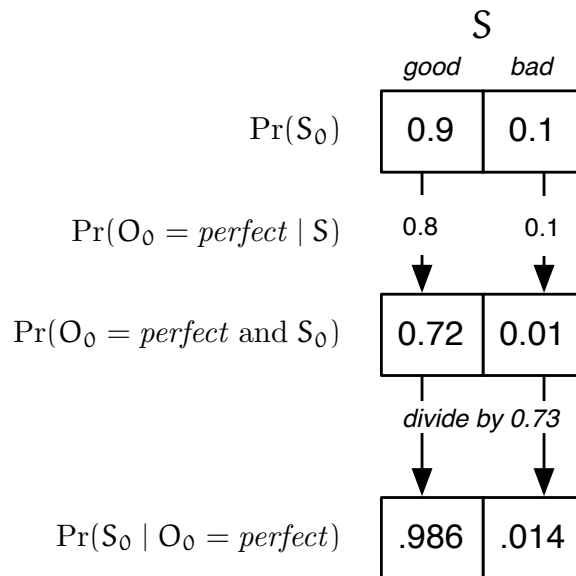
What information do we gain from knowing that the machine's first copy was perfect? We can use Bayesian reasoning to form the joint distribution between the state and observations in general, and then condition on the observation we actual received. The joint distribution has the form

		O		
		<i>perfect</i>	<i>smudged</i>	<i>black</i>
S	<i>good</i>	0.72	0.09	0.09
	<i>bad</i>	0.01	0.07	0.02

Now, conditioning on the actual observation, $O_0 = \textit{perfect}$, we extract the column corresponding to the observation, $\{0.72, 0.01\}$, and divide by the sum, 0.73, to get the distribution

$$\Pr(S_0 \mid O_0 = \textit{perfect}) = \{\textit{good} : 0.986, \textit{bad} : 0.014\}$$

Here is a schematic version of this update rule, which is a good way to think about computing it by hand. Rather than creating the whole joint distribution and then conditioning by selecting out a single column, we just create the column we know we're going to need (based on the observation we got):



Because we will use it in later calculations, we will define B'_0 as an abbreviation:

$$B'_0 = \Pr(S_0 \mid O_0 = \textit{perfect}) ;$$

that is, our belief that the system is in state s on the 0th step, after having taken the actual observation o_0 into account. This update strategy computes $B'_0(s)$ for all s , which we'll need in order to do further calculations.

Now, we can think about the consequences of the passage of one step of time. We'd like to compute $\Pr(S_1 \mid O_0 = \textit{perfect})$. Note that we are not yet thinking about the observation we'll get on step 1; just what we know about the machine on step 1 having observed a perfect copy at step 0.

What matters is the probability of making transitions between states at time 0 and states at time 1. We can make this relationship clear by constructing the joint probability distribution on S_0 and S_1 (it is actually conditioned on $O_0 = \textit{perfect}$). So, $\Pr(S_0, S_1 \mid O_0 = \textit{perfect})$ can be constructed from $\Pr(S_0 \mid O_0 = \textit{perfect})$ (which, it is important to remember, is a distribution on S_0) and $\Pr(S_1 \mid S_0)$, which is our transition model:

		S ₁	
		good	bad
S ₀	good	0.690	0.296
	bad	0.001	0.012

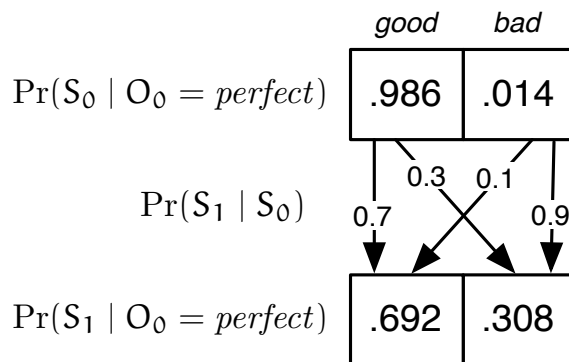
Now, we have never really known the value of S_0 for sure, and we can never really know it; we'd like to concentrate all of our information in a representation of our knowledge about S_1 . We can do that by computing the marginal distribution on S_1 from this joint. Summing up the columns, we get

$$\Pr(S_1 \mid O_0 = \textit{perfect}) = \{\textit{good} : 0.692, \textit{bad} : 0.308\}$$

This is an application of the law of total probability.

We'll give this distribution, $\Pr(S_1 \mid O_0 = \textit{perfect})$, that is, everything we know about the machine after the first observation and transition, the abbreviation B_1 .

Here is a schematic version of the transition update:

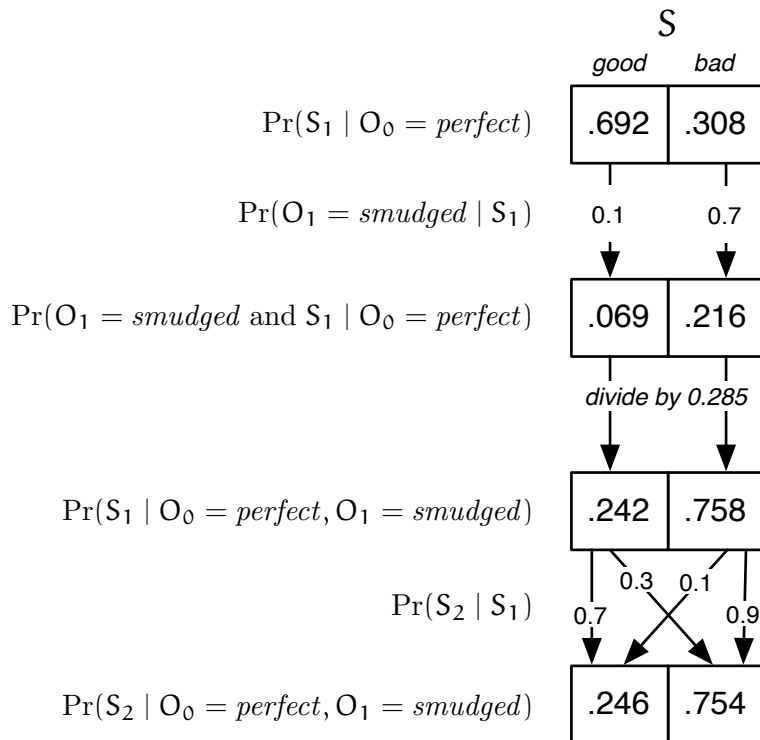


You can see that there are two ways the copier could be in a good state at time 1: because it was in a good state at time 0 (probability 0.986) and made a transition to a good state at time 1 (probability 0.7) or because it was in a bad state at time 0 (probability 0.014) and made a transition to a good state at time 1 (probability 0.1). So, the resulting probability of being in a good state is $0.986 \cdot 0.7 + 0.014 \cdot 0.1 = 0.692$. The reasoning for ending in a bad state is the same.

7.7.2 Our second copy

Now, let's imagine we print another page, and it's smudged. We want to compute $\Pr(S_2 \mid O_0 = \textit{perfect}, O_1 = \textit{smudged})$. This can be computed in two stages as before. We start with B_1 , which already has the information that $O_0 = \textit{perfect}$ incorporated into it. Our job will be to fold in the information that $O_1 = \textit{smudged}$, and then the passage of another step of time.

1. Construct a joint distribution from B_1 and $\Pr(O_1 \mid S_1)$, and condition on $O_1 = \textit{smudged}$ to get B'_1 . This is a Bayesian reasoning step.
2. Construct a joint distribution from B'_1 and $\Pr(S_2 \mid S_1)$ and marginalize out S_1 to get B_2 . This is a law-of-total-probability step.



Ow. Now we're pretty sure our copy machine is no good. Planned obsolescence strikes again!

7.8 General state estimation

Now we'll write out the state-update procedure for SSMs in general. As a reminder, here are the random variables involved:

- State at each time S_0, \dots, S_T
- Observation at each time O_0, \dots, O_T
- Input at each time I_0, \dots, I_T

Here are the components of the model:

- **Initial distribution** $\Pr(S_0)$
- **Observation distribution** $\Pr(O_t \mid S_t)$. Because the process is time-invariant, this is the same for all t .
- **Transition distribution** $\Pr(S_{t+1} \mid S_t, I_t)$. Because the process is time-invariant, this is the same for all t . Think of i as selecting a particular conditional transition distribution to be used in the update.

Now, here is the update procedure. Assume that we have a *belief state* at time t , corresponding to $\Pr(S_t \mid O_{0..t-1} = o_{0..t-1})$. Then we proceed in two steps:

- **Observation update, given o_t :**

$$\begin{aligned} & \Pr(S_t \mid O_{0..t} = o_{0..t}, I_{0..t} = i_{0..t}) \\ &= \frac{\Pr(O_t = o_t \mid S_t) \Pr(S_t \mid O_{0..t-1} = o_{0..t-1}, I_{0..t-1} = i_{0..t-1})}{\Pr(O_t = o_t \mid O_{0..t-1} = o_{0..t-1}, I_{0..t-1} = i_{0..t-1})}. \end{aligned}$$

- **Transition update, given i_t :**

$$\begin{aligned} \Pr(S_{t+1} \mid O_{0..t} = o_{0..t}, I_{0..t} = i_{0..t}) \\ = \sum_r \Pr(S_{t+1} \mid S_t = r, I_t = i_t) \Pr(S_t = r \mid O_{0..t} = o_{0..t}, I_{0..t-1} = i_{0..t-1}) . \end{aligned}$$

A very important thing to see about these definitions is that they enable us to build what is known as a *recursive* state estimator. (Unfortunately, this is a different use of the term “recursive” than we’re used to from programming languages). If we define our belief at time t ,

$$B_t = \Pr(S_t \mid O_{0..t-1} = o_{0..t-1}, I_{0..t-1} = i_{0..t-1})$$

and our belief after making the observation update to be

$$B'_t = \Pr(S_t \mid O_{0..t} = o_{0..t}, I_{0..t-1} = i_{0..t-1}) ,$$

then after each observation and transition, we can update our belief state, to get a new B_t . Then, we can forget the particular observation we had, and just use the B_t and o_t to compute B_{t+1} .

Algorithmically, we can run a loop of the form:

- **Condition on actual observation $O_t = o_t$.**

$$B'(s) = \frac{\Pr(O_t = o_t \mid S_t = s)B(s)}{\sum_r \Pr(O_t = o_t \mid S_t = r)B(r)}$$

- **Make transition based on input $I_t = i_t$.**

$$B(s) = \sum_r \Pr(S_{t+1} = s \mid S_t = r, I_t = i_t)B'(r)$$

where B is initialized to be our initial belief about the state of the system.

7.8.1 Python

We can implement this state estimation process straightforwardly in Python. The nicest way to think of a state estimator is, itself, as a state machine. The state of the state estimator is B , the belief state, which is a probability distribution over the states of the SSM whose state we are estimating. The state estimator takes an SSM model at initialization time. At each step, the input to the state estimator is an (o, i) pair, specifying the observation from and the input to the machine on the current step; the output of the state estimator is a probability distribution, represented as a `dist.DDist` over the states of the underlying system.

The initial state of the state estimator is the initial state distribution of the underlying machine:

```
class StateEstimator(sm.SM):
    def __init__(self, model):
        self.model = model
        self.startState = model.startDistribution
```

Now, to get the next values of the state estimator, we perform the observation and transition updates, as described mathematically above. We observe that the observation update is an instance

of the Bayes evidence procedure, with the current belief as the prior, the observation model as the conditional distribution, and the actual observation o as the value we're conditioning on. We also observe that the transition update is an instance of the law of total probability: we select the appropriate state transition distribution depending on the input i , then use it to make a joint distribution over S_t and S_{t+1} , then marginalize out S_t , because we want to represent all of our information as a distribution on S_{t+1} . The output of the state estimator is the same as its state.

Now, we can do state estimation on our copy machine. We might have this sequence of observations and inputs:

```
copyData = [('perfect', 'copy'), ('smudged', 'copy')]
```

which means that we saw a perfect page, made a copy, saw a smudged page, and made another copy. Now, we can make a state estimator, using the model we defined in the previous section, and feed the data into the estimator:

```
>>> copyMachineSE = se.StateEstimator(copyMachine)
>>> copyMachineSE.transduce(copyData)
[DDist(bad: 0.308219, good: 0.691781), DDist(bad: 0.754327, good: 0.245673)]
```

Note that these are the same distributions we got when we did the state estimation by hand.

Chapter 8

Long-term decision-making and search

In the lab exercises of this course, we have implemented several brains for our robots. We used wall-following to navigate through the world and we used various linear controllers to drive down the hall and to control the robot head. In the first case, we just wrote a program that we hoped would do a good job. When we were studying linear controllers, we, as designers, made models of the controller's behavior in the world and tried to prove whether it would behave in the way we wanted it to, taking into account a longer-term pattern of behavior.

Often, we will want a system to generate complex long-term patterns of behavior, but we will not be able to write a simple control rule to generate those behavior patterns. In that case, we'd like the system to evaluate alternatives for itself, but instead of evaluating single actions, it will have to evaluate whole sequences of actions, deciding whether they're a good thing to do given the current state of the world.

Let's think of the problem of navigating through a city, given a road map, and knowing where we are. We can't usually decide whether to turn left at the next intersection without deciding on a whole path.

As always, the first step in the process will be to come up with a formal model of a real-world problem that abstracts away the irrelevant detail. So, what, exactly, is a path? The car we're driving will actually follow a trajectory through continuous space(time), but if we tried to plan at that level of detail we would fail miserably. Why? First, because the space of possible trajectories through two-dimensional space is just too enormous. Second, because when we're trying to decide which roads to take, we don't have the information about where the other cars will be on those roads, which will end up having a huge effect on the detailed trajectory we'll end up taking.

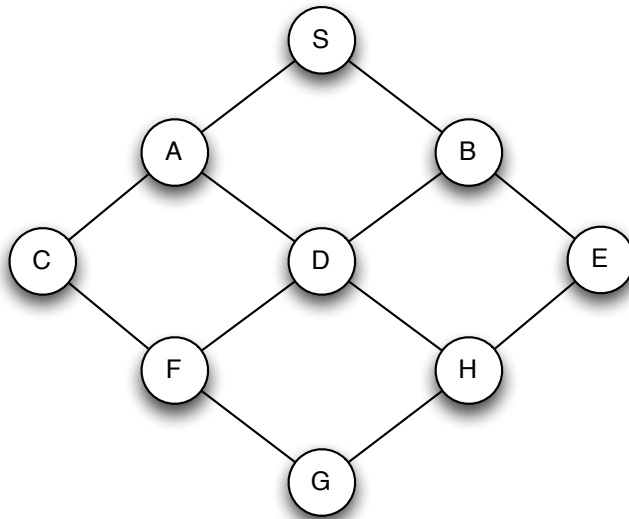
So, we can divide the problem into two levels: planning in advance which turns we'll make at which intersections, but deciding 'on-line', while we're driving, exactly how to control the steering wheel and the gas to best move from intersection to intersection, given the current circumstances (other cars, stop-lights, etc.).

We can make an abstraction of the driving problem to include road intersections and the way they're connected by roads. Then, given a start and a goal intersection, we could consider all possible paths between them, and choose the one that is best.

What criteria might we use to evaluate a path? There are all sorts of reasonable ones: distance, time, gas mileage, traffic-related frustration, scenery, etc. Generally speaking, the approach we'll outline below can be extended to any criterion that is additive: that is, your happiness with the whole path is the sum of your happiness with each of the segments. We'll start with the simple

criterion of wanting to find a path with the fewest “steps”; in this case, it will be the path that traverses the fewest intersections. Then in [section 8.5](#) we will generalize our methods to handle problems where different actions have different costs.

One possible algorithm for deciding on the best path through a map of the road intersections in this (very small) world



would be to enumerate all the paths, evaluate each one according to our criterion, and then return the best one. The problem is that there are *lots* of paths. Even in our little domain, with 9 intersections, there are 210 paths from the intersection labeled S to the one labeled G.

We can get a much better handle on this problem, by formulating it as an instance of a graph search (or a “state-space search”) problem, for which there are simple algorithms that perform well.

8.1 State-space search

We’ll model a state-space search problem formally as

- a (possibly infinite) set of *states* the system can be in;
- a *starting state*, which is an element of the set of states;
- a *goal test*, which is a procedure that can be applied to any state, and returns True if that state can serve as a goal;⁴⁷
- a *successor function*, which takes a state and an action as input, and returns the new state that will result from taking the action in the state; and
- a *legal action list*, which is just a list of actions that can be legally executed in this domain.

The decision about what constitutes an *action* is a modeling decision. It could be to drive to the next intersection, or to drive a meter, or a variety of other things, depending on the domain. The

⁴⁷ Although in many cases we have a particular goal state (such as the intersection in front of my house), in other cases, we may have the goal of going to any gas station, which can be satisfied by many different intersections.

only requirement is that it terminate in a well-defined next state (and that, when it is time to execute the plan, we will know how to execute the action.)

We can think of this model as specifying a *labeled graph* (in the computer scientist's sense), in which the states are the nodes, action specifies which of the arcs leading out of a node is to be selected, and the successor function specifies the node at the end of each of the arcs.

So, for the little world above, we might make a model in which

- The set of states is the intersections {'S', 'A', 'B', 'C', 'D', 'E', 'F', 'G', 'H'}.
- The starting state is 'S'.
- The goal test is something like:

```
lambda x: x == 'H'
```

- The legal actions in this domain are the numbers 0, 1, ..., n-1, where n is the maximum number of successors in any of the states.
- The map can be defined using a dictionary:

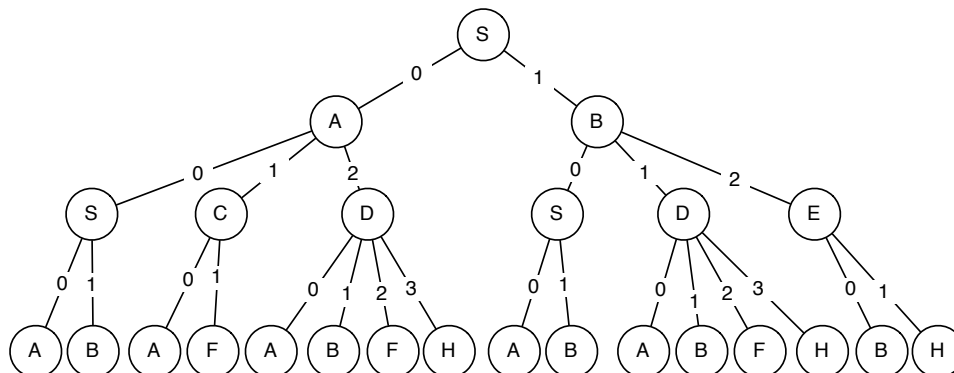
```
map1 = {'S' : ['A', 'B'],
        'A' : ['S', 'C', 'D'],
        'B' : ['S', 'D', 'E'],
        'C' : ['A', 'F'],
        'D' : ['A', 'B', 'F', 'H'],
        'E' : ['B', 'H'],
        'F' : ['C', 'D', 'G'],
        'H' : ['D', 'E', 'G'],
        'G' : ['F', 'H']}
```

where each key is a state, and its value is a list of states that can be reached from it in one step. Now we can define the successor function as

```
def map1successors(s, a):
    return map1[s][a]
```

but with an additional test to be sure that if we attempt to take an action that doesn't exist in s, it just results in state s. So, for example, the successor reached from state 'A' by taking action 1 is state 'C'.

We can think of this structure as defining a *search tree*, like this:



It has the starting state, *S*, at the root node, and then each node has its successor states as children. Layer *k* of this tree contains all possible paths of length *k* through the graph.

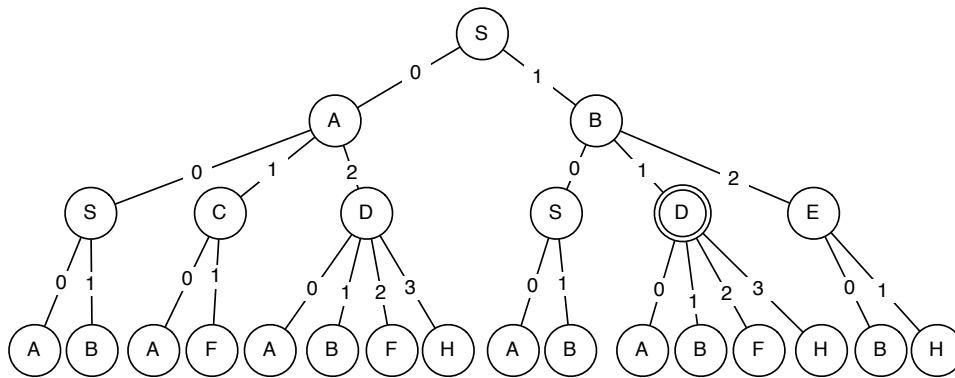
8.1.1 Representing search trees

We will need a way to represent the tree as a Python data structure as we construct it during the search process. We will start by defining a class to represent a *search node*, which is one of the circles in the tree.

Each search node represents:

- The state of the node;
- the action that was taken to arrive at the node; and
- the search node *from which* this node can be reached.

We will call the node from which a node can be reached its *parent* node. So, for example, in the figure below



we will represent the node with double circles around it with its state, 'D', the action that reached it, 1, and its parent node, which is the node labeled 'B' above it.

Note that *states and nodes are not the same thing!* In this tree, there are many nodes labeled by the same state; they represent different paths to and through the state.

Here is a Python class representing a search node. It's pretty straightforward.

```
class SearchNode:
    def __init__(self, action, state, parent):
        self.state = state
        self.action = action
        self.parent = parent
```

There are a couple of other useful methods for this class. First, the `path` method, returns a list of pairs (`a`, `s`) corresponding to the path starting at the top (root) of the tree, going down to this node. It works its way up the tree, until it reaches a node whose parent is `None`.

```
def path(self):
    if self.parent == None:
        return [(self.action, self.state)]
    else:
        return self.parent.path() + [(self.action, self.state)]
```

The path corresponding to our double-circled node is `((None, 'S'), (1, 'B'), (1, 'D'))`. Another helper method that we will find useful is the `inPath` method, which takes a state, and returns `True` if the state occurs anywhere in the path from the root to the node.

```
def inPath(self, s):
    if s == self.state:
        return True
    elif self.parent == None:
        return False
    else:
        return self.parent.inPath(s)
```

8.1.2 Basic search algorithm

We'll describe a sequence of search algorithms of increasing sophistication and efficiency. An ideal algorithm will take a problem description as input and return a path from the start to a goal state, if one exists, and return `None`, if it does not. Some algorithms will not be capable of finding a path in all cases.

How can we systematically search for a path to the goal? There are two plausible strategies:

- Start down a path, keep trying to extend it until you get stuck, in which case, go back to the last choice you had, and go a different way. This is how kids often solve mazes. We'll call it *depth-first* search.
- Go layer by layer through the tree, first considering all paths of length 1, then all of length 2, etc. We'll call this *breadth-first* search.

Both of the search strategies described above can be implemented using a procedure with this basic structure:

```
def search(initialState, goalTest, actions, successor):
    if goalTest(initialState):
        return [(None, initialState)]
    agenda = EmptyAgenda()
    add(SearchNode(None, initialState, None), agenda)
    while not empty(agenda):
        parent = getElement(agenda)
        for a in actions(parent.state):
            newS = successor(parent.state, a)
            newN = SearchNode(a, newS, parent)
            if goalTest(newS):
                return newN.path()
            else:
                add(newN, agenda)
    return None
```

We start by checking to see if the initial state is a goal state. If so, we just return a path consisting of the initial state.

Otherwise, we have real work to do. We make the *root* node of the tree. It has no parent, and there was no action leading to it, so all it needs to specify is its state, which is `initialState`, so it is created with

```
SearchNode(None, initialState, None)
```

During the process of constructing the search tree, we will use a data structure, called an *agenda*, to keep track of which nodes in the partially-constructed tree are on the fringe, ready to be expanded, by adding their children to the tree. We initialize the agenda to contain the root node. Now, we enter a loop that will run until the agenda is empty (we have no more paths to consider), but could stop sooner.

Inside the loop, we select a node from the agenda (more on how we decide which one to take out in a bit) and *expand it*. To expand a node, we determine which actions can be taken from the state that is stored in the node, and *visit* the *successor* states that can be reached via the actions.

When we visit a state, we make a new search node (`newN`, in the code) that has the node we are in the process of expanding as the parent, and that remembers the state being visited and the action that brought us here from the parent.

Next, we check to see if the new state satisfies the goal test. If it does, we're done! We return the path associated with the new node.

If this state it doesn't satisfy the goal test, then we add the new node to the agenda. We continue this process until we find a goal state or the agenda becomes empty. This is not quite yet an algorithm, though, because we haven't said anything about what it means to add and extract nodes from the agenda. And, we'll find, that it will do some very stupid things, in its current form.

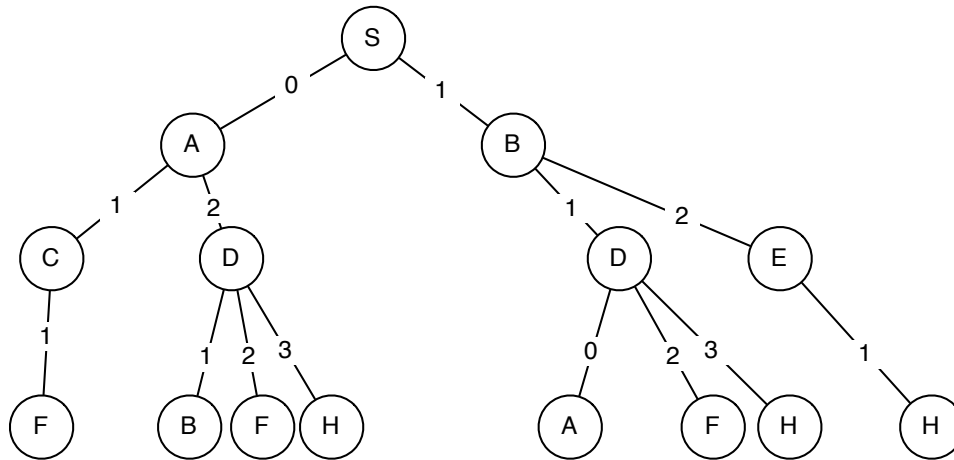
We'll start by curing the stupidities, and then return to the question of how best to select nodes from the agenda.

8.1.3 Basic pruning, or How not to be completely stupid

If you examine the full search tree, you can see that some of the paths it contains are completely ridiculous. It can never be reasonable, if we're trying to find the shortest path between two states, to go back to a state we have previously visited on that same path. So, to avoid trivial infinite loops, we can adopt the following rule:

Pruning Rule 1. Don't consider any path that visits the same state twice.

If we can apply this rule, then we will be able to remove a number of branches from the tree, as shown here:



It is relatively straightforward to modify our code to implement this rule:

```
def search(initialState, goalTest, actions, successor):
    if goalTest(initialState):
        return [(None, initialState)]
    agenda = [SearchNode(None, initialState, None)]
    while agenda != []:
        parent = getElement(agenda)
        for a in actions(parent.state):
            newS = successor(parent.state, a)
            newN = SearchNode(a, newS, parent)
            if goalTest(newS):
                return newN.path()
            elif parent.inPath(newS):
                pass
            else:
                add(newN, agenda)
    return None
```

The red text above shows the code we've added to our basic algorithm. It just checks to see whether the current state already exists on the path to the node we're expanding and, if so, it doesn't do anything with it.

The next pruning rule doesn't make a difference in the current domain, but can have a big effect in other domains:

Pruning Rule 2. If there are multiple actions that lead from a state r to a state s , consider only one of them.

To handle this in the code, we have to keep track of which new states we have reached in expanding this node, and if we find another way to reach one of those states, we just ignore it. The changes to the code for implementing this rule are shown in red:

```
def search(initialState, goalTest, actions, successor):
    if goalTest(initialState):
        return [(None, initialState)]
```

```

agenda = [SearchNode(None, initialState, None)]
while agenda != []:
    parent = getElement(agenda)
    newChildStates = []
    for a in actions(parent.state):
        newS = successor(parent.state, a)
        newN = SearchNode(a, newS, parent)
        if goalTest(newS):
            return newN.path()
        elif newS in newChildStates:
            pass
        elif parent.inPath(newS):
            pass
        else:
            newChildStates.append(newS)
            add(newN, agenda)
return None

```

Each time we pick a new node to expand, we make a new empty list, `newChildStates`, and keep track of all of the new states we have reached from this node.

Now, we have to think about how to extract nodes from the agenda.

8.1.4 Stacks and Queues

In designing algorithms, we frequently make use of two simple data structures: stacks and queues. You can think of them both as abstract data types that support two operations: push and pop. The push operation adds an element to the stack or queue, and the pop operation removes an element. The difference between a stack and a queue is what element you get back when you do a pop.

- **stack:** When you pop a stack, you get back the element that you most recently put in. A stack is also called a LIFO, for *last in, first out*.
- **queue:** When you pop a queue, you get back the element that you put in earliest. A queue is also called a FIFO, for *first in, first out*.

In Python, we can use lists to represent both stacks and queues. If `data` is a list, then `data.pop(0)` removes the first element from the list and returns it, and `data.pop()` removes the last element and returns it.

Here is a class representing stacks as lists. It always adds new elements to the end of the list, and pops items off of the same end, ensuring that the most recent items get popped off first.

```

class Stack:
    def __init__(self):
        self.data = []
    def push(self, item):
        self.data.append(item)
    def pop(self):
        return self.data.pop()
    def isEmpty(self):
        return self.data == []

```

Here is a class representing stacks as lists. It always adds new elements to the end of the list, and pops items off of the front, ensuring that the oldest items get popped off first.

```
class Queue:
    def __init__(self):
        self.data = []
    def push(self, item):
        self.data.append(item)
    def pop(self):
        return self.data.pop(0)
    def isEmpty(self):
        return self.data == []
```

We will use stacks and queues to implement our search algorithms.

8.1.5 Depth-First Search

Now we can easily describe depth-first search by saying that it's an instance of the generic search procedure described above, but in which the agenda is a *stack*: that is, we always expand the node we most recently put into the agenda.

The code listing below shows our implementation of depth-first search.

```
def depthFirstSearch(initialState, goalTest, actions, successor):
    agenda = Stack()
    if goalTest(initialState):
        return [(None, initialState)]
    agenda.push(SearchNode(None, initialState, None))
    while not agenda.isEmpty():
        parent = agenda.pop()
        newChildStates = []
        for a in actions(parent.state):
            newS = successor(parent.state, a)
            newN = SearchNode(a, newS, parent)
            if goalTest(newS):
                return newN.path()
            elif newS in newChildStates:
                pass
            elif parent.inPath(newS):
                pass
            else:
                newChildStates.append(newS)
                agenda.push(newN)
    return None
```

You can see several operations on the agenda. We:

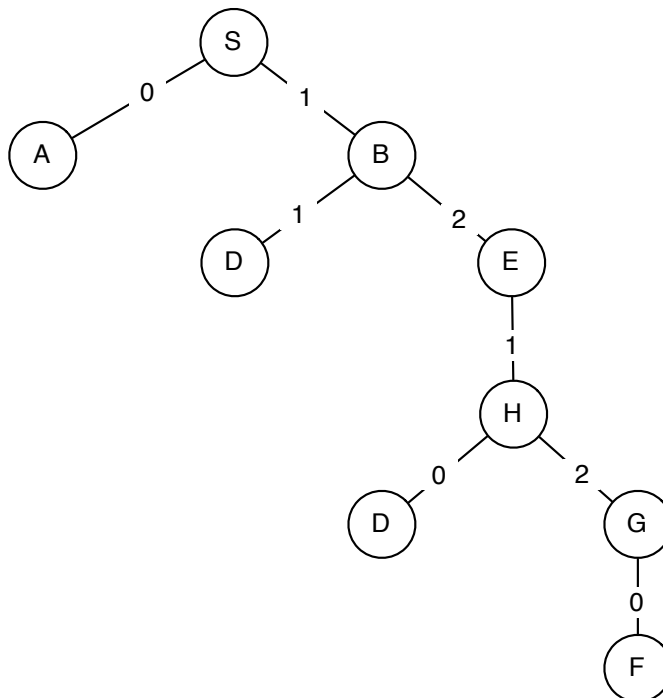
- Create an empty Stack instance, and let that be the agenda.
- Push the initial node onto the agenda.
- Test to see if the agenda is empty.
- Pop the node to be expanded off of the agenda.
- Push newly visited nodes onto the agenda.

Because the agenda is an instance of the `Stack` class, subsequent operations on the agenda ensure that it will act like a stack, and guarantee that children of the most recently expanded node will be chosen for expansion next.

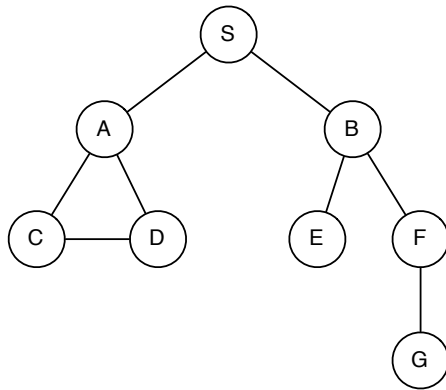
So, let's see how this search method behaves on our city map, with start state `S` and goal state `F`. Here is a trace of the algorithm (you can get this in the code we distribute by setting `verbose = True` before you run it.)

```
depthFirst('S', lambda x: x == 'F', mapLegalActions, map1successors)
agenda: Stack([S])
  expanding: S
agenda: Stack([S-0->A, S-1->B])
  expanding: S-1->B
agenda: Stack([S-0->A, S-1->B-1->D, S-1->B-2->E])
  expanding: S-1->B-2->E
agenda: Stack([S-0->A, S-1->B-1->D, S-1->B-2->E-1->H])
  expanding: S-1->B-2->E-1->H
agenda: Stack([S-0->A, S-1->B-1->D, S-1->B-2->E-1->H-0->D, S-1->B-2->E-1->H-2->G])
  expanding: S-1->B-2->E-1->H-2->G
8 states visited
[(None, 'S'), (1, 'B'), (2, 'E'), (1, 'H'), (2, 'G'), (0, 'F')]
```

You can see that in this world, the search never needs to “backtrack”, that is, to go back and try expanding an older path on its agenda. It is always able to push the current path forward until it reaches the goal. Here is the search tree generated during the depth-first search process.



Here is another city map (it's a weird city, we know, but maybe a bit like trying to drive in Boston):



In this city, depth-first search behaves a bit differently (trying to go from S to D this time):

```

depthFirst('S', lambda x: x == 'D', map2LegalActions, map2successors)
agenda: Stack([S])
expanding: S
agenda: Stack([S-0->A, S-1->B])
expanding: S-1->B
agenda: Stack([S-0->A, S-1->B-1->E, S-1->B-2->F])
expanding: S-1->B-2->F
agenda: Stack([S-0->A, S-1->B-1->E, S-1->B-2->F-1->G])
expanding: S-1->B-2->F-1->G
agenda: Stack([S-0->A, S-1->B-1->E])
expanding: S-1->B-1->E
agenda: Stack([S-0->A])
expanding: S-0->A
7 states visited
[(None, 'S'), (0, 'A'), (2, 'D')]

```

In this case, it explores all possible paths down in the left branch of the world, and then has to backtrack up and over to the right branch.

Here are some important properties of depth-first search:

- It will run forever if we don't apply pruning rule 1, potentially going back and forth from one state to another, forever.
- It may run forever in an infinite domain (as long as the path it's on has a new successor that hasn't been previously visited, it can go down that path forever; we'll see an example of this in the last section).
- It doesn't necessarily find the shortest path (as we can see from the very first example).
- It is generally efficient in the amount of space it requires to store the agenda, which will be a constant factor times the depth of the path it is currently considering (we'll explore this in more detail in [section 8.4](#).)

8.1.6 Breadth-First Search

To change to breadth-first search, we need to choose the oldest, rather than the newest paths from the agenda to expand. All we have to do is change the agenda to be a queue instead of a stack, and everything else stays the same, in the code.

```
def breadthFirstSearch(initialState, goalTest, actions, successor):
    agenda = Queue()
    if goalTest(initialState):
        return [(None, initialState)]
    agenda.push(SearchNode(None, initialState, None))
    while not agenda.isEmpty():
        parent = agenda.pop()
        newChildStates = []
        for a in actions(parent.state):
            newS = successor(parent.state, a)
            newN = SearchNode(a, newS, parent)
            if goalTest(newS):
                return newN.path()
            elif newS in newChildStates:
                pass
            elif parent.inPath(newS):
                pass
            else:
                newChildStates.append(newS)
                agenda.push(newN)
    return None
```

Here is how breadth-first search works, looking for a path from S to F in our first city:

```
>>> breadthFirst('S', lambda x: x == 'F', map1LegalActions, map1successors)
agenda: Queue([S])
    expanding: S
agenda: Queue([S-0->A, S-1->B])
    expanding: S-0->A
agenda: Queue([S-1->B, S-0->A-1->C, S-0->A-2->D])
    expanding: S-1->B
agenda: Queue([S-0->A-1->C, S-0->A-2->D, S-1->B-1->D, S-1->B-2->E])
    expanding: S-0->A-1->C
7 states visited
[(None, 'S'), (0, 'A'), (1, 'C'), (1, 'F')]
```

We can see it proceeding systematically through paths of length two, then length three, finding the goal among the length-three paths.

Here are some important properties of breadth-first search:

- Always returns a shortest (least number of steps) path to a goal state, if a goal state exists in the set of states reachable from the start state.
- It may run forever if there is no solution and the domain is infinite.
- It requires more space than depth-first search.

8.1.7 Dynamic programming

Let's look at breadth-first search in the first city map example, but this time with goal state G:

```
>>> breadthFirst('S', lambda x: x == 'G', map1LegalActions, map1successors)
agenda: Queue([S])
    expanding: S
agenda: Queue([S-0->A, S-1->B])
```

```

    expanding: S-0->A
agenda: Queue([S-1->B, S-0->A-1->C, S-0->A-2->D])
    expanding: S-1->B
agenda: Queue([S-0->A-1->C, S-0->A-2->D, S-1->B-1->D, S-1->B-2->E])
    expanding: S-0->A-1->C
agenda: Queue([S-0->A-2->D, S-1->B-1->D, S-1->B-2->E, S-0->A-1->C-1->F])
    expanding: S-0->A-2->D
agenda: Queue([S-1->B-1->D, S-1->B-2->E, S-0->A-1->C-1->F, S-0->A-2->D-1->B, S-0->A-2->D-2->F, S-0->A-2->D-3->H])
    expanding: S-1->B-1->D
agenda: Queue([S-1->B-2->E, S-0->A-1->C-1->F, S-0->A-2->D-1->B, S-0->A-2->D-2->F, S-0->A-2->D-3->H, S-1->B-1->D-0->A, S-1->B-1->D-2->F, S-1->B-1->D-3->H])
    expanding: S-1->B-2->E
agenda: Queue([S-0->A-1->C-1->F, S-0->A-2->D-1->B, S-0->A-2->D-2->F, S-0->A-2->D-3->H, S-1->B-1->D-0->A, S-1->B-1->D-2->F, S-1->B-1->D-3->H, S-1->B-2->E-1->H])
    expanding: S-0->A-1->C-1->F
16 states visited
[(None, 'S'), (0, 'A'), (1, 'C'), (1, 'F'), (2, 'G')]

```

The first thing that is notable about this trace is that it ends up visiting 16 states in a domain with 9 different states. The issue is that it is exploring multiple paths to the same state. For instance, it has both `S-0->A-2->D` and `S-1->B-1->D` in the agenda. Even worse, it has both `S-0->A` and `S-1->B-1->D-0->A` in there! We really don't need to consider all of these paths. We can make use of the following example of the dynamic programming principle:

The shortest path from X to Z that goes through Y is made up of the shortest path from X to Y and the shortest path from Y to Z.

So, as long as we find the shortest path from the start state to some intermediate state, we don't need to consider any other paths between those two states; there is no way that they can be part of the shortest path between the start and the goal. This insight is the basis of a new pruning principle:

Pruning Rule 3. Don't consider any path that visits a state that you have already visited via some other path.

In breadth-first search, because of the orderliness of the expansion of the layers of the search tree, we can guarantee that the first time we visit a state, we do so along the shortest path. So, we'll keep track of the states that we have visited so far, by using a dictionary, called `visited` that has an entry for every state we have visited.⁴⁸ Then, if we are considering adding a new node to the tree that goes to a state we have already visited, we just ignore it. This test can take the place of the test we used to have for pruning rule 1; it's clear that if the path we are considering already contains this state, then the state has been visited before. Finally, we have to remember, whenever we add a node to the agenda, to add the corresponding state to the visited list.

Here is our breadth-first search code, modified to take advantage of dynamic programming.

⁴⁸ An alternative representation would be just to keep a Python `set` of visited nodes.

```

def breadthFirstDP(initialState, goalTest, actions, successor):
    agenda = Queue()
    if goalTest(initialState):
        return [(None, initialState)]
    agenda.push(SearchNode(None, initialState, None))
    visited = {initialState: True}
    while not agenda.isEmpty():
        parent = agenda.pop()
        for a in actions(parent.state):
            newS = successor(parent.state, a)
            newN = SearchNode(a, newS, parent)
            if goalTest(newS):
                return newN.path()
            elif visited.has_key(newS):
                pass
            else:
                visited[newS] = True:
                agenda.push(newN)
    return None

```

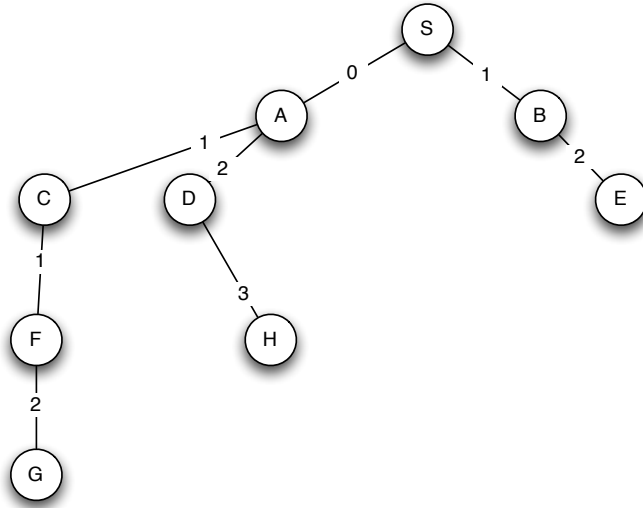
So, let's see how this performs on the task of going from S to G in the first city map:

```

>>> breadthFirstDP('S', lambda x: x == 'G', map1LegalActions, map1successors)
agenda: Queue([S])
  expanding: S
agenda: Queue([S-0->A, S-1->B])
  expanding: S-0->A
agenda: Queue([S-1->B, S-0->A-1->C, S-0->A-2->D])
  expanding: S-1->B
agenda: Queue([S-0->A-1->C, S-0->A-2->D, S-1->B-2->E])
  expanding: S-0->A-1->C
agenda: Queue([S-0->A-2->D, S-1->B-2->E, S-0->A-1->C-1->F])
  expanding: S-0->A-2->D
agenda: Queue([S-1->B-2->E, S-0->A-1->C-1->F, S-0->A-2->D-3->H])
  expanding: S-1->B-2->E
agenda: Queue([S-0->A-1->C-1->F, S-0->A-2->D-3->H])
  expanding: S-0->A-1->C-1->F
8 states visited
[(None, 'S'), (0, 'A'), (1, 'C'), (1, 'F'), (2, 'G')]

```

As you can see, this results in visiting significantly fewer states. Here is the tree generated by this process:



In bigger problems, this effect will be amplified hugely, and will make the difference between whether the algorithm can run in a reasonable amount of time, and not.

We can make the same improvement to depth-first search; we just need to use a stack instead of a queue in the algorithm above. It still will not guarantee that the shortest path will be found, but will guarantee that we never visit more paths than the actual number of states. The only change to breadth-first search with dynamic programming is that the new states are added to the beginning of the agenda.

8.1.8 Configurable search code

Because all of our search algorithms (breadth-first and depth-first, with and without dynamic programming) are all so similar, and we don't like to repeat code, we provide (in file `search.py`) a single, configurable search procedure. It also prints out some information as it goes, if you have the `verbose` or `somewhatVerbose` variables set to `True`, and has a limit on the maximum number of nodes it will expand (to keep from going into an infinite loop).

```

def search(initialState, goalTest, actions, successor,
           depthFirst = False, DP = True, maxNodes = 10000):
    if depthFirst:
        agenda = Stack()
    else:
        agenda = Queue()

    startNode = SearchNode(None, initialState, None)
    if goalTest(initialState):
        return startNode.path()
    agenda.push(startNode)
    if DP: visited = {initialState: True}
    count = 1
    while not agenda.isEmpty() and maxNodes > count:
        n = agenda.pop()
        newStates = []
        for a in actions:

```

```

newS = successor(n.state, a)
newN = SearchNode(a, newS, n)
if goalTest(newS):
    return newN.path()
elif newS in newStates:
    pass
elif ((not DP) and n.inPath(newS)) or \
      (DP and visited.has_key(newS)):
    pass
else:
    count += 1
    if DP: visited[newS] = True
    newStates.append(newS)
    agenda.push(newN)
return None

```

8.2 Connection to state machines

We can use state machines as a convenient representation of state-space search problems. Given a state machine, in its initial state, what sequence of inputs can we feed to it to get it to enter a done state? This is a search problem, analogous to determining the sequence of actions that can be taken to reach a goal state.

The `getNextValues` method of a state machine can serve as the successor function in a search (the inputs to the machine are the actions). Our standard machines do not have a notion of legal actions; but we will add an attribute called `legalInputs`, which is a list of values that are legal inputs to the machine (these are the actions, from the planning perspective) to machines that we want to use with a search.

The `startState` attribute can serve as the initial state in the search and the `done` method of the machine can serve as the goal test function.

Then, we can plan a sequence of actions to go from the start state to one of the done states using this function, where `smToSearch` is an instance of `sm.SM`.

```

def smSearch(smToSearch, initialState = None, goalTest = None, maxNodes = 10000,
            depthFirst = False, DP = True):
    if initialState == None:
        initialState = smToSearch.startState
    if goalTest == None:
        goalTest = smToSearch.done
    return search(initialState, goalTest, smToSearch.legalInputs,
                  # This returns the next state
                  lambda s, a: smToSearch.getNextValues(s, a)[0],
                  maxNodes = maxNodes,
                  depthFirst=depthFirst, DP=DP)

```

It is mostly clerical: it allows us to specify a different initial state or goal test if we want to, and it extracts the appropriate functions out of the state machine and passes them into the search procedure. Also, because `getNextValues` returns both a state and an output, we have to wrap it inside a function that just selects out the next state and returns it.

8.3 Numeric search domain

Many different kinds of problems can be formulated in terms of finding the shortest path through a space of states. A famous one, which is very appealing to beginning calculus students, is to take a derivative of a complex equation by finding a sequence of operations that takes you from the starting expression to one that doesn't contain any derivative operations. We'll explore a different simple one here:

- The states are the integers.
- The initial state is some integer; let's say 1.
- The legal actions are to apply the following operations: $\{2n, n + 1, n - 1, n^2, -n\}$.
- The goal test is `lambda x: x == 10`.

So, the idea would be to find a short sequence of operations to move from 1 to 10.

Here it is, formalized as state machine in Python:

```
class NumberTestSM(sm.SM):
    startState = 1
    legalInputs = ['x*2', 'x+1', 'x-1', 'x**2', '-x']
    def __init__(self, goal):
        self.goal = goal
    def nextState(self, state, action):
        if action == 'x*2':
            return state*2
        elif action == 'x+1':
            return state+1
        elif action == 'x-1':
            return state-1
        elif action == 'x**2':
            return state**2
        elif action == '-x':
            return -state
    def getNextValues(self, state, action):
        nextState = self.nextState(state, action)
        return (nextState, nextState)
    def done(self, state):
        return state == self.goal
```

First of all, this is a bad domain for applying depth-first search. Why? Because it will go off on a gigantic chain of doubling the starting state, and never find the goal. We can run breadth-first search, though. Without dynamic programming, here is what happens (we have `set verbose = False` and `somewhatVerbose = True` in the search file):

```
>>> smSearch(NumberTestSM(10), initialState = 1, depthFirst = False, DP = False)
expanding: 1
expanding: 1-x*2->2
expanding: 1-x-1->0
expanding: 1--x->-1
expanding: 1-x*2->2-x*2->4
expanding: 1-x*2->2-x+1->3
expanding: 1-x*2->2--x->-2
expanding: 1-x-1->0-x-1->-1
```

```

expanding: 1--x->-1-x*2->-2
expanding: 1--x->-1-x+1->0
expanding: 1-x*2->2-x*2->4-x*2->8
expanding: 1-x*2->2-x*2->4-x+1->5
33 states visited
[(None, 1), ('x*2', 2), ('x*2', 4), ('x+1', 5), ('x*2', 10)]

```

We find a nice short path, but visit 33 states. Let's try it with DP:

```

>>> smSearch(NumberTestSM(10), initialState = 1, depthFirst = False, DP = True)
expanding: 1
expanding: 1-x*2->2
expanding: 1-x-1->0
expanding: 1--x->-1
expanding: 1-x*2->2-x*2->4
expanding: 1-x*2->2-x+1->3
expanding: 1-x*2->2--x->-2
expanding: 1-x*2->2-x*2->4-x*2->8
expanding: 1-x*2->2-x*2->4-x+1->5
17 states visited
[(None, 1), ('x*2', 2), ('x*2', 4), ('x+1', 5), ('x*2', 10)]

```

We find the same path, but visit noticeably fewer states. If we change the goal to 27, we find that we visit 564 states without DP and 119, with. If the goal is 1027, then we visit 12710 states without DP and 1150 with DP, which is getting to be a very big difference.

To experiment with depth-first search, we can make a version of the problem where the state space is limited to the integers in some range. We do this by making a subclass of the `NumberTestSM`, which remembers the maximum legal value, and uses it to restrict the set of legal inputs for a state (any input that would cause the successor state to go out of bounds just results in staying at the same state, and it will be pruned.)

```

class NumberTestFiniteSM(NumberTestSM):
    def __init__(self, goal, maxVal):
        self.goal = goal
        self.maxVal = maxVal
    def getNextValues(self, state, action):
        nextState = self.nextState(state, action)
        if abs(nextState) < self.maxVal:
            return (nextState, nextState)
        else:
            return (state, state)

```

Here's what happens if we give it a range of -20 to +20 to work in:

```

>>> smSearch(NumberTestFiniteSM(10, 20), initialState = 1, depthFirst = True,
              DP = False)
expanding: 1
expanding: 1--x->-1
expanding: 1--x->-1-x+1->0
expanding: 1--x->-1-x*2->-2
expanding: 1--x->-1-x*2->-2--x->2
expanding: 1--x->-1-x*2->-2--x->2-x+1->3
expanding: 1--x->-1-x*2->-2--x->2-x+1->3--x->-3

```

```

expanding: 1--x->-1-x*2->-2--x->2-x+1->3--x->-3-x**2->9
20 states visited
[(None, 1), ('-x', -1), ('x*2', -2), ('-x', 2), ('x+1', 3), ('-x', -3), ('x**2', 9), ('x+1', 10)]

```

We generate a much longer path!

We can see from trying lots of different searches in this space that (a) the DP makes the search much more efficient and (b) that the difficulty of these search problems varies incredibly widely.

8.4 Computational complexity

To finish up this segment, let's consider the computational complexity of these algorithms. As we've already seen, there can be a huge variation in the difficulty of a problem that depends on the exact structure of the graph, and is very hard to quantify in advance. It can sometimes be possible to analyze the average case running time of an algorithm, if you know some kind of distribution over the problems you're likely to encounter. We'll just stick with the traditional *worst-case analysis*, which tries to characterize the approximate running time of the worst possible input for the algorithm.

First, we need to establish a bit of notation. Let

- b be the *branching factor* of the graph; that is, the number of successors a node can have. If we want to be truly worst-case in our analysis, this needs to be the maximum branching factor of the graph.
- d be the *maximum depth* of the graph; that is, the length of the longest path in the graph. In an infinite space, this could be infinite.
- l be the *solution depth* of the problem; that is, the length of the shortest path from the start state to the shallowest goal state.
- n be the *state space size* of the graph; that is the total number of states in the domain.

Depth-first search, in the worst case, will search the entire search tree. It has d levels, each of which has b times as many paths as the previous one. So, there are b^d paths on the d^{th} level. The algorithm might have to visit all of the paths at all of the levels, which is about b^{d+1} states. But the amount of storage it needs for the agenda is only $b \cdot d$.

Breadth-first search, on the other hand, only needs to search as deep as the depth of the best solution. So, it might have to visit as many as b^{l+1} nodes. The amount of storage required for the agenda can be as bad as b^l , too.

So, to be clear, consider the numeric search problem. The branching factor $b = 5$, in the worst case. So, if we have to find a sequence of 10 steps, breadth-first search could potentially require visiting as many as $5^{11} = 48828125$ nodes!

This is all pretty grim. What happens when we consider the DP version of breadth-first search? We can promise that every state in the state space is visited at most once. So, it will visit at most n states. Sometimes n is *much* smaller than b^l (for instance, in a road network). In other cases, it can be much larger (for instance, when you are solving an easy (short solution path) problem embedded in a very large space). Even so, the DP version of the search will visit fewer states, except in the very rare case in which there are never multiple paths to the same state (the graph

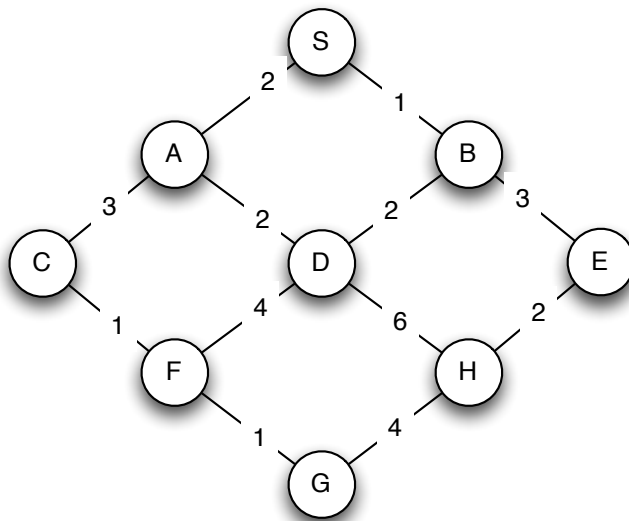
is actually a tree). For example, in the numeric search problem, the shortest path from 1 to 91 is 9 steps long, but using DP it only requires visiting 1973 states, rather than $5^{10} = 9765625$.

8.5 Uniform cost search

In many cases, the arcs in our graph will actually have different costs. In a road network, we would really like to find the shortest path in miles (or in time to traverse), and different road segments have different lengths and times. To handle a problem like this, we need to extend our representation of search problems, and add a new algorithm to our repertoire.

We will extend our notion of a successor function, so that it takes a state and an action, as before, but now it returns a pair (*newS*, *cost*), which represents the resulting state, as well as the cost that is incurred in traversing that arc. To guarantee that all of our algorithms are well behaved, we will require that all costs be positive (not zero or negative).

Here is our original city map, now with distances associated with the roads between the cities.



We can describe it in a dictionary, this time associating a cost with each resulting state, as follows:

```
map1dist = {'S' : [('A', 2), ('B', 1)],
            'A' : [('S', 2), ('C', 3), ('D', 2)],
            'B' : [('S', 1), ('D', 2), ('E', 3)],
            'C' : [('A', 3), ('F', 1)],
            'D' : [('A', 2), ('B', 2), ('F', 4), ('H', 6)],
            'E' : [('B', 3), ('H', 2)],
            'F' : [('C', 1), ('D', 4), ('G', 1)],
            'H' : [('D', 6), ('E', 2), ('G', 4)],
            'G' : [('F', 1), ('H', 4)]}
```

When we studied breadth-first search, we argued that it found the shortest path, in the sense of having the fewest nodes, by seeing that it investigate all of the length 1 paths, then all of the length 2 paths, etc. This orderly enumeration of the paths guaranteed that when we first encountered a goal state, it would be via a shortest path. The idea behind *uniform cost search* is basically the

same: we are going to investigate paths through the graph, in the order of the sum of the costs on their arcs. If we do this, we guarantee that the first time we extract a node with a given state from the agenda, it will be via a shortest path, and so the first time we extract a node with a goal state from the agenda, it will be an optimal solution to our problem.

Priority Queue

Just as we used a stack to manage the agenda for depth-first search and a queue to manage the agenda for bread-first search, we will need to introduce a new data structure, called a *priority queue* to manage the agenda for uniform-cost search. A priority queue is a data structure with the same basic operations as stacks and queues, with two differences:

- Items are pushed into a priority queue with a numeric score, called a *cost*.
- When it is time to pop an item, the item in the priority queue with the least *cost* is returned and removed from the priority queue.

There are many interesting ways to implement priority queues so that they are very computationally efficient. Here, we show a very simple implementation that simply walks down the entire contents of the priority queue to find the least-cost item for a pop operation. Its data attribute consists of a list of (*cost*, *item*) pairs. It calls the `argmaxIndex` procedure from our utility package, which takes a list of items and a scoring function, and returns a pair consisting of the index of the list with the highest scoring item, and the score of that item. Note that, because `argmaxIndex` finds the item with the *highest score*, and we want to extract the item with the *least cost*, our scoring function is the *negative* of the cost.

```
class PQ:
    def __init__(self):
        self.data = []
    def push(self, item, cost):
        self.data.append((cost, item))
    def pop(self):
        (index, cost) = util.argmaxIndex(self.data, lambda (c, x): -c)
        return self.data.pop(index)[1] # just return the data item
    def isEmpty(self):
        return self.data == []
```

UC Search

Now, we're ready to study the uniform-cost search algorithm itself. We will start with a simple version that doesn't do any pruning or dynamic programming, and then add those features back in later. First, we have to extend our definition of a `SearchNode`, to incorporate costs. So, when we create a new search node, we pass in an additional parameter `actionCost`, which represents the cost just for the action that moves from the parent node to the state. Then, we create an attribute `self.cost`, which encodes the cost of this entire path, from the starting state to the last state in the path. We compute it by adding the path cost of the parent to the cost of this last action, as shown by the red text below.

```
class SearchNode:
    def __init__(self, action, state, parent, actionCost):
        self.state = state
        self.action = action
        self.parent = parent
        if self.parent:
            self.cost = self.parent.cost + actionCost
        else:
            self.cost = actionCost
```

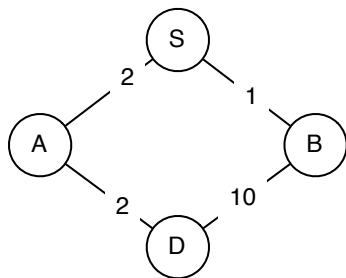
Now, here is the search algorithm. It looks a lot like our standard search algorithm, but there are two important differences:

- The agenda is a priority queue.
- Instead of testing for a goal state when we put an element *into* the agenda, as we did in breadth-first search, we test for a goal state when we take an element *out of* the agenda. This is crucial, to ensure that we actually find the shortest path to a goal state.

```
def ucSearch(initialState, goalTest, actions, successor):
    startNode = SearchNode(None, initialState, None, 0)
    if goalTest(initialState):
        return startNode.path()
    agenda = PQ()
    agenda.push(startNode, 0)
    while not agenda.isEmpty():
        n = agenda.pop()
        if goalTest(n.state):
            return n.path()
        for a in actions:
            (newS, cost) = successor(n.state, a)
            if not n.inPath(newS):
                newN = SearchNode(a, newS, n, cost)
                agenda.push(newN, newN.cost)
    return None
```

Example

Consider the following simple graph:



Let's simulate the uniform-cost search algorithm, and see what happens when we try to start from S and go to D:

- The agenda is initialized to contain the starting node. The agenda is shown as a list of cost, node pairs.

```
agenda: PQ([(0, S)])
```

- The least-cost node, S, is extracted and expanded, adding two new nodes to the agenda. The notation S-0->A means that the path starts in state S, takes action 0, and goes to state A.

```
0 :   expanding: S
agenda: PQ([(2, S-0->A), (1, S-1->B)])
```

- The least-cost node, S-1->B, is extracted, and expanded, adding one new node to the agenda. Note, that, at this point, we have discovered a path to the goal: S-1->B-1->D is a path to the goal, with cost 11. But we cannot be sure that it is the shortest path to the goal, so we simply put it into the agenda, and wait to see if it gets extracted before any other path to a goal state.

```
1 :   expanding: S-1->B
agenda: PQ([(2, S-0->A), (11, S-1->B-1->D)])
```

- The least-cost node, S-0->A is extracted, and expanded, adding one new node to the agenda. At this point, we have two different paths to the goal in the agenda.

```
2 :   expanding: S-0->A
agenda: PQ([(11, S-1->B-1->D), (4, S-0->A-1->D)])
```

- Finally, the least-cost node, S-0->A-1->D is extracted. It is a path to the goal, so it is returned as the solution.

```
5 states visited; Solution cost: 4
[(None, 'S'), (0, 'A'), (1, 'D')]
```

Dynamic programming

Now, we just need to add dynamic programming back in, but we have to do it slightly differently. We promise that, once we have *expanded* a node, that is, taken it out of the agenda, then we have found the shortest path to that state, and we need not consider any further paths that go through that state. So, instead of remembering which nodes we have visited (put onto the agenda) we will remember nodes we have *expanded* (gotten out of the agenda), and never visit or expand a node that has already been expanded. In the code below, the first test ensures that we don't expand a node that goes to a state that we have already found the shortest path to, and the second test ensures that we don't put any additional paths to such a state into the agenda.

```
def ucSearch(initialState, goalTest, actions, successor):
    startNode = SearchNode(None, initialState, None, 0)
    if goalTest(initialState):
        return startNode.path()
    agenda = PQ()
    agenda.push(startNode, 0)
    expanded = { }
    while not agenda.isEmpty():
        n = agenda.pop()
```

```

    if not expanded.has_key(n.state):
        expanded[n.state] = True
        if goalTest(n.state):
            return n.path()
        for a in actions:
            (newS, cost) = successor(n.state, a)
            if not expanded.has_key(newS):
                newN = SearchNode(a, newS, n, cost)
                agenda.push(newN, newN.cost)
return None

```

Here is the result of running this version of uniform cost search on our bigger city graph with distances:

```

mapDistTest(mapldist, 'S', 'G')
agenda: PQ([(0, S)])
0 : expanding: S
agenda: PQ([(2, S-0->A), (1, S-1->B)])
1 : expanding: S-1->B
agenda: PQ([(2, S-0->A), (3, S-1->B-1->D), (4, S-1->B-2->E)])
2 : expanding: S-0->A
agenda: PQ([(3, S-1->B-1->D), (4, S-1->B-2->E), (5, S-0->A-1->C), (4, S-0->A-2->D)])
3 : expanding: S-1->B-1->D
agenda: PQ([(4, S-1->B-2->E), (5, S-0->A-1->C), (4, S-0->A-2->D), (7, S-1->B-1->D-2->F),
(9, S-1->B-1->D-3->H)])
4 : expanding: S-1->B-2->E
agenda: PQ([(5, S-0->A-1->C), (4, S-0->A-2->D), (7, S-1->B-1->D-2->F), (9, S-1->B-1->D-3->H),
(6, S-1->B-2->E-1->H)])
agenda: PQ([(5, S-0->A-1->C), (7, S-1->B-1->D-2->F), (9, S-1->B-1->D-3->H), (6, S-1->B-2->E-1->H)])
5 : expanding: S-0->A-1->C
agenda: PQ([(7, S-1->B-1->D-2->F), (9, S-1->B-1->D-3->H), (6, S-1->B-2->E-1->H), (6, S-0->A-1->C-1->F)])
6 : expanding: S-1->B-2->E-1->H
agenda: PQ([(7, S-1->B-1->D-2->F), (9, S-1->B-1->D-3->H), (6, S-0->A-1->C-1->F), (10, S-1->B-2->E-1->H-2->G)])
6 : expanding: S-0->A-1->C-1->F
agenda: PQ([(7, S-1->B-1->D-2->F), (9, S-1->B-1->D-3->H), (10, S-1->B-2->E-1->H-2->G), (7, S-0->A-1->C-1->F-2->G)])
agenda: PQ([(9, S-1->B-1->D-3->H), (10, S-1->B-2->E-1->H-2->G), (7, S-0->A-1->C-1->F-2->G)])
13 states visited; Solution cost: 7
[(None, 'S'), (0, 'A'), (1, 'C'), (1, 'F'), (2, 'G')]

```

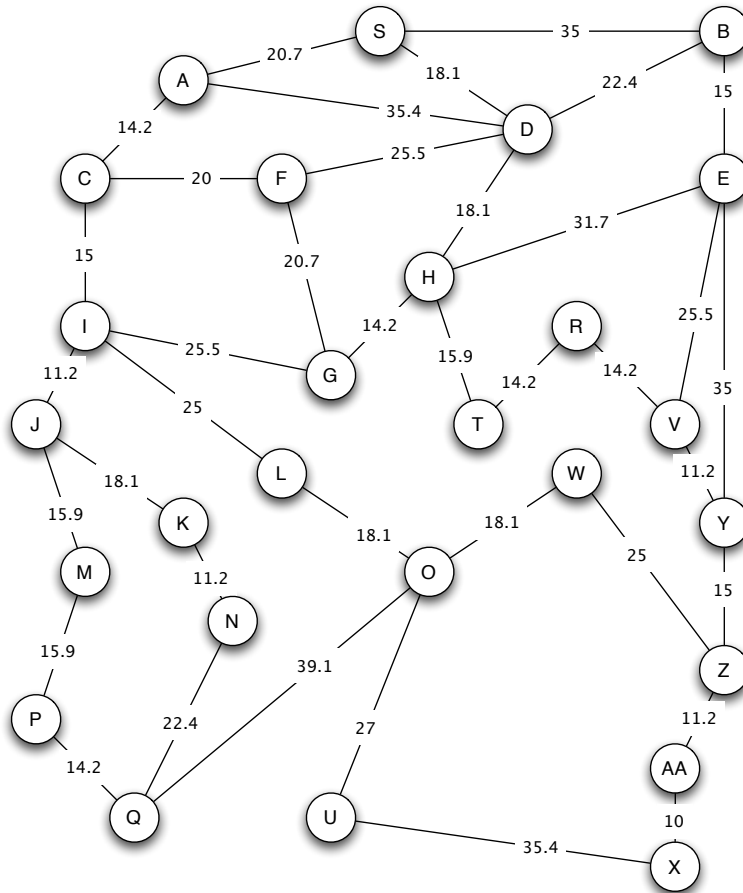
8.5.1 Connection to state machines

When we use a state machine to specify a domain for a cost-based search, we only need to make a small change: the `getNextValues` method of a state machine can still serve as the successor function in a search (the inputs to the machine are the actions). We usually think of `getNextValues` as returning the next state and the output: now, we will modify that interpretation slightly, and think of it as returning the next state and the incremental cost of taking the action that transitions to that next state. This has the same form as the the `ucSearch.search` procedure expects a

successor function to have, so we don't need to change anything about the `smSearch` procedure we have already defined.

8.6 Search with heuristics

Ultimately, we'd like to be able to solve huge state-space search problems, such as those solved by a GPS that can plan long routes through a complex road network. We'll have to add something to uniform-cost search to solve such problems efficiently. Let's consider the city below, where the actual distances between the intersections are shown on the arcs:



If we use uniform cost search to find a path from G to X, we expand states in the following order (the number at the beginning of each line is the length of the path from G to the state at the end of the path):

```
>>> bigTest('G', 'X')
0 :   expanding: G
14.2 :   expanding: G-2->H
20.7 :   expanding: G-1->F
25.5 :   expanding: G-0->I
30.1 :   expanding: G-2->H-2->T
32.3 :   expanding: G-2->H-0->D
36.7 :   expanding: G-0->I-3->J
```

```

40.5 : expanding: G-0->I-0->C
44.3 : expanding: G-2->H-2->T-1->R
45.9 : expanding: G-2->H-1->E
50.4 : expanding: G-2->H-0->D-0->S
50.5 : expanding: G-0->I-2->L
52.6 : expanding: G-0->I-3->J-2->M
54.7 : expanding: G-2->H-0->D-2->B
54.7 : expanding: G-0->I-0->C-0->A
54.8 : expanding: G-0->I-3->J-1->K
58.5 : expanding: G-2->H-2->T-1->R-1->V
66.0 : expanding: G-0->I-3->J-1->K-1->N
68.5 : expanding: G-0->I-3->J-2->M-1->P
68.6 : expanding: G-0->I-2->L-1->O
69.7 : expanding: G-2->H-2->T-1->R-1->V-2->Y
82.7 : expanding: G-0->I-3->J-2->M-1->P-1->Q
84.7 : expanding: G-2->H-2->T-1->R-1->V-2->Y-2->Z
86.7 : expanding: G-0->I-2->L-1->O-1->W
95.6 : expanding: G-0->I-2->L-1->O-2->U
95.9 : expanding: G-2->H-2->T-1->R-1->V-2->Y-2->Z-2->AA

```

39 nodes visited; 27 states expanded; solution cost: 105.9

```
[(None, 'G'), (2, 'H'), (2, 'T'), (1, 'R'), (1, 'V'), (2, 'Y'), (2, 'Z'), (2, 'AA'), (1, 'X')]
```

This search process works its way out, radially, from G, expanding nodes in contours of increasing path length. That means that, by the time the search expands node X, it has expanded every single node. This seems kind of silly: if you were looking for a good route from G to X, it's unlikely that states like S and B would ever come into consideration.

Heuristics

What is it about state B that makes it seem so irrelevant? Clearly, it's far away from where we want to go. We can incorporate this idea into our search algorithm using something called a *heuristic function*. A heuristic function takes a state as an argument and returns a numeric estimate of the total cost that it will take to reach the goal from there. We can modify our search algorithm to be biased toward states that are closer to the goal, in the sense that the heuristic function has a smaller value on them.

In a path-planning domain, such as our example, a reasonable heuristic is the actual Euclidean distance between the current state and the goal state; this makes sense because the states in this domain are actual locations on a map.

A*

If we modify the uniform-cost search algorithm to take advantage of a heuristic function, we get an algorithm called A* (pronounced 'a star'). It is given below, with the differences highlighted in red. The *only* difference is that, when we insert a node into the priority queue, we do so with a cost that is $\text{newN.cost} + \text{heuristic}(\text{newS})$. That is, it is the sum of the actual cost of the path from the start state to the current state, and the estimated cost to go from the current state to the goal.

```

def ucSearch(initialState, goalTest, actions, successor, heuristic):
    startNode = SearchNode(None, initialState, None, 0)
    if goalTest(initialState):
        return startNode.path()
    agenda = PQ()
    agenda.push(startNode, 0)
    expanded = { }
    while not agenda.isEmpty():
        n = agenda.pop()
        if not expanded.has_key(n.state):
            expanded[n.state] = True
            if goalTest(n.state):
                return n.path()
            for a in actions:
                (newS, cost) = successor(n.state, a)
                if not expanded.has_key(newS):
                    newN = SearchNode(a, newS, n, cost)
                    agenda.push(newN, newN.cost + heuristic(newS))
    return None

```

Example

Now, we can try to search in the big map for a path from G to X, using, as our heuristic function, the distance between the state of interest and X. Here is a trace of what happens (with the numbers rounded to increase readability):

- We get the start node out of the agenda, and add its children. Note that the costs are the actual path cost *plus* the heuristic estimate.

```

0 : expanding: G
agenda: PQ([(107, G-0->I), (101, G-1->F), (79, G-2->H)])

```

- The least cost path is G-2->H, so we extract it, and add its successors.

```

14.2 : expanding: G-2->H
agenda: PQ([(107, G-0->I), (101, G-1->F), (109, G-2->H-0->D), (116, G-2->H-1->E), (79, G-2->H-2->T)])

```

- Now, we can see the heuristic function really having an effect. The path G-2->H-2->T has length 30.1, and the path G-1->F has length 20.7. But when we add in the heuristic cost estimates, the path to T has a lower cost, because it seems to be going in the right direction. Thus, we select G-2->H-2->T to expand next:

```

30.1 : expanding: G-2->H-2->T
agenda: PQ([(107, G-0->I), (101, G-1->F), (109, G-2->H-0->D), (116, G-2->H-1->E), (100, G-2->H-2->T-1->R)])

```

- Now the path G-2->H-2->T-1->R looks best, so we expand it.

```

44.3 : expanding: G-2->H-2->T-1->R
agenda: PQ([(107, G-0->I), (101, G-1->F), (109, G-2->H-0->D), (116, G-2->H-1->E), (103.5, G-2->H-2->T-1->R-1->V)])

```

- Here, something interesting happens. The node with the least estimated cost is G-1->F. It's going in the wrong direction, but if we were to be able to fly straight from F to X, then that would be a good way to go. So, we expand it:

20.7 : expanding: G-1->F

agenda: PQ([(107, G-0->I), (109, G-2->H-0->D), (116, G-2->H-1->E), (103.5, G-2->H-2->T-1->R-1->V), (123, G-1->F-0->D), (133, G-1->F-1->C)])

- Continuing now, basically straight to the goal, we have:

58.5 : expanding: G-2->H-2->T-1->R-1->V

agenda: PQ([(107, G-0->I), (109, G-2->H-0->D), (116, G-2->H-1->E), (123, G-1->F-0->D), (133, G-1->F-1->C), (154, G-2->H-2->T-1->R-1->V-1->E), (105, G-2->H-2->T-1->R-1->V-2->Y)])

69.7 : expanding: G-2->H-2->T-1->R-1->V-2->Y

agenda: PQ([(107, G-0->I), (109, G-2->H-0->D), (116, G-2->H-1->E), (123, G-1->F-0->D), (133, G-1->F-1->C), (154, G-2->H-2->T-1->R-1->V-1->E), (175, G-2->H-2->T-1->R-1->V-2->Y-0->E), (105, G-2->H-2->T-1->R-1->V-2->Y-2->Z)])

84.7 : expanding: G-2->H-2->T-1->R-1->V-2->Y-2->Z

agenda: PQ([(107, G-0->I), (109, G-2->H-0->D), (116, G-2->H-1->E), (123, G-1->F-0->D), (133, G-1->F-1->C), (154, G-2->H-2->T-1->R-1->V-1->E), (175, G-2->H-2->T-1->R-1->V-2->Y-0->E), (151, G-2->H-2->T-1->R-1->V-2->Y-2->Z-1->W), (106, G-2->H-2->T-1->R-1->V-2->Y-2->Z-2->AA)])

95.9 : expanding: G-2->H-2->T-1->R-1->V-2->Y-2->Z-2->AA

agenda: PQ([(107, G-0->I), (109, G-2->H-0->D), (116, G-2->H-1->E), (123, G-1->F-0->D), (133, G-1->F-1->C), (154, G-2->H-2->T-1->R-1->V-1->E), (175, G-2->H-2->T-1->R-1->V-2->Y-0->E), (151, G-2->H-2->T-1->R-1->V-2->Y-2->Z-1->W), (106, G-2->H-2->T-1->R-1->V-2->Y-2->Z-2->AA-1->X)])

18 nodes visited; 10 states expanded; solution cost: 105.9

[(None, 'G'), (2, 'H'), (2, 'T'), (1, 'R'), (1, 'V'), (2, 'Y'), (2, 'Z'), (2, 'AA'), (1, 'X')]

Using A* has roughly halved the number of nodes visited and expanded. In some problems it can result in an enormous savings, but, as we'll see in the next section, it depends on the heuristic we use.

Good and bad heuristics

In order to think about what makes a heuristic good or bad, let's imagine what the perfect heuristic would be. If we were to magically know the distance, via the shortest path in the graph, from each node to the goal, then we could use that as a heuristic. It would lead us directly from start to goal, without expanding any extra nodes. But, of course, that's silly, because it would be at least as hard to compute the heuristic function as it would be to solve the original search problem.

So, we would like our heuristic function to give an estimate that is as close as possible to the true shortest-path-length from the state to the goal, but also to be relatively efficient to compute.

An important additional question is: if we use a heuristic function, are we still guaranteed to find the shortest path through our state space? The answer is: yes, if the heuristic function is *admissible*. A heuristic function is admissible if it is guaranteed to be an *underestimate* of the actual cost of the optimal path to the goal. To see why this is important, consider a state *s* from which the goal can actually be reached in 10 steps, but for which the heuristic function gives a

value of 100. Any path to that state will be put into the agenda with a total cost of 90 more than the true cost. That means that if a path is found that is as much as 89 units more expensive than the optimal path, it will be accepted and returned as a result of the search.

It is important to see that if our heuristic function always returns value 0, it is admissible. And, in fact, with that heuristic, the A^* algorithm reduces to uniform cost search.

In the example of navigating through a city, we used the Euclidean distance between cities, which, if distance is our cost, is clearly admissible; there's no shorter path between any two points.

Exercise 8.1. Would the so-called 'Manhattan distance', which is the sum of the absolute differences of the x and y coordinates be an admissible heuristic in the city navigation problem, in general? Would it be admissible in Manhattan?

Exercise 8.2. If we were trying to minimize travel time on a road network (and so the estimated time to travel each road segment was the cost), what would be an appropriate heuristic function?

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