Chapter 2: Introduction to Electrodynamics

2.1 Maxwell's differential equations in the time domain

Whereas the Lorentz force law characterizes the observable effects of electric and magnetic fields on charges, Maxwell's equations characterize the origins of those fields and their relationships to each other. The simplest representation of Maxwell's equations is in differential form, which leads directly to waves; the alternate integral form is presented in Section 2.4.3.

The differential form uses the vector *del operator* ∇:

$$\nabla \equiv \hat{x}\frac{\partial}{\partial \mathrm{x}} + \hat{y}\frac{\partial}{\partial \mathrm{y}} + \hat{z}\frac{\partial}{\partial \mathrm{z}} \tag{2.1.1}$$

where $\hat{x}$, $\hat{y}$, and $\hat{z}$ are defined as unit vectors in cartesian coordinates. Relations involving ∇ are summarized in Appendix D. Here we use the conventional vector *dot product*[1] and *cross product*[2] of ∇ with the electric and magnetic field vectors where, for example:

$$\overline{\mathrm{E}} = \hat{x}\mathrm{E}_\mathrm{x} + \hat{y}\mathrm{E}_\mathrm{y} + \hat{z}\mathrm{E}_\mathrm{z} \tag{2.1.2}$$

$$\nabla \bullet \overline{\mathrm{E}} \equiv \frac{\partial \mathrm{E}_\mathrm{x}}{\partial \mathrm{x}} + \frac{\partial \mathrm{E}_\mathrm{y}}{\partial \mathrm{y}} + \frac{\partial \mathrm{E}_\mathrm{z}}{\partial \mathrm{z}} \tag{2.1.3}$$

We call $\nabla \bullet \overline{\mathrm{E}}$ the *divergence* of $\overline{\mathrm{E}}$ because it is a measure of the degree to which the vector field $\overline{\mathrm{E}}$ diverges or flows outward from any position. The cross product is defined as:

$$\begin{aligned} \nabla \times \overline{\mathrm{E}} \; &\equiv \hat{x}\left(\frac{\partial \mathrm{E}_\mathrm{z}}{\partial \mathrm{y}} - \frac{\partial \mathrm{E}_\mathrm{y}}{\partial \mathrm{z}}\right) + \hat{y}\left(\frac{\partial \mathrm{E}_\mathrm{x}}{\partial \mathrm{z}} - \frac{\partial \mathrm{E}_\mathrm{z}}{\partial \mathrm{x}}\right) + \hat{z}\left(\frac{\partial \mathrm{E}_\mathrm{y}}{\partial \mathrm{x}} - \frac{\partial \mathrm{E}_\mathrm{x}}{\partial \mathrm{y}}\right) \\ &= \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial \mathrm{x} & \partial/\partial \mathrm{y} & \partial/\partial \mathrm{z} \\ \mathrm{E}_\mathrm{x} & \mathrm{E}_\mathrm{y} & \mathrm{E}_\mathrm{z} \end{vmatrix} \end{aligned} \tag{2.1.4}$$

which is often called the *curl* of $\overline{\mathrm{E}}$. Figure 2.1.1 illustrates when the divergence and curl are zero or non-zero for five representative field distributions.

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[1] The dot product of $\overline{\mathrm{A}}$ and $\overline{\mathrm{B}}$ can be defined as $\overline{\mathrm{A}} \bullet \overline{\mathrm{B}} = \mathrm{A}_\mathrm{x}\mathrm{B}_\mathrm{x} + \mathrm{A}_\mathrm{y}\mathrm{B}_\mathrm{y} + \mathrm{A}_\mathrm{z}\mathrm{B}_\mathrm{z} = |\mathrm{A}||\mathrm{B}|\cos\theta$, where θ is the angle between the two vectors.

[2] The cross product of $\overline{\mathrm{A}}$ and $\overline{\mathrm{B}}$ can be defined as $\overline{\mathrm{A}} \times \overline{\mathrm{B}} = \underline{\hat{x}}\left(\mathrm{A}_\mathrm{y}\mathrm{B}_\mathrm{z} - \mathrm{A}_\mathrm{z}\mathrm{B}_\mathrm{y}\right) + \hat{y}\left(\mathrm{A}_\mathrm{z}\mathrm{B}_\mathrm{x} - \mathrm{A}_\mathrm{x}\mathrm{B}_\mathrm{z}\right) + \hat{z}\left(\mathrm{A}_\mathrm{x}\mathrm{B}_\mathrm{y} - \mathrm{A}_\mathrm{y}\mathrm{B}_\mathrm{x}\right)$; its magnitude is $|\overline{\mathrm{A}}| \bullet |\overline{\mathrm{B}}| \sin\theta$. Alternatively, $\overline{\mathrm{A}} \times \overline{\mathrm{B}} = \det|[\mathrm{A_x},\mathrm{A_y},\mathrm{A_z}],[\mathrm{B_x},\mathrm{B_y},\mathrm{B_z}],[\,\hat{x},\hat{y},\hat{z}\,]|$.

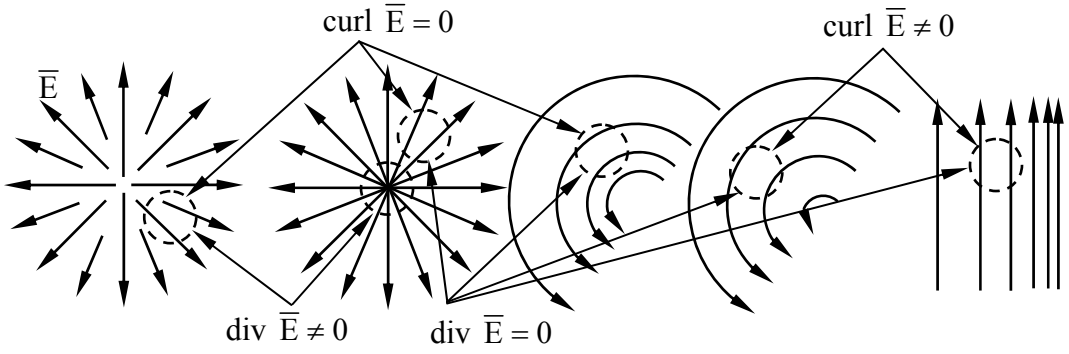

Figure 2.1.1 Fields with zero or non-zero divergence or curl.

The differential form of *Maxwell's equations* in the time domain are:

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \textit{Faraday's Law} \qquad (2.1.5)$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \textit{Ampere's Law} \qquad (2.1.6)$$

$$\nabla \bullet \overline{D} = \rho \qquad \textit{Gauss's Law} \qquad (2.1.7)$$

$$\nabla \bullet \overline{B} = 0 \qquad \textit{Gauss's Law} \qquad (2.1.8)$$

The field variables are defined as:

$\overline{E}$	*electric field*	[volts/meter; Vm^{-1}]	(2.1.9)
$\overline{H}$	*magnetic field*	[amperes/meter; Am^{-1}]	(2.1.10)
$\overline{B}$	*magnetic flux density*	[Tesla; T]	(2.1.11)
$\overline{D}$	*electric displacement*	[coulombs/m^2; Cm^{-2}]	(2.1.12)
$\overline{J}$	*electric current density*	[amperes/m^2; Am^{-2}]	(2.1.13)
ρ	*electric charge density*	[coulombs/m^3; Cm^{-3}]	(2.1.14)

These four Maxwell equations invoke one scalar and five vector quantities comprising 16 variables. Some variables only characterize how matter alters field behavior, as discussed later in Section 2.5. In vacuum we can eliminate three vectors (9 variables) by noting:

$$\overline{D} = \varepsilon_o \overline{E} \qquad \text{(constitutive relation for } \overline{D}\text{)} \qquad (2.1.15)$$

$$\overline{B} = \mu_o \overline{H} \qquad \text{(constitutive relation for } \overline{B}\text{)} \qquad (2.1.16)$$

$$\bar{J} = \rho \bar{v} = \sigma \overline{E} \qquad \text{(constitutive relation for } \bar{J}\text{)} \qquad (2.1.17)$$

where $\varepsilon_o = 8.8542 \times 10^{-12}$ [farads m^{-1}] is the *permittivity* of vacuum, $\mu_o = 4\pi \times 10^{-7}$ [henries m^{-1}] is the *permeability* of vacuum[3], $\bar{v}$ is the velocity of the local net charge density ρ, and σ is the *conductivity* of a medium [Siemens m^{-1}]. If we regard the electrical sources ρ and $\bar{J}$ as given, then the equations can be solved for all remaining unknowns. Specifically, we can then find $\overline{E}$ and $\overline{H}$, and thus compute the forces on all charges present. Except for special cases we shall avoid solving problems where the electromagnetic fields and the motions of ρ are interdependent.

The constitutive relations for vacuum, $\overline{D} = \varepsilon_0 \overline{E}$ and $\overline{B} = \mu_0 \overline{H}$, can be generalized to $\overline{D} = \varepsilon \overline{E}$, $\overline{B} = \mu \overline{H}$, and $\bar{J} = \sigma \overline{E}$ for simple media. Media are discussed further in Section 2.5.

Maxwell's equations require conservation of charge. By taking the divergence of Ampere's law (2.1.6) and noting the vector identity $\nabla \bullet (\nabla \times \overline{A}) = 0$, we find:

$$\nabla \bullet (\nabla \times \overline{H}) = 0 = \nabla \bullet \frac{\partial \overline{D}}{\partial t} + \nabla \bullet \bar{J} \qquad (2.1.18)$$

Then, by reversing the sequence of the derivatives in (2.1.18) and substituting Gauss's law $\nabla \bullet \overline{D} = \rho$ (2.1.7), we obtain the differential expression for *conservation of charge*:

$$\nabla \bullet \bar{J} = -\frac{\partial \rho}{\partial t} \qquad \text{(conservation of charge)} \qquad (2.1.19)$$

The integral expression can be derived from the differential expression by using *Gauss's divergence theorem*, which relates the integral of $\nabla \bullet \overline{G}$ over any volume V to the integral of $\overline{G} \bullet \hat{n}$ over the surface area A of that volume, where the surface normal unit vector $\hat{n}$ points outward:

$$\iiint_V \nabla \bullet \overline{G} \; dv = \oiint_A \overline{G} \bullet \hat{n} \; da \qquad \text{(Gauss's divergence theorem)} \qquad (2.1.20)$$

Thus the integral expression for conservation of charge is:

$$\frac{d}{dt} \iiint_V \rho \; dv = -\oiint_A \bar{J} \bullet \hat{n} \; da \qquad \text{(conservation of charge)} \qquad (2.1.21)$$

[3] The constant $4\pi \times 10^{-7}$ is exact and enters into the definition of an ampere.

which says that if no net current $\overline{J}$ flows through the walls A of a volume V, then the total charge inside must remain constant.

Example 2.1A

If the electric field in vacuum is $\overline{E} = \hat{x}E_o \cos(\omega t - ky)$, what is $\overline{H}$?

Solution: From Faraday's law (2.1.5): $\mu_o(\partial\overline{H}/\partial t) = -(\nabla\times\overline{E}) = \hat{z}\,\partial E_x/\partial y = \hat{z}\,kE_o \sin(\omega t - ky)$, using (2.1.4) for the curl operator. Integration of this equation with respect to time yields: $\overline{H} = -\hat{z}\left(kE_o/\mu_o\omega\right)\cos(\omega t - ky)$.

Example 2.1B

Does the electric field in vacuum $\overline{E} = \hat{x}E_o \cos(\omega t - kx)$ satisfy Maxwell's equations? Under what circumstances would this $\overline{E}$ satisfy the equations?

Solution: This electric field does not satisfy Gauss's law for vacuum, which requires $\nabla\bullet\overline{D} = \rho = 0$. It satisfies Gauss's law only for non-zero charge density: $\rho = \nabla\bullet\overline{D} = \varepsilon_o\partial E_x/\partial x = \partial\left[\varepsilon_o E_o \cos(\omega t - kx)\right]/\partial x = k\varepsilon_o E_o \sin(\omega t - kx) \neq 0$. To satisfy the remaining Maxwell equations and conservation of charge (2.1.19) there must also be a current $\overline{J} \neq 0$ corresponding to $\rho : \overline{J} = \sigma\overline{E} = \hat{x}\sigma E_o \cos(\omega t - kx)$, where (2.1.17) simplified the computation.

2.2 Electromagnetic waves in the time domain

Perhaps the greatest triumph of Maxwell's equations was their ability to predict in a simple way the existence and velocity of electromagnetic waves based on simple laboratory measurements of the permittivity and permeability of vacuum. In vacuum the charge density ρ = $\overline{J}$ = 0, and so Maxwell's equations become:

$$\nabla\times\overline{E} = -\mu_o\frac{\partial\overline{H}}{\partial t} \qquad \text{(Faraday's law in vacuum)} \qquad (2.2.1)$$

$$\nabla\times\overline{H} = \varepsilon_o\frac{\partial\overline{E}}{\partial t} \qquad \text{(Ampere's law in vacuum)} \qquad (2.2.2)$$

$$\nabla\bullet\overline{E} = 0 \qquad \text{(Gauss's law in vacuum)} \qquad (2.2.3)$$

$$\nabla\bullet\overline{H} = 0 \qquad \text{(Gauss's law in vacuum)} \qquad (2.2.4)$$

We can eliminate $\overline{H}$ from these equations by computing the curl of Faraday's law, which introduces $\nabla\times\overline{H}$ on its right-hand side so Ampere's law can be substituted:

$$\nabla\times(\nabla\times\overline{E}) = -\mu_o\frac{\partial(\nabla\times\overline{H})}{\partial t} = -\mu_o\varepsilon_o\frac{\partial\overline{E}^2}{\partial t^2} \tag{2.2.5}$$

Using the well known vector identity (see Appendix D):

$$\nabla\times(\nabla\times\overline{A}) = \nabla(\nabla\bullet\overline{A}) - \nabla^2\overline{A} \qquad \text{(“well-known vector identity”)} \tag{2.2.6}$$

and then using (2.2.3) to eliminate $\nabla\bullet\overline{E}$, (2.2.5) becomes the electromagnetic *wave equation*, often called the *Helmholtz wave equation*:

$$\nabla^2\overline{E} - \mu_o\varepsilon_o\frac{\partial^2\overline{E}}{\partial t^2} = 0 \qquad \text{(Helmholtz wave equation)} \tag{2.2.7}$$

where:

$$\nabla^2\overline{E} \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\hat{x}E_x + \hat{y}E_y + \hat{z}E_z\right) \tag{2.2.8}$$

The solutions to this wave equation (2.2.7) are any fields $\overline{E}(\bar{r},t)$ for which the second spatial derivative $(\nabla^2\overline{E})$ equals a constant times the second time derivative $(\partial^2\overline{E}/\partial t^2)$. The *position vector* $\bar{r} \equiv \hat{x}x + \hat{y}y + \hat{z}z$. The wave equation is therefore satisfied by any arbitrary $\overline{E}(\bar{r},t)$ having identical dependence on space and time within a constant multiplier. For example, arbitrary functions of the arguments (z - ct), (z + ct), or (t ± z/c) have such an identical dependence and are among the valid solutions to (2.2.7), where c is some constant to be determined. One such solution is:

$$\overline{E}(\bar{r},t) = \overline{E}(z-ct) = \hat{x}E_x(z-ct) \tag{2.2.9}$$

where the arbitrary function $E_x(z - ct)$ might be that illustrated in Figure 2.2.1 at time t = 0 and again at some later time t. Note that as time advances within the argument (z - ct), z must advance with ct in order for that argument, or $\overline{E}$ at any point of interest on the waveform, to remain constant.

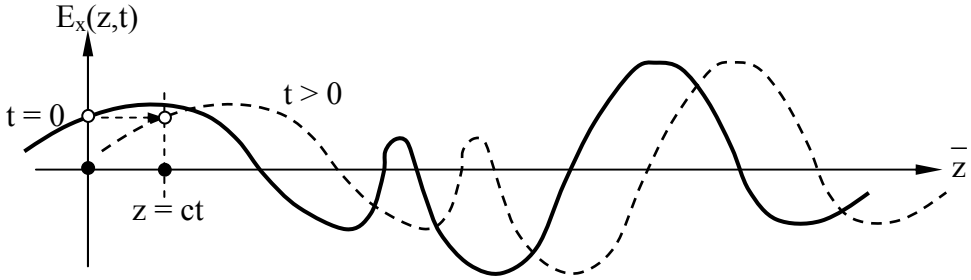

Figure 2.2.1 Arbitrary electromagnetic wave propagating in the +z direction.

We can test this candidate solution (2.2.9) by substituting it into the wave equation (2.2.7), yielding:

$$\begin{aligned}\nabla^2\overline{E}(z-ct) &= \frac{\partial^2\left[\overline{E}(z-ct)\right]}{\partial z^2} \equiv \overline{E}''(z-ct) \\ &= \mu_o\varepsilon_o\frac{\partial^2\left[\overline{E}(z-ct)\right]}{\partial t^2} = \mu_o\varepsilon_o(-c)^2\,\overline{E}''(z-ct)\end{aligned} \tag{2.2.10}$$

where we define $\overline{A}'(q)$ as the first derivative of $\overline{A}$ with respect to its argument q and $\overline{A}''(q)$ as its second derivative. Equation (2.2.10) is satisfied if:

$$c = \frac{1}{\sqrt{(\mu_o\varepsilon_o}} \quad [m/s] \tag{2.2.11}$$

where we define c as the *velocity of light* in vacuum:

$$c = 2.998\times10^8 \ [m\ s^{-1}] \qquad \text{(velocity of light)} \tag{2.2.12}$$

Figure 2.2.1 illustrates how an arbitrary $\overline{E}(z,t)$ can propagate by translating at velocity c. However, some caution is warranted when $\overline{E}(z,t)$ is defined. Although our trial solution (2.2.9) satisfies the wave equation (2.2.7), it may not satisfy Gauss's laws. For example, consider the case where:

$$\overline{E}(z,t) = \hat{z}E_z(z-ct) \tag{2.2.13}$$

Then Gauss's law $\nabla\bullet\overline{E} = 0$ is not satisfied:

$$\nabla\bullet\overline{E} = \frac{\partial\overline{E}_z}{\partial z} \neq 0 \text{ for arbitrary } \overline{E}(z) \tag{2.2.14}$$

In contrast, if $\overline{E}(z,t)$ is oriented perpendicular to the direction of propagation (in the $\hat{x}$ and/or $\hat{y}$ directions for z-directed propagation), then all Maxwell's equations are satisfied and the solution is valid. In the case $\overline{E}(z,t) = \hat{y}E_y(z-ct)$, independent of x and y, we have a *uniform plane wave* because the fields are uniform with respect to two of the coordinates (x,y) so that $\partial\overline{E}/\partial x = \partial\overline{E}/\partial y = 0$. Since this electric field is in the y direction, it is said to be y-polarized; by convention, *polarization* of a wave refers to the direction of its electric vector. Polarization is discussed further in Section 2.3.4.

Knowing $\overline{E}(z,t) = \hat{y}E_y(z-ct)$ for this example, we can now find $\overline{H}(z,t)$ using Faraday's law (2.2.1):

$$\frac{\partial \overline{H}}{\partial t} = -\frac{(\nabla \times \overline{E})}{\mu_o} \tag{2.2.15}$$

We can evaluate the curl of $\overline{E}$ using (2.1.4) and knowing $E_x = E_z = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$:

$$\nabla \times \overline{E} = \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -\hat{x}\frac{\partial E_y}{\partial z} \tag{2.2.16}$$

Then, by integrating (2.2.15) over time it becomes:

$$\begin{aligned} \overline{H}(z,t) &= -\int_{-\infty}^{t} \frac{(\nabla \times \overline{E})}{\mu_o} dt = \hat{x}\frac{1}{\mu_o}\int_{-\infty}^{t} \frac{\partial E_y(z-ct)}{\partial z} dt \\ &= -\hat{x}\frac{1}{c\mu_o} E_y(z-ct) = -\hat{x}\sqrt{\frac{\varepsilon_o}{\mu_o}} E_y(z-ct) \end{aligned} \tag{2.2.17}$$

$$\overline{H}(z,t) = \sqrt{\frac{\varepsilon_o}{\mu_o}}\hat{z} \times \overline{E}(z,t) = \hat{z} \times \frac{\overline{E}(z,t)}{\eta_o} \tag{2.2.18}$$

where we used the velocity of light $c = 1/\sqrt{\varepsilon_o \mu_o}$, and defined $\eta_o = \sqrt{\mu_o/\varepsilon_o}$.

Thus $\overline{E}$ and $\overline{H}$ in a uniform plane wave are very simply related. Their directions are orthogonal to each other and to the direction of propagation, and the magnitude of the electric field is $(\mu_o/\varepsilon_o)^{0.5}$ times that of the magnetic field; this factor $\eta_o = \sqrt{\mu_o/\varepsilon_o}$ is known as the *characteristic impedance of free space* and equals ~377 ohms. That is, for a single uniform plane wave in free space,

$$|\overline{E}|/|\overline{H}| = \eta_o = \sqrt{\mu_o/\varepsilon_o} \cong 377 \ \text{[ohms]} \tag{2.2.19}$$

Electromagnetic waves can propagate in any arbitrary direction in space with arbitrary time behavior. That is, we are free to define $\hat{x}$, $\hat{y}$, and $\hat{z}$ in this example as being in any three orthogonal directions in space. Because Maxwell's equations are linear in field strength, superposition applies and any number of plane waves propagating in arbitrary directions with arbitrary polarizations can be superimposed to yield valid electromagnetic solutions. Exactly which superposition is the valid solution in any particular case depends on the boundary conditions and the initial conditions for that case, as discussed later in Chapter 9 for a variety of geometries.

Example 2.2A

Show that $\overline{E} = \hat{y}E_o(t + z/c)$ satisfies the wave equation (2.2.7). In which direction does this wave propagate?

Solution: $\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)\overline{E} = \hat{y}\frac{1}{c^2}\left[E_o''(t + z/c) - E_o''(t + z/c)\right] = 0$; Q.E.D[4]. Since the argument (t + z/c) remains constant as t increases only if z/c decreases correspondingly, the wave is propagating in the -z direction.

2.3 Maxwell's equations, waves, and polarization in the frequency domain

2.3.1 Sinusoidal waves

Linear systems are easily characterized by the magnitude and phase of each output as a function of the frequency at which the input is sinusoidally stimulated. This simple characterization is sufficient because sinusoids of different frequencies can be superimposed to construct any arbitrary input waveform[5], and the output of a linear system is the superposition of its responses to each superimposed input. Systems with multiple inputs and outputs can be characterized in the same way. Nonlinear systems are more difficult to characterize because their output frequencies generally include harmonics of their inputs.

Fortunately free space is a linear system, and therefore it is fully characterized by its response to sinusoidal plane waves. For example, the arbitrary z-propagating x-polarized uniform plane wave of (2.2.9) and Figure 2.2.1 could be sinusoidal and represented by:

$$\overline{E}(\bar{r}, t) = \hat{x}E_o \cos\left[k(z - ct)\right] \tag{2.3.1}$$

$$\overline{H}(\bar{r}, t) = \hat{y}\sqrt{\varepsilon_o/\mu_o}\,E_o \cos\left[k(z - ct)\right] \tag{2.3.2}$$

where the *wave amplitude* E_o is a constant and the factor k is related to frequency, as shown below.

It is more common to represent sinusoidal waves using the argument (ωt - kz) so that their frequency and spatial dependences are more evident. The *angular frequency* ω is simply related to frequency f [Hz]:

$$\omega = 2\pi f \ \text{[radians s}^{-1}\text{]} \qquad \text{(angular frequency)} \tag{2.3.3}$$

[4] Q.E.D. is the abbreviation for the Latin phrase "*quod erat demonstratum*" or "that which was to be demonstrated."

[5] The Fourier transform pair (10.4.17) and (10.4.18) relate arbitrary pulse waveforms h(t) to their corresponding spectra H(f), where each frequency f has its own magnitude and phase represented by H(f).

and the *spatial frequency* k, often called the *wavenumber*, is simply related to ω and wavelength λ [m], which is the length of one period in space:

$$k = 2\pi/\lambda = \omega/c \quad [\text{radians m}^{-1}] \qquad \text{(wave number)} \qquad (2.3.4)$$

The significance and dimensions of ω and k are directly analogous; they are radians s^{-1} and radians m^{-1}, respectively.

Therefore we can alternatively represent the wave of (2.3.1) and (2.3.2) as:

$$\overline{E}(\bar{r},t) = \hat{x}E_o \cos(\omega t - kz) \quad [\text{v m}^{-1}] \qquad (2.3.5)$$

$$\overline{H}(z,t) = \hat{y}\sqrt{\varepsilon_o/\mu_o}\,E_o \cos(\omega t - kz) \quad [\text{A m}^{-1}] \qquad (2.3.6)$$

Figure 2.3.1 suggests the form of this wave. Its *wavelength* is λ, the length of one cycle, where:

$$\lambda = c/f \quad [\text{m}] \qquad \text{(wavelength)} \qquad (2.3.7)$$

The figure illustrates how these electric and magnetic fields are in phase but orthogonal to each other and to the direction of propagation. When the argument (ωt – kz) equals zero, the fields are maximum, consistent with cos(ωt - kz).

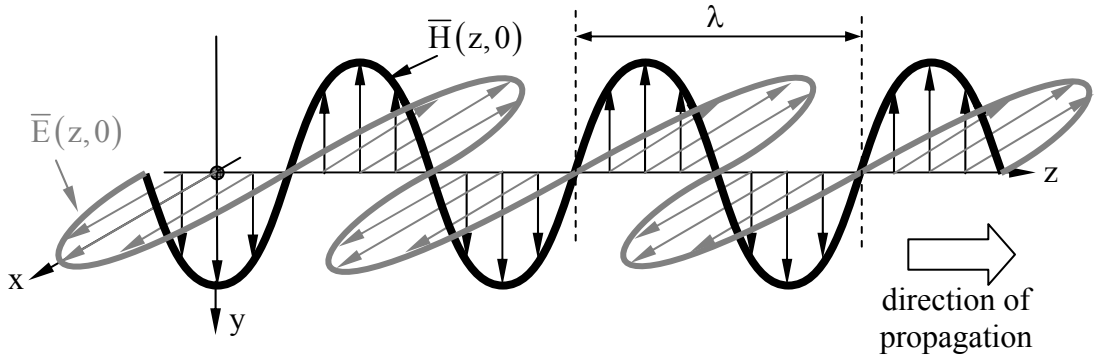

Figure 2.3.1 +z propagating y-polarized uniform plane wave of wavelength λ.

This notation makes it easy to characterize uniform plane waves propagating in other directions as well. For example:

$$\overline{E}(\bar{r},t) = \hat{x}E_o \cos(\omega t + kz) \qquad \text{(x-polarized wave in -z direction)} \qquad (2.3.8)$$

$$\bar{E}(\bar{r},t) = \hat{y}E_o\cos(\omega t - kz) \qquad \text{(y-polarized wave in +z direction)} \qquad (2.3.9)$$

$$\bar{E}(\bar{r},t) = \hat{y}E_o\cos(\omega t - kx) \qquad \text{(y-polarized wave in +x direction)} \qquad (2.3.10)$$

$$\bar{E}(\bar{r},t) = \hat{z}E_o\cos(\omega t + kx) \qquad \text{(z-polarized wave in -x direction)} \qquad (2.3.11)$$

2.3.2 Maxwell's equations in the complex-frequency domain

Electromagnetic fields are commonly characterized in the frequency domain in terms of their magnitudes and phases as a function of position $\bar{r}$ for frequency f. For example, the $\hat{x}$ component of a general sinusoidally varying $\bar{E}$ might be:

$$\bar{E}(\bar{r},t) = \hat{x}E_x(\bar{r})\cos\left[\omega t + \phi(\bar{r})\right] \qquad (2.3.12)$$

This might become $\bar{E}(\bar{r},t) = \hat{x}E_x\cos(\omega t - kz)$ for a uniform plane wave propagating in the +z direction.

It is generally more convenient to express phase using *complex notation* (see Appendix B). The x-component of the wave of (2.3.12) can also be represented as:

$$E_x(\bar{r},t) = R_e\left\{E_x(\bar{r})e^{j\left(\omega t+\phi_x(\bar{r})\right)}\right\} = \hat{x}R_e\left\{\underline{E}_x(\bar{r})e^{j\omega t}\right\} \qquad (2.3.13)$$

where the spatial and frequency parts of $E_x(\bar{r},t)$ have been separated and $\underline{E}_x(\bar{r}) = \left|\underline{E}_x(\bar{r})\right|e^{j\phi_x(\bar{r})}$ is called a *phasor*. The simplicity will arise later when we omit $R_e\{[\ ]e^{j\omega t}\}$ from our expressions as "understood", so only the phasors remain. The underbar under $\underline{E}_x$ indicates $\underline{E}_x$ is not a function of time, but rather is a complex quantity with a real part and an imaginary part, where:

$$\underline{E}_x(\bar{r}) = R_e\{\underline{E}_x(\bar{r})\} + jI_m\{\underline{E}_x(\bar{r})\} = \left|\underline{E}_x(\bar{r})\right|e^{j\phi_x(\bar{r})} \qquad (2.3.14)$$

and $\phi_x(\bar{r}) = \tan^{-1}\left(\mathrm{Im}\{\underline{E}_x(\bar{r})\}/\mathrm{Re}\{\underline{E}_x(\bar{r})\}\right)$. A general vector can also be a phasor, e.g., $\underline{\bar{E}}(\bar{r}) = \hat{x}\underline{E}_x(\bar{r}) + \hat{y}\underline{E}_y(\bar{r}) + \hat{z}\underline{E}_z(\bar{r})$, where $\bar{E}(\bar{r},t) = R_e\{\underline{\bar{E}}(\bar{r})e^{j\omega t}\}$.

We can use such phasors to simplify Maxwell's equations. For example, we can express Faraday's law (2.2.1) as:

$$\nabla\times R_e\{\underline{\bar{E}}(\bar{r})e^{j\omega t}\} = -\partial R_e\{\underline{\bar{B}}(\bar{r})e^{j\omega t}\}/\partial t = R_e\{\nabla\times\bar{E}(\bar{r})e^{j\omega t}\} = R_e\{-j\omega\underline{\bar{B}}(\bar{r})e^{j\omega t}\} \qquad (2.3.15)$$

The other Maxwell equations can be similarly transformed, which suggests that the notation $\mathrm{R_e}\{[\ ]e^{j\omega t}\}$ can be omitted and treated as understood. For example, removing this redundant notation from (2.3.15) results in: $\nabla\times\overline{\underline{E}}=-j\omega\overline{\underline{B}}$. Any problem solution expressed as a phasor, e.g. $\overline{\underline{E}}(\bar{r})$, can be converted back into a time-domain expression by the operator $\mathrm{R_e}\{[\ ]e^{j\omega t}\}$. These omissions of the understood notation result in the complex or *time-harmonic Maxwell equations*:

$$\nabla\times\overline{\underline{E}}=-j\omega\overline{\underline{B}} \qquad \text{(Faraday's law)} \qquad (2.3.16)$$

$$\nabla\times\overline{\underline{H}}=\overline{\underline{J}}+j\omega\overline{\underline{D}} \qquad \text{(Ampere's law)} \qquad (2.3.17)$$

$$\nabla\bullet\overline{\underline{D}}=\underline{\rho} \qquad \text{(Gauss's law)} \qquad (2.3.18)$$

$$\nabla\bullet\overline{\underline{B}}=0 \qquad \text{(Gauss's law)} \qquad (2.3.19)$$

Note that these equations are the same as before [i.e., (2.2.1–4)], except that we have simply replaced the operator $\partial/\partial t$ with $j\omega$ and placed an underbar under all variables, signifying that they are now phasors.

We can immediately derive the time-harmonic equation for conservation of charge (2.1.19) by computing the divergence of (2.3.17), noting that $\nabla\bullet(\nabla\times\overline{A})=0$ for any $\overline{A}$, and substituting $\nabla\bullet\overline{\underline{D}}=\rho$ (2.3.18):

$$\nabla\bullet\overline{\underline{J}}+j\omega\underline{\rho}=0 \qquad (2.3.20)$$

Example 2.3A

Convert the following expressions into their time-domain equivalents: $j\omega\nabla\times\overline{Q}=\overline{R}j$, $\overline{R}e^{-jkz}$, and $\overline{\underline{E}}=\hat{x}3+\hat{y}j4$.

Solution: $-\omega(\nabla\times\overline{Q})\sin(\omega t)=-\overline{R}\sin\omega t$, $\overline{R}\cos(\omega t-kz)$, and $3\hat{x}\cos\omega t-4\hat{y}\sin\omega t$.

Example 2.3B

Convert the following expressions into their complex frequency-domain equivalents: $A\cos(\omega t+kz)$, and $B\sin(\omega t+\phi)$.

Solution: Ae^{+jkz}, and $-jBe^{j\phi}=-jB\cos\phi+B\sin\phi$.

2.3.3 Sinusoidal uniform plane waves

We can readily derive from Maxwell's equations the time-harmonic Helmholtz wave equation for vacuum (2.2.7) by substituting jω for ∂/∂t or, as we did earlier, by taking the curl of Faraday's law, using the well known vector identity (2.2.6) and Gauss's law, replacing $\underline{\overline{B}}$ by $\mu_o \underline{\overline{H}}$, and using Ampere's law to replace $\nabla \times \underline{\overline{H}}$. In both cases the Helmholtz wave equation becomes:

$$\left(\nabla^2 + \omega^2 \mu_o \varepsilon_o\right) \underline{\overline{E}} = 0 \qquad \text{(wave equation)} \qquad (2.3.21)$$

As before, the solution $\underline{\overline{E}}(\bar{r})$ to the wave equation can be any arbitrary function of space $(\bar{r})$ such that its second spatial derivative $\left(\nabla^2 \underline{\overline{E}}\right)$ equals a constant $\left(-\omega^2 \varepsilon_o \mu_o\right)$ times that same function $\underline{\overline{E}}(\bar{r})$. One solution with these properties is the time-harmonic version of the time-domain expression $\overline{E}(\bar{r}, t) = \hat{y} E_o \cos(\omega t - kz)$:

$$\underline{\overline{E}}(\bar{r}) = \hat{y} E_o e^{-jkz} \quad \left[\text{v m}^{-1}\right] \qquad (2.3.22)$$

Substituting (2.3.22) into the wave equation (2.3.21) yields:

$$\left(\left[\partial^2 / \partial z^2\right] + \omega^2 \mu_o \varepsilon_o\right) \underline{\overline{E}} = \left(\left[-jk\right]^2 + \omega^2 \mu_o \varepsilon_o\right) \underline{\overline{E}} = 0 \qquad (2.3.23)$$

which is satisfied if the wavenumber k is:

$$k = \omega \sqrt{\mu_o \varepsilon_o} = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda} \quad \left[\text{radians m}^{-1}\right] \qquad (2.3.24)$$

It is now an easy matter to find the magnetic field that corresponds to (2.3.22) by using Faraday's law (2.3.16), $\underline{\overline{B}} = \mu_o \underline{\overline{H}}$, and the definition of the "∇×" operator (2.1.1):

$$\begin{aligned} \underline{\overline{H}}(\bar{r}) &= -\frac{\left(\nabla \times \underline{\overline{E}}\right)}{j\omega\mu_o} = \frac{1}{j\omega\mu_o} \frac{\hat{x} \partial E_y}{\partial z} = -\frac{\hat{x} k E_o e^{-jkz}}{\omega \mu_o} \\ &= -\hat{x} \frac{1}{\eta_o} E_o e^{-jkz} \quad \left[\text{Am}^{-1}\right] \end{aligned} \qquad (2.3.25)$$

As before, $\underline{\overline{E}}$ and $\underline{\overline{H}}$ are orthogonal to each other and to the direction of propagation, and $\left|\underline{\overline{E}}\right| = \eta_o \left|\underline{\overline{H}}\right|$.

As another example, consider a z-polarized wave propagating in the -x direction; then:

$$\underline{\overline{E}}(\bar{r}) = \hat{z}E_o e^{+jkx} \ , \ \underline{\overline{H}}(\bar{r}) = \hat{y}E_o e^{jkx}/\eta_o \tag{2.3.26}$$

It is easy to convert phasor expressions such as (2.3.26) into time-domain expressions. We simply divide the phasor expressions into their real and imaginary parts, and note that the real part varies as cos(ωt - kz) and the imaginary part varies as sin(ωt - kz). Thus the fields in (2.3.22) could be written instead as a real time-domain expression:

$$\overline{E}(\bar{r},t) = \hat{y}E_o \cos(\omega t - kz) \tag{2.3.27}$$

Had the electric field solution been instead the phasor $\hat{y}jE_o e^{-jkz}$, the time domain expression $R_e\{\underline{\overline{E}}(\bar{r})e^{j\omega t}\}$ would then be:

$$\overline{E}(\bar{r},t) = -\hat{y}E_o \sin(\omega t - kz) \tag{2.3.28}$$

The conversion of complex phasors to time-domain expressions, and vice-versa, is discussed further in Appendix B.

2.3.4 Wave polarization

Complex notation simplifies the representation of wave *polarization*, which characterizes the behavior of the sinusoidally varying electric field vector as a function of time. It is quite distinct from the polarization of media discussed in Section 2.5.3. Previously we have seen waves for which the time-varying electric vector points only in the ±x, ±y, or ±z directions, corresponding to x, y, or z polarization, respectively. By superimposing such waves at the same frequency and propagating in the same direction we can obtain any other desired time-harmonic polarization. Linear polarization results when the oscillating electric vector points only along a single direction in the plane perpendicular to the direction of propagation, while elliptical polarization results when the x and y components of the electric vector are out of phase so that the tip of the electric vector traces an ellipse in the same plane. Circular polarization results only when the phase difference between x and y is 90 degrees and the two amplitudes are equal. These various polarizations for $+\hat{z}$ propagation are represented below at z = 0 in the time domain and as phasors, and in Figure 2.3.2.

$$\overline{E}(t) = \hat{y}E_o \cos\omega t \qquad \underline{\overline{E}} = \hat{y}E_o \qquad \text{(y-polarized)} \tag{2.3.29}$$

$$\overline{E}(t) = \hat{x}E_o \cos\omega t \qquad \underline{\overline{E}} = \hat{x}E_o \qquad \text{(x-polarized)} \tag{2.3.30}$$

$$\overline{E}(t) = (\hat{x}+\hat{y})E_o \cos\omega t \qquad \underline{\overline{E}} = (\hat{x}+\hat{y})E_o \qquad \text{(45°-polarized)} \tag{2.3.31}$$

$$\overline{E}(t) = E_o(\hat{x}\cos\omega t + \hat{y}\sin\omega t) \qquad \underline{\overline{E}} = (\hat{x}-j\hat{y})E_o \qquad \text{(right-circular)} \tag{2.3.32}$$

$$\overline{E}(t) = E_o(\hat{x}\cos\omega t + 1.5\hat{y}\sin\omega t) \qquad \underline{\overline{E}} = (\hat{x}-1.5j\hat{y})E_o \qquad \text{(right-elliptical)} \tag{2.3.33}$$

$$\overline{E}(t) = E_o\left[\hat{x}\cos\omega t + \hat{y}\cos(\omega t + 20°)\right] \qquad \underline{E} = \left(\hat{x} + e^{0.35j}\hat{y}\right)E_o \qquad \text{(left-elliptical)} \qquad (2.3.34)$$

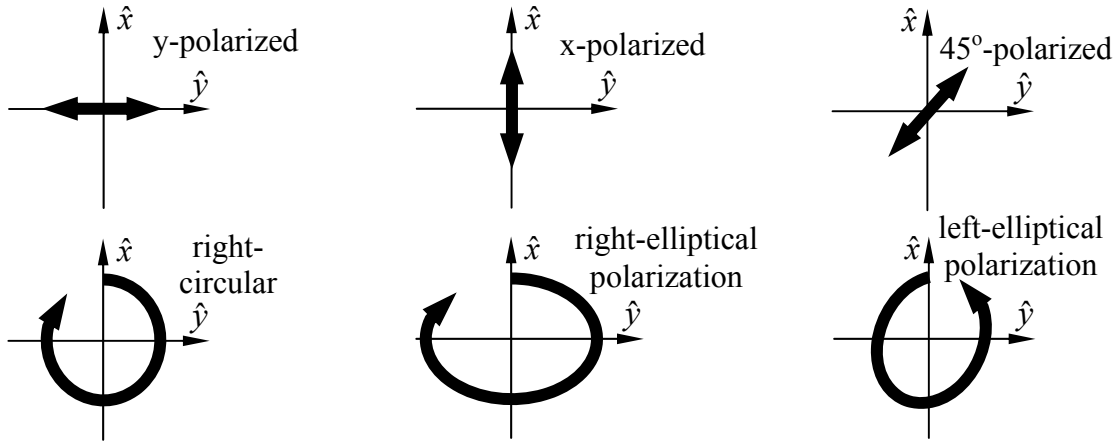

Figure 2.3.2 Polarization ellipses for +z-propagating plane waves (into the page).

The Institute of Electrical and Electronics Engineers (IEEE) has defined polarization as right-handed if the electric vector traces a right-handed ellipse in the x-y plane for a wave propagating in the +z direction, as suggested in Figure 2.3.3. That is, for *right-handed polarization* the fingers of the right hand circle in the direction taken by the electric vector while the thumb points in the direction of propagation. This legal definition is opposite that commonly used in physics, where that alternative definition is consistent with the handedness of the "screw" formed by the instantaneous three-dimensional loci of the tips of the electric vectors comprising a wave.

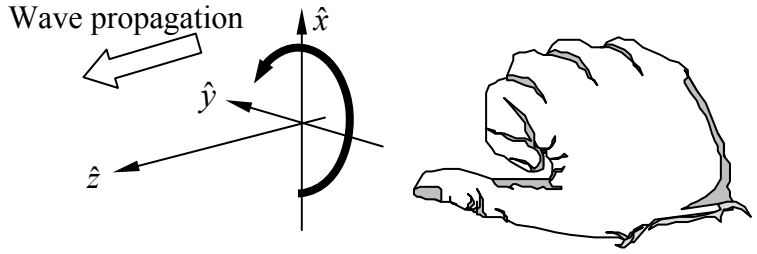

Figure 2.3.3 IEEE definition of right-handed polarization.

Example 2.3C

If $\overline{E} = \underline{E}_o e^{-jkz}$, what polarizations correspond to: $\underline{E}_o = \hat{y}$, $\underline{E}_o = \hat{x} + 2\hat{y}$, and $\underline{E}_o = \hat{x} - j\hat{y}$?

Solution: y polarization, linear polarization at angle $\tan^{-1}2$ relative to the x-z plane, and right-circular polarization.

2.4 Relation between integral and differential forms of Maxwell's equations

2.4.1 Gauss's divergence theorem

Two theorems are very useful in relating the differential and integral forms of Maxwell's equations: Gauss's divergence theorem and Stokes theorem. Gauss's divergence theorem (2.1.20) states that the integral of the normal component of an arbitrary analytic vector field $\overline{A}$ over a surface S that bounds the volume V equals the volume integral of $\nabla \bullet \overline{A}$ over V. The theorem can be derived quickly by recalling (2.1.3):

$$\nabla \bullet \overline{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{2.4.1}$$

Therefore $\nabla \bullet \overline{A}$ at the position x_o, y_o, z_o can be found using (2.4.1) in the limit where Δx, Δy, and Δz approach zero:

$$\begin{aligned} \nabla \bullet \overline{A} = \lim_{\Delta i \to 0} \Big\{ & \Big[A_x\left(x_o + \Delta x/2\right) - A_x\left(x_o - \Delta x/2\right) \Big] / \Delta x \\ & + \Big[A_y\left(y_o + \Delta y/2\right) - A_y\left(y_o - \Delta y/2\right) \Big] / \Delta y \\ & + \Big[A_z\left(z_o + \Delta z/2\right) - A_z\left(z_o - \Delta z/2\right) \Big] / \Delta z \Big\} \end{aligned} \tag{2.4.2}$$

$$\begin{aligned} = \lim_{\Delta i \to 0} \Big\{ & \Delta y \Delta z \Big[A_x\left(x_o + \Delta x/2\right) - A_x\left(x_o - \Delta x/2\right) \Big] \\ & + \Delta x \Delta z \Big[A_y\left(y_o + \Delta y/2\right) - A_y\left(y_o - \Delta y/2\right) \Big] \\ & + \Delta x \Delta y \Big[A_z\left(z_o + \Delta z/2\right) - A_z\left(z_o - \Delta z/2\right) \Big] \Big\} / \Delta x \Delta y \Delta z \end{aligned} \tag{2.4.3}$$

$$= \lim_{\Delta v \to 0} \left\{ \oiint_{S_c} \overline{A} \bullet \hat{n} \, da / \Delta v \right\} \tag{2.4.4}$$

where $\hat{n}$ is the unit normal vector for an incremental cube of dimensions Δx, Δy, Δz; da is its differential surface area; S_c is its surface area; and Δv is its volume, as suggested in Figure 2.4.1(a).

We may now stack an arbitrary number of such infinitesimal cubes to form a volume V such as that shown in Figure 2.4.1(b). Then we can sum (2.4.4) over all these cubes to obtain:

$$\lim_{\Delta v \to 0} \sum_i \left(\nabla \bullet \overline{A}\right) \Delta v_i = \lim_{\Delta v \to 0} \sum_i \left\{ \oiint_{S_c} \overline{A} \bullet \hat{n} \, da_i \right\} \tag{2.4.5}$$

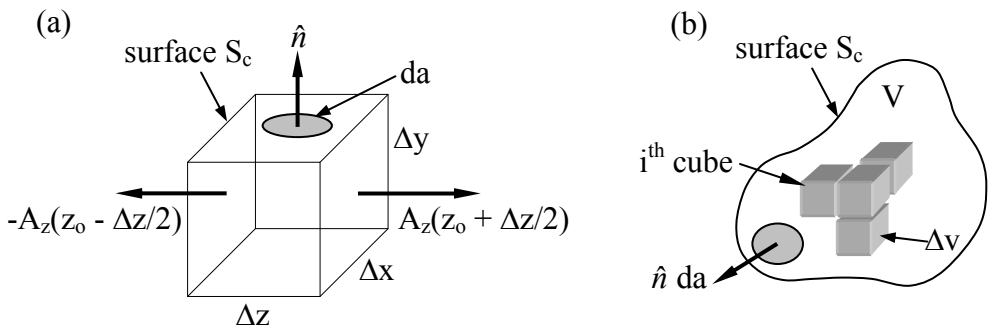

Figure 2.4.1 Derivation of Gauss's divergence theorem.

Since all contributions to $\sum_i \left\{ \oiint_S \overline{A} \bullet \hat{n}\, da_i \right\}$ from interior-facing adjacent cube faces cancel, the only remaining contributions from the right-hand side of (2.4.5) are from the outer surface of the volume V. Proceeding to the limit, we obtain *Gauss's divergence theorem*:

$$\iiint_V (\nabla \bullet \overline{A})\, dv = \oiint_S (\overline{A} \bullet \hat{n})\, da \tag{2.4.6}$$

2.4.2 Stokes' theorem

Stokes' theorem states that the integral of the curl of a vector field over a bounded surface equals the line integral of that vector field along the contour C bounding that surface. Its derivation is similar to that for Gauss's divergence theorem (Section 2.4.1), starting with the definition of the z component of the curl operator [from Equation (2.1.4)]:

$$(\nabla \times \overline{A})_z \equiv \hat{z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \tag{2.4.7}$$

$$\begin{aligned} = \hat{z} \lim_{\Delta x, \Delta y \to 0} \Big\{ & \left[A_y(x_o + \Delta x/2) - A_y(x_o - \Delta x/2) \right] / \Delta x \\ & - \left[A_x(y_o + \Delta y/2) - A_x(y_o - \Delta y/2) \right] / \Delta y \Big\} \end{aligned} \tag{2.4.8}$$

$$\begin{aligned} = \hat{z} \lim_{\Delta x, \Delta y \to 0} \Big\{ & \Delta y \left[A_y(x_o + \Delta x/2) - A_y(x_o - \Delta x/2) \right] / \Delta x \Delta y \\ & - \Delta x \left[A_x(y_o + \Delta y/2) - A_x(y_o - \Delta y/2) \right] / \Delta x \Delta y \Big\} \end{aligned} \tag{2.4.9}$$

Consider a surface in the x-y plane, perpendicular to $\hat{z}$ and $\hat{n}$, the local surface normal, as illustrated in Figure 2.4.2(a).

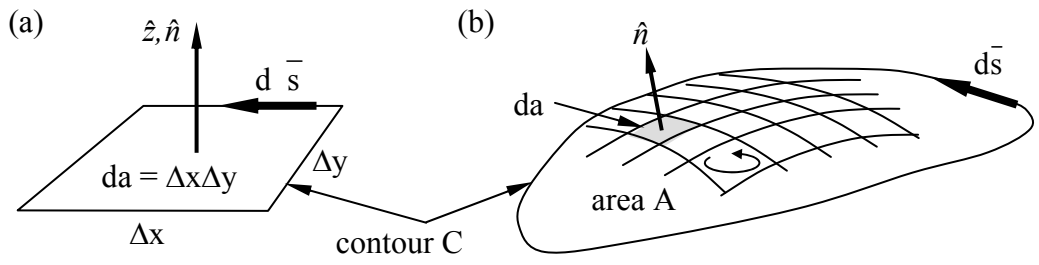

Figure 2.4.2 Derivation of Stokes' theorem.

Then (2.4.9) applied to ΔxΔy becomes:

$$\Delta x \Delta y \left(\nabla \times \overline{A}\right) \bullet \hat{n} = \oint_C \overline{A} \bullet d\bar{s} \tag{2.4.10}$$

where $d\bar{s}$ is a vector differential length [m] along the contour C bounding the incremental area defined by ΔxΔy = da. The contour C is transversed in a right-hand sense relative to $\hat{n}$. We can assemble such infinitesimal areas to form surfaces of arbitrary shapes and area A, as suggested in Figure 2.4.2(b). When we sum (2.4.10) over all these infinitesimal areas da, we find that all contributions to the right-hand side interior to the area A cancel, leaving only the contributions from contour C along the border of A. Thus (2.4.10) becomes *Stokes' theorem*:

$$\iint_A \left(\nabla \times \overline{A}\right) \bullet \hat{n}\ da = \oint_C \overline{A} \bullet d\bar{s} \tag{2.4.11}$$

where the relation between the direction of integration around the loop and the orientation of $\hat{n}$ obey the right-hand rule (if the right-hand fingers curl in the direction of $d\bar{s}$, then the thumb points in the direction $\hat{n}$).

2.4.3 Maxwell's equations in integral form

The differential form of Maxwell's equations (2.1.5–8) can be converted to integral form using Gauss's divergence theorem and Stokes' theorem. Faraday's law (2.1.5) is:

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{2.4.12}$$

Applying Stokes' theorem (2.4.11) to the curved surface A bounded by the contour C, we obtain:

$$\iint_A \left(\nabla \times \overline{E}\right) \bullet \hat{n}\ da = \oint_C \overline{E} \bullet d\bar{s} = -\iint_A \frac{\partial \overline{B}}{\partial t} \bullet \hat{n}\ da \tag{2.4.13}$$

This becomes the integral form of Faraday's law:

$$\oint_C \bar{E} \bullet d\bar{s} = -\frac{\partial}{\partial t} \iint_A \bar{B} \bullet \hat{n}\, da \qquad (\textit{Faraday's Law}) \qquad (2.4.14)$$

A similar application of Stokes' theorem to the differential form of Ampere's law yields its integral form:

$$\oint_C \bar{H} \bullet d\bar{s} = \iint_A \left[\bar{J} + \frac{\partial \bar{D}}{\partial t} \right] \bullet \hat{n}\, da \qquad (\textit{Ampere's Law}) \qquad (2.4.15)$$

Gauss's divergence theorem (2.1.20) can be similarly applied to Gauss's laws to yield their integral form:

$$\iiint_V (\nabla \bullet \bar{D})\, dv = \iiint_V \rho\, dv = \oiint_A (\bar{D} \bullet \hat{n})\, da \qquad (2.4.16)$$

This conversion procedure thus yields the integral forms of Gauss's laws. That is, we can integrate $\bar{D} \bullet \hat{n}$ and $\bar{B} \bullet \hat{n}$ in the differential equations over the surface A that bounds the volume V:

$$\oiint_A (\bar{D} \bullet \hat{n})\, da = \iiint_V \rho\, dv \qquad (\textit{Gauss's Law} \text{ for charge}) \qquad (2.4.17)$$

$$\oiint_A (\bar{B} \bullet \hat{n})\, da = 0 \qquad (\textit{Gauss's Law} \text{ for } \bar{B}) \qquad (2.4.18)$$

Finally, conservation of charge (1.3.19) can be converted to integral form as were Gauss's laws:

$$\oiint_A (\bar{J} \bullet \hat{n})\, da = -\iiint_V \frac{\partial \rho}{\partial t}\, dv \qquad (\textit{conservation of charge}) \qquad (2.4.19)$$

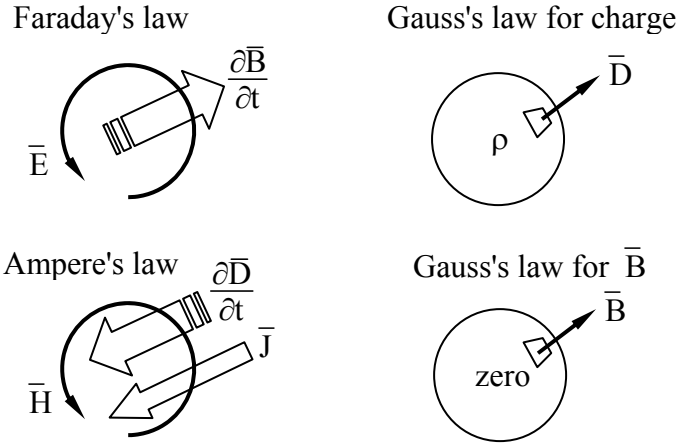

Figure 2.4.3 Maxwell's equations in sketch form.

The four sketches of Maxwell's equations presented in Figure 2.4.3 may facilitate memorization; they can be interpreted in either differential or integral form because they capture the underlying physics.

Example 2.4A

Using Gauss's law, find $\overline{E}$ at distance r from a point charge q.

Solution: The spherical symmetry of the problem requires $\overline{E}$ to be radial, and Gauss's law requires $\int_A \varepsilon_o \overline{E} \bullet \hat{r}\, dA = \int_V \rho\, dv = q = 4\pi r^2 \varepsilon_o E_r$, so $\overline{E} = \hat{r} E_r = \hat{r} q / 4\pi\varepsilon_o r^2$.

Example 2.4B

What is $\overline{H}$ at r = 1 cm from a line current $\overline{I} = \hat{z}$ [amperes] positioned at r = 0?

Solution: Because the geometry of this problem is cylindrically symmetric, so is the solution. Using the integral form of Ampere's law (2.4.15) and integrating in a right-hand sense around a circle of radius r centered on the current and in a plane orthogonal to it, we obtain $2\pi r H = I$, so $\overline{H} = \hat{\theta}\, 100/2\pi$ [A m^{-1}].

2.5 Electric and magnetic fields in media

2.5.1 Maxwell's equations and media

The great success of Maxwell's equations lies partly in their simple prediction of electromagnetic waves and their simple characterization of materials in terms of conductivity σ [Siemens m^{-1}], permittivity ε [Farads m^{-1}], and permeability μ [Henries m^{-1}]. In vacuum we find $\sigma = 0$, $\varepsilon = \varepsilon_o$, and $\mu = \mu_o$, where $\varepsilon_o = 8.8542\times10^{-12}$ and $\mu_o = 4\pi\times10^{-7}$. For reference, Maxwell's equations are:

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{2.5.1}$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \tag{2.5.2}$$

$$\nabla \bullet \overline{D} = \rho \tag{2.5.3}$$

$$\nabla \bullet \overline{B} = 0 \tag{2.5.4}$$

The electromagnetic properties of most media can be characterized by the *constitutive relations*:

$$\overline{D} = \varepsilon \overline{E} \tag{2.5.5}$$

$$\bar{B} = \mu \bar{H} \tag{2.5.6}$$

$$\bar{J} = \sigma \bar{E} \tag{2.5.7}$$

In contrast, the nano-structure of media can be quite complex and requires quantum mechanics for its full explanation. Fortunately, simple classical approximations to atoms and molecules suffice to understand the origins of σ, ε, and μ, as discussed below in that sequence.

2.5.2 Conductivity

Conduction in metals and *n-type semiconductors*[6] involves free electrons moving many atomic diameters before they lose momentum by interacting with atoms or other particles. Acceleration induced by the small applied electric field inside the conductor restores electron velocities to produce an equilibrium current. The total current density $\bar{J}$ [A m^{-2}] is proportional to the product of the average electron velocity $\bar{v}$ [m s^{-1}] and the number density n [m^{-3}] of free electrons. A related conduction process occurs in ionic liquids, where both negative and positive ions can carry charge long distances.

In *metals* there is approximately one free electron per atom, and in warm n-type semiconductors there is approximately one free electron per donor atom, where the sparse donor atoms are easily ionized thermally. Since, for non-obvious reasons, the average electron velocity $\langle \bar{v} \rangle$ is proportional to $\bar{E}$, therefore $\bar{J} = -en_3 \langle \bar{v} \rangle = \sigma \bar{E}$, as stated in (2.5.7). As the conductivity σ approaches infinity the electric field inside a conductor approaches zero for any given current density $\bar{J}$.

Warm donor atoms in n-type semiconductors can be easily ionized and contribute electrons to the conduction band where they move freely. Only certain types of impurity atoms function as donors--those that are most easily ionized. As the density of donor atoms approaches zero and as temperature declines, the number of free electrons and the conductivity approach very low values that depend on temperature and any alternative ionization mechanisms that are present.

In *p-type semiconductors* the added impurity atoms readily trap nearby free electrons to produce a negative ion; this results in a corresponding number of positively ionized semiconductor atoms that act as "*holes*". As a result any free electrons typically move only short distances before they are trapped by one of these holes. Moreover, the threshold energy required to move an electron from a neutral atom to an adjacent positive ion is usually less than the available thermal energy, so such transfers occur rapidly, causing the hole to move quickly from

[6] n-type semiconductors (e.g., silicon, germanium, gallium arsenide, indium phosphide, and others) are doped with a tiny percentage of donor atoms that ionize easily at temperatures of interest, releasing mobile electrons into the conduction band of the semiconductor where they can travel freely across the material until they recombine with another ionized donor atom. The conduction band is not a place; it refers instead to an energy and wave state of electrons that enables them to move freely. The conductivity of semiconductors therefore increases with temperature; they become relatively insulating at low temperatures, depending on the ionization potentials of the impurity atoms.

place to place over long distances. Thus holes are the *dominant charge carriers* in p-type semiconductors, whereas electrons dominate in n-type semiconductors.

More broadly, *semiconductors* have a *conduction band* in which free electrons can propagate long distances; this band is separated by an energy of one or a few electron volts from the *valence band* in which electrons cannot move. The conduction band is not a location, it is a family of possible electron wave states. When electrons are excited from the valence band to the conduction band by some energetic process, they become free to move in response to electric fields. Semiconductor conductivity is approximately proportional to the number of free electrons or holes produced by the scarce impurity atoms, and therefore to the doping density of those impurity atoms. Easily ionized impurity atoms are the principal mechanism by which electrons enter the conduction band, and impurities that readily trap adjacent electrons are the principal mechanism by which holes enter and move in the valence band. Semiconductors are discussed further in Section 8.2.4. The current leakage processes in insulators vaguely resemble electron and hole conduction in semiconductors, and can include weak surface currents as well as bulk conduction; microscopic flaws can also increase conductivity. The conductivities of typical materials are listed in Table 2.5.1.

Table 2.5.1 Nominal conductivities σ of common materials [Siemens m^{-1}].

paraffin	$10^{-14} - 10^{-16}$	sea water	3-5
glass	10^{-12}	iron	10^{7}
dry earth	$10^{-4} - 10^{-5}$	copper	5.8×10^{7}
distilled water	2×10^{-4}	silver	6.14×10^{7}

In some exotic materials the conductivity is a function of direction and can be represented by the 3×3 matrix $\overline{\overline{\sigma}}$; such materials are not addressed here, but Section 2.5.3 addresses similar issues in the context of permittivity ε.

Some materials exhibit *superconductivity*, or infinite conductivity. In these materials pairs of electrons become loosely bound magnetically and move as a unit called a *Cooper pair*. Quantum mechanics prevents these pairs from colliding with the lattice and losing energy. Because the magnetic binding energy for these pairs involves electron spins, it is quite small. Normal conductivity returns above a threshold *critical temperature* at which the pairs are shaken apart, and it also returns above some threshold *critical magnetic field* at which the magnetic bonds coupling the electrons break as the electron spins all start to point in the same direction. Materials having critical temperatures above 77K (readily obtained in cryogenic refrigerators) are difficult to achieve. The finite number of such pairs at any temperature and magnetic field limits the current to some maximum value; moreover that current itself produces a magnetic field that can disrupt pairs. Even a few pairs can move so as to reduce electric fields to zero by short-circuiting the normal electrons until the maximum current carrying capacity of those pairs is exceeded. If the applied fields have frequency $f > 0$, then the Cooper pairs behave much like collisionless electrons in a plasma and therefore the applied electric field can penetrate that plasma to its skin depth, as discussed in Section 9.8. Those penetrating electric fields interact

with a small number of normal electrons to produce tiny losses in superconductors that increase with frequency.

2.5.3 Permittivity

The *permittivity* ε_o of free space is 8.854×10^{-12} farads/meter, where $\bar{D} = \varepsilon_o \bar{E}$. The permittivity ε of any material deviates from ε_o for free space if applied electric fields induce *electric dipoles* in the medium; such dipoles alter the applied electric field seen by neighboring atoms. Electric fields generally distort atoms because $\bar{E}$ pulls on positively charged nuclei ($f = q\bar{E}$ [N]) and repels the surrounding negatively charged electron clouds. The resulting small offset $\bar{d}$ of each atomic nucleus of charge +q relative to the center of its associated electron cloud produces a tiny electric dipole in each atom, as suggested in Figure 2.5.1(a). In addition, most asymmetric molecules are permanently polarized, such as H_2O or NH_3, and can rotate within fluids or gases to align with an applied field. Whether the dipole moments are induced, or permanent and free to rotate, the result is a complete or partial alignment of dipole moments as suggested in Figure 2.5.1(b).

These polarization charges generally cancel inside the medium, as suggested in Figure 2.5.1(b), but the immobile atomic dipoles on the outside surfaces of the medium are not fully cancelled and therefore contribute the *surface polarization charge* ρ_{sp}.

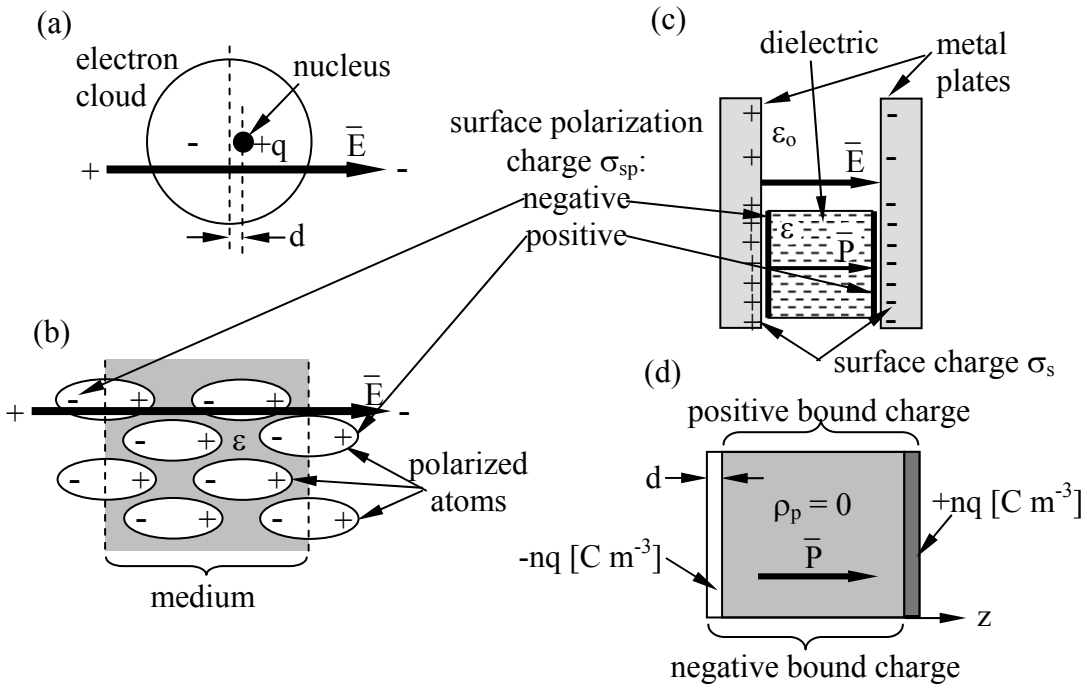

Figure 2.5.1 Polarized media.

Figure 2.5.1(c) suggests how two charged plates might provide an electric field $\bar{E}$ that polarizes a dielectric slab having permittivity $\varepsilon > \varepsilon_o$. [The electric field $\bar{E}$ is the same in vacuum

as it is inside the dielectric (assuming no air gaps) because the path integral of $\overline{E} \bullet d\overline{s}$ from plate to plate equals their voltage difference V in both cases. The electric displacement vector $\overline{D}_e = \varepsilon \overline{E}$ and therefore differs.] We associate the difference between $\overline{D}_o = \varepsilon_o \overline{E}$ (vacuum) and $\overline{D}_\varepsilon = \varepsilon \overline{E}$ (dielectric) with the *electric polarization vector* $\overline{P}$, where:

$$\overline{D} = \varepsilon \overline{E} = \varepsilon_o \overline{E} + \overline{P} = \varepsilon_o \overline{E}(1 + \chi) \tag{2.5.8}$$

The *polarization vector* $\overline{P}$ is defined by (2.5.8) and is normally parallel to $\overline{E}$ in the same direction, as shown in Figure 2.5.1(c); it points from the negative surface polarization charge to the positive surface polarization charge (unlike $\overline{E}$, which points from positive charges to negative ones). As suggested in (2.5.8), $\overline{P} = \overline{E}\varepsilon_o \chi$, where χ is defined as the dimensionless *susceptibility* of the dielectric. Because nuclei are bound rather tightly to their electron clouds, χ is generally less than 3 for most atoms, although some molecules and crystals, particularly in fluid form, can exhibit much higher values. It is shown later in (2.5.13) that $\overline{P}$ simply equals the product of the number density n of these dipoles and the average vector *electric dipole moment* of each atom or molecule, $\overline{p} = q\overline{d}$, where $\overline{d}$ is the offset (meters) of the positive charge relative to the negative charge:

$$\overline{P} = nq\overline{d} \quad \left[\text{C m}^{-2}\right] \tag{2.5.9}$$

Gauss's law relates $\overline{D}$ to *charge density* ρ [C m^{-2}], but we now have two types of density: that of free charges ρ_f, including ions and surface charges on conductors, and that of any locally un-neutralized polarization charge ρ_p bound within charge-neutral atoms or molecules. Gauss's law says:

$$\nabla \bullet \overline{D} = \rho_f \tag{2.5.10}$$

where ρ_f is the free charge density [C m^{-3}].

We can derive a relation similar to (2.5.10) for $\overline{P}$ by treating materials as distributed bound positive and negative charges with vacuum between them; the net bound charge density is designated the *polarization charge density* ρ_p [C m^{-3}]. Then in the vacuum between the charges $\overline{D} = \varepsilon_o \overline{E}$ and (2.5.10) becomes:

$$\varepsilon_o \nabla \bullet \overline{E} = \rho_f + \rho_p \tag{2.5.11}$$

From $\nabla \bullet$ (2.5.8), we obtain $\nabla \bullet \overline{D} = \varepsilon_o \nabla \bullet \overline{E} + \nabla \bullet \overline{P} = \rho_f$. Combining this with (2.5.11) yields:

$$\nabla \bullet \overline{P} = -\rho_p \tag{2.5.12}$$

The negative sign in (2.5.12) is consistent with the fact that $\bar{P}$, unlike $\bar{E}$, is directed from negative to positive polarization charge, as noted earlier.

Outside a polarized dielectric the polarization $\bar{P}$ is zero, as suggested by Figure 2.5.1(d). Note that the net polarization charge density is ±nq for only an atomic-scale distance d at the boundaries of the dielectric, where we model the positive and negative charge distributions within the medium as continuous uniform rectilinear slabs of density ± nq [C m^{-3}]. These two slabs are offset by the distance d. If $\bar{P}$ is in the z direction and arises from n dipole moments $\bar{p} = q\bar{d}$ per cubic meter, where $\bar{d}$ is the offset [m] of the positive charge relative to the negative charge, then (2.5.12) can be integrated over a volume V that encloses a unit area of the left-hand face of a polarized dielectric [see Figure 2.5.1(d)] to yield the polarization vector $\bar{P}$ inside the dielectric:

$$\bar{P} = \int_V \nabla \bullet \bar{P}\, dv = -\int_V \rho_p\, dv = nq\bar{d} \qquad (2.5.13)$$

The first equality of (2.5.13) involving $\bar{P}$ follows from Gauss's divergence theorem: $\int_V \nabla \bullet \bar{P}\, dv = \int_A \bar{P} \bullet \hat{z}\, da = P_z$ if A = 1. Therefore, $\bar{P} = nq\bar{d}$, proving (2.5.9).

When the electric displacement $\bar{D}$ varies with time it becomes *displacement current*, $\partial \bar{D}/\partial t$ [A/m^2], analogous to $\bar{J}$, as suggested by Ampere's law: $\nabla \times \bar{H} = \bar{J} + \partial \bar{D}/\partial t$. For reference, Table 2.5.2 presents the *dielectric constant*s $\varepsilon/\varepsilon_o$ for some common materials near 1 MHz, after Von Hippel (1954).

Table 2.5.2 Dielectric constants $\varepsilon/\varepsilon_o$ of common materials near 1 MHz.

vacuum	1.0	fused quartz	3.78
fir wood	1.8 – 2.0	ice	4.15
Teflon, petroleum	2.1	pyrex glass	5.1
vaseline	2.16	aluminum oxide	8.8
paper	2 – 3	ethyl alcohol	24.5
polystyrene	2.55	water	81.0
sandy soil	2.59	titaniumdioxide	100.0

Most dielectric materials are lossy because oscillatory electric fields dither the directions and magnitudes of the induced electric dipoles, and some of this motion heats the dielectric. This effect can be represented by a frequency-dependent complex permittivity $\underline{\varepsilon}$, as discussed further in Section 9.5. Some dielectrics have direction-dependent permittivities that can be represented by $\bar{\bar{\varepsilon}}$; such anisotropic materials are discussed in Section 9.6. Lossy anisotropic materials can be characterized by $\bar{\bar{\underline{\varepsilon}}}$.

Some special dielectric media are spontaneously polarized, even in the absence of an externally applied $\bar{E}$. This occurs for highly polarizable media where orientation of one electric dipole in the media can motivate its neighbors to orient themselves similarly, forming domains

of atoms, molecules, or crystal unit cells that are all polarized the same. Such spontaneously polarized domains are illustrated for magnetic materials in Figure 2.5.2. As in the case of ferromagnetic domains, in the absence of externally applied fields, domain size is limited by the buildup of stored field energy external to the domain; adjacent domains are oriented so as to largely cancel each other. Such ferroelectric materials have large effective values of ε, although $\overline{D}$ saturates if $\overline{E}$ is sufficient to produce ~100% alignment of polarization. They can also exhibit hysteresis as do the ferromagnetic materials discussed in Section 2.5.4.

Example 2.5A

What are the free and polarization charge densities ρ_f and ρ_p in a medium having conductivity $\sigma = \sigma_o/(1+z)$, permittivity $\varepsilon = 3\varepsilon_o$, and current density $\overline{J} = \hat{z}J_o$?

Solution: $\overline{J} = \sigma\overline{E}$, so $\overline{E} = \hat{z}J_o(1+z)/\sigma_o = \varepsilon_o\overline{E} + \overline{P}$.

From (2.5.10) $\rho_f = \nabla \bullet \overline{D} = (\partial/\partial z)\left[3\varepsilon_o J_o(1+z)/\sigma_o\right] = 3\varepsilon_o J_o/\sigma_o \quad \left[\text{C m}^{-3}\right]$.

From (2.5.12) $\rho_p = -\nabla \bullet \overline{P} = -\nabla \bullet (\varepsilon - \varepsilon_o)\overline{E} = -(\partial/\partial z)2\varepsilon_o J_o(1+z)/\sigma_o$

$= 2\varepsilon_o J_o/\sigma_o \quad \left[\text{C m}^{-3}\right]$.

2.5.4 Permeability

The *permeability* μ_o of free space is $4\pi 10^{-7}$ Henries/meter by definition, where $\overline{B} = \mu\overline{H}$. The permeability μ of matter includes the additional contributions to $\overline{B}$ from atomic magnetic dipoles aligning with the applied $\overline{H}$. These magnetic dipoles and their associated magnetic fields arise either from electrons orbiting about atomic nuclei or from spinning charged particles, where such moving charge is current. All electrons and protons have spin $\pm 1/2$ in addition to any orbital spin about the nucleus, and the net spin of an atom or nucleus can be non-zero. Their magnetic fields are linked to their equivalent currents by Ampere's law, $\nabla \times \overline{H} = \overline{J} + \partial\overline{D}/\partial t$. Quantum theory describes how these magnetic moments are quantized at discrete levels, and for some devices quantum descriptions are necessary. In this text we average over large numbers of atoms, so that $\overline{B} = \mu\overline{H}$ accurately characterizes media, and quantum effects can be ignored.

In any medium the cumulative contribution of induced magnetic dipoles to $\overline{B}$ is characterized by the *magnetization* $\overline{M}$, which is defined by:

$$\overline{B} = \mu\overline{H} = \mu_o(\overline{H} + \overline{M}) = \mu_o\overline{H}(1+\chi_m) \tag{2.5.14}$$

where χ_m is the *magnetic susceptibility* of the medium. Because of quantum effects χ_m for *diamagnetic* materials is slightly negative so that $\mu < \mu_o$; examples include silver, copper, and water, as listed in Table 2.5.3. The table also lists representative *paramagnetic* materials, which have slightly positive magnetic susceptibilities, and ferromagnetic materials, which have very large susceptibilities (e.g., cobalt, etc.).

Table 2.5.3 Approximate relative permeabilities μ/μ_o of common materials.

bismuth	0.99983	aluminum	1.00002
silver	0.99998	cobalt	250
lead	0.999983	nickel	600
copper	0.999991	mild steel	2000
water	0.999991	iron	5000
vacuum	1.000000	mumeltal	100,000
air	1.0000004	supermalloy	1,000,000

The sharp difference between normal materials with $\mu \cong \mu_o$ and *ferromagnetic* materials having $\mu >> \mu_o$ is due to the spontaneous alignment of atomic magnetic dipoles in the same direction so as to increase the applied field, reorienting the remaining dipoles. That is, if the susceptibility of a material is above some threshold, then the atomic magnetic dipoles spontaneously align over regions of size limited by grain structure or energy considerations, as suggested in Figure 2.5.2(a and b). These regions of nearly perfect alignment are called *magnetic domains*. These domains are normally quite small (perhaps micron-size) so as to minimize the stored magnetic energy μH^2. In this regime, if only energy considerations control domain size, then the sizes of those domains oriented in the general direction of the applied magnetic field grow as that field increases, while other domains shrink, as suggested in Figure 2.5.2(c). Since domain walls cannot easily move across grain walls, the granular structure of the material can be engineered to control magnetic properties. If domain walls move easily, the magnetic susceptability χ_m is large.

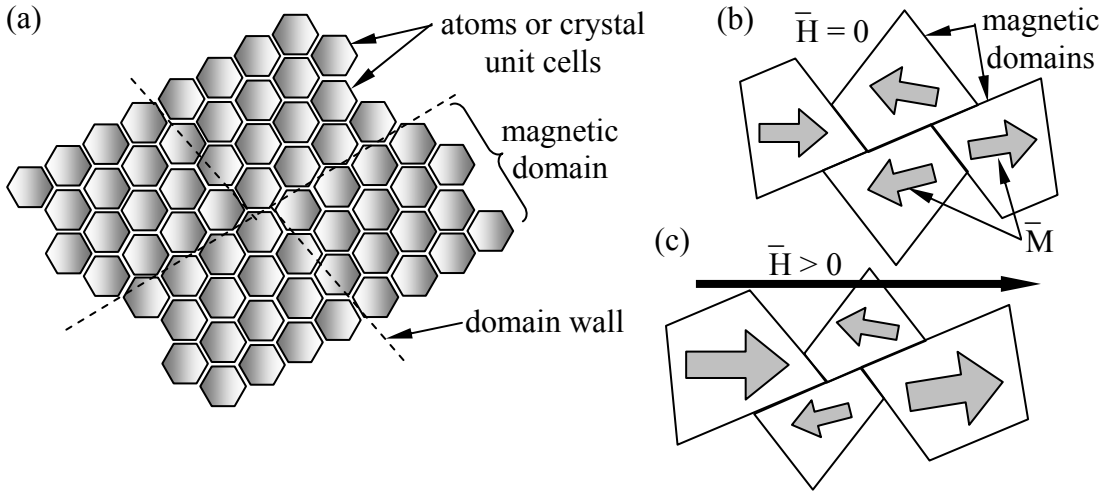

Figure 2.5.2 Magnetic domains in ferromagnetic materials.

At sufficiently high magnetic fields all domains will expand to their maximum size and/or rotate in the direction of $\bar{H}$. This corresponds to the maximum value of $\bar{M}$ and *magnetic saturation*. The resulting typical non-linear behavior of the *magnetization curve* relating B and H for ferromagnetic materials is suggested in Figure 2.5.3(a). The slope of the B vs. H curve is

~μ near the origin and ~μ_o beyond saturation. If the domains resist change and dissipate energy when doing so, the *hysteresis curve* of Figure 2.5.3(b) results. It can be shown that the area enclosed by the figure is the energy dissipated per cycle as the applied $\overline{H}$ oscillates.

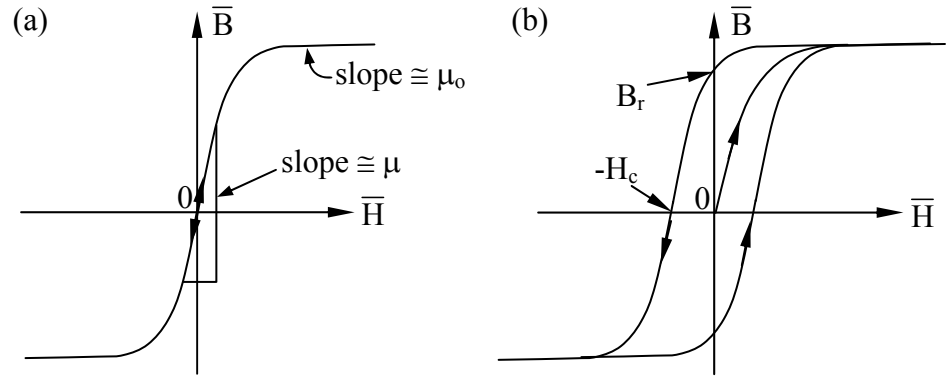

Figure 2.5.3 Magnetization curve and hysterisis loop for a ferromagnetic material.

Hard magnetic materials have large values of *residual flux density* B_r and *magnetic coercive force* or *coercivity* H_c, as illustrated. B_r corresponds to the magnetic strength B of this *permanent magnet* with no applied H. The magnetic energy density $W_m = \overline{B} \bullet \overline{H}/2 \cong 0$ inside permanent magnets because $\overline{H} = 0$, while $W_m = \mu_o H^2/2$ [J m^{-3}] outside. To *demagnetize* a permanent magnet we can apply a magnetic field H of magnitude H_c, which is the field strength necessary to drive B to zero.

If we represent the magnetic dipole moment of an atom by m, where $\overline{m}$ for a current loop of magnitude I and area $\hat{n}$A is $\hat{n}$IA [A m][7], then it can be shown that the total magnetization $\overline{M}$ of a medium is $n\overline{m}$ [A m^{-1}] via the same approach used to derive $\overline{P} = n\overline{p}$ (2.5.13) for the polarization of dielectrics; n is the number of dipoles per m^3.

Example 2.5B

Show how the power dissipated in a hysteretic magnetic material is related to the area circled in Figure 2.5.3(b) as H oscillates. For simplicity, approximate the loop in the figure by a rectangle bounded by $\pm H_o$ and $\pm B_o$.

Solution: We seek the energy dissipated in the material by one traverse of this loop as H goes from $+H_o$ to $-H_o$ and back to $+H_o$. The energy density $W_m = BH/2$ when B = 0 at t = 0 is $W_m = 0$; $W_m \rightarrow B_o H_o/2$ [J m^{-3}] as $B \rightarrow B_o$. As H returns to 0 while $B = B_o$, this energy is dissipated and cannot be recovered by an external circuit because any voltage induced in that circuit would be $\propto \partial B/\partial t = 0$. As $H \rightarrow -H_o$, $W_m \rightarrow B_o H_o/2$; this energy can be recovered by an external circuit later as $B \rightarrow 0$ because $\partial B/\partial t \neq 0$. As $B \rightarrow -B_o$, $W_m \rightarrow B_o H_o/2$, which is lost later as $H \rightarrow 0$ with $\partial B/\partial t = 0$. The energy stored

[7] $\hat{n}$ is the unit vector normal to the tiny area A enclosed by the current I, using the right-hand-rule.

as $H \rightarrow H_o$ with $B = -B_o$ is again recoverable as $B \rightarrow 0$ with $H = H_o$. Thus the minimum energy dissipated during one loop traverse is vB_oH_o[J], where v is material volume. If the drive circuits do not recapture the available energy but dissipate it, the total energy dissipated per cycle is doubled.

2.6 Boundary conditions for electromagnetic fields

2.6.1 Introduction

Maxwell's equations characterize macroscopic matter by means of its permittivity ε, permeability μ, and conductivity σ, where these properties are usually represented by scalars and can vary among media. Section 2.5 discussed media for which ε, μ, and σ are represented by matrices, complex quantities, or other means. This Section 2.6 discusses how Maxwell's equations strongly constrain the behavior of electromagnetic fields at boundaries between two media having different properties, where these constraint equations are called *boundary conditions*. Section 2.6.2 discusses the boundary conditions governing field components perpendicular to the boundary, and Section 2.6.3 discusses the conditions governing the parallel field components. Section 2.6.4 then treats the special case of fields adjacent to perfect conductors.

One result of these boundary conditions is that waves at boundaries are generally partially transmitted and partially reflected with directions and amplitudes that depend on the two media and the incident angles and polarizations. Static fields also generally have different amplitudes and directions on the two sides of a boundary. Some boundaries in both static and dynamic situations also possess surface charge or carry surface currents that further affect the adjacent fields.

2.6.2 Boundary conditions for perpendicular field components

The boundary conditions governing the perpendicular components of $\overline{E}$ and $\overline{H}$ follow from the integral forms of Gauss's laws:

$$\oiint_S (\overline{D} \bullet \hat{n})\, da = \iiint_V \rho \, dv \qquad (\textit{Gauss's Law} \text{ for } \overline{D}) \qquad (2.6.1)$$

$$\oiint_S (\overline{B} \bullet \hat{n})\, da = 0 \qquad (\textit{Gauss's Law} \text{ for } \overline{B}) \qquad (2.6.2)$$

We may integrate these equations over the surface S and volume V of the thin infinitesimal pillbox illustrated in Figure 2.6.1. The pillbox is parallel to the surface and straddles it, half being on each side of the boundary. The thickness δ of the pillbox approaches zero faster than does its surface area S, where S is approximately twice the area A of the top surface of the box.

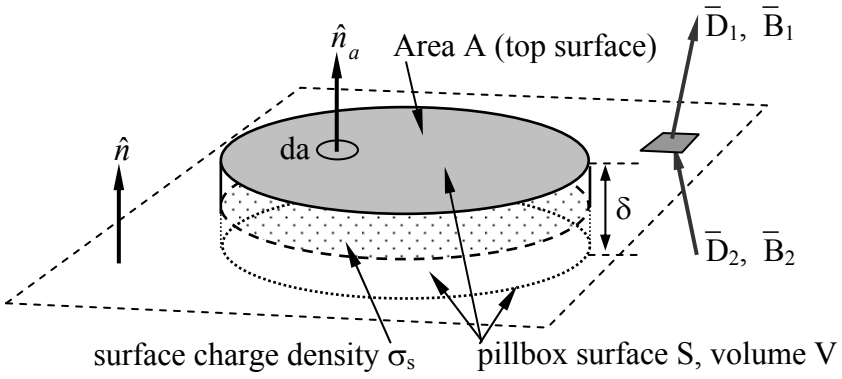

Figure 2.6.1 Elemental volume for deriving boundary conditions for perpendicular field components.

Beginning with the boundary condition for the perpendicular component $D_\perp$, we integrate Gauss's law (2.6.1) over the pillbox to obtain:

$$\oiint_S \left(\overline{D} \bullet \hat{n}_a\right) da \cong \left(D_{1\perp} - D_{2\perp}\right) A = \iiint_V \rho \, dv = \rho_s A \tag{2.6.3}$$

where ρ_s is the surface charge density [Coulombs m^{-2}]. The subscript s for surface charge ρ_s distinguishes it from the volume charge density ρ [C m^{-3}]. The pillbox is so thin ($\delta \to 0$) that: 1) the contribution to the surface integral of the sides of the pillbox vanishes in comparison to the rest of the integral, and 2) only a surface charge q can be contained within it, where $\rho_s = q/A = \lim \rho\delta$ as the charge density $\rho \to \infty$ and $\delta \to 0$. Thus (2.6.3) becomes $D_{1\perp} - D_{2\perp} = \rho_s$, which can be written as:

$$\hat{n} \bullet \left(\overline{D}_1 - \overline{D}_2\right) = \rho_s \qquad \text{(boundary condition for } \overline{D}_\perp\text{)} \tag{2.6.4}$$

where $\hat{n}$ is the unit vector normal to the boundary and points into medium 1. Thus the perpendicular component of the electric displacement vector $\overline{D}$ changes value at a boundary in accord with the surface charge density ρ_s.

Because Gauss's laws are the same for electric and magnetic fields, except that there are no magnetic charges, the same analysis for the magnetic flux density $\overline{B}$ in (2.6.2) yields a similar boundary condition:

$$\hat{n} \bullet \left(\overline{B}_1 - \overline{B}_2\right) = 0 \qquad \text{(boundary condition for } \overline{B}_\perp\text{)} \tag{2.6.5}$$

Thus the perpendicular component of $\overline{B}$ must be continuous across any boundary.

2.6.3 Boundary conditions for parallel field components

The boundary conditions governing the parallel components of $\overline{E}$ and $\overline{H}$ follow from Faraday's and Ampere's laws:

$$\oint_C \overline{E} \bullet d\bar{s} = -\frac{\partial}{\partial t}\iint_A \overline{B} \bullet \hat{n}\, da \qquad (\textit{Faraday's Law}) \qquad (2.6.6)$$

$$\oint_C \overline{H} \bullet d\bar{s} = \iint_A \left[\bar{J} + \frac{\partial \overline{D}}{\partial t}\right] \bullet \hat{n}\, da \qquad (\textit{Ampere's Law}) \qquad (2.6.7)$$

We can integrate these equations around the elongated rectangular contour C that straddles the boundary and has infinitesimal area A, as illustrated in Figure 2.6.2. We assume the total height δ of the rectangle is much less than its length W, and circle C in a right-hand sense relative to the surface normal $\hat{n}_a$.

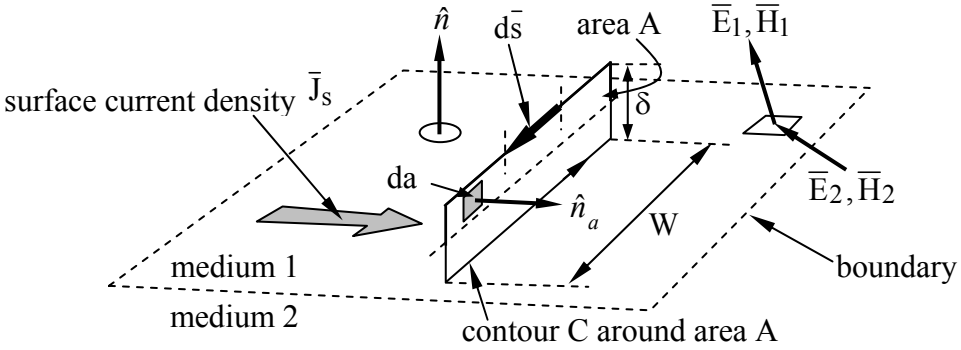

Figure 2.6.2 Elemental contour for deriving boundary conditions for parallel field components.

Beginning with Faraday's law, (2.6.6), we find:

$$\oint_C \overline{E} \bullet d\bar{s} \cong \left(\overline{E}_{1//} - \overline{E}_{2//}\right) W = -\frac{\partial}{\partial t}\iint_A \overline{B} \bullet \hat{n}_a da \to 0 \qquad (2.6.8)$$

where the integral of $\overline{B}$ over area A approaches zero in the limit where δ approaches zero too; there can be no impulses in $\overline{B}$. Since $W \neq 0$, it follows from (2.6.8) that $E_{1//} - E_{2//} = 0$, or more generally:

$$\hat{n} \times \left(\overline{E}_1 - \overline{E}_2\right) = 0 \qquad (\text{boundary condition for } \overline{E}_{//}) \qquad (2.6.9)$$

Thus the parallel component of $\overline{E}$ must be continuous across any boundary.

A similar integration of Ampere's law, (2.6.7), under the assumption that the contour C is chosen to lie in a plane perpendicular to the surface current $\bar{J}_s$ and is traversed in the right-hand sense, yields:

$$\oint_C \bar{H} \bullet d\bar{s} = \left(\bar{H}_{1//} - \bar{H}_{2//}\right) W$$
$$= \iint_A \left[\bar{J} + \frac{\partial \bar{D}}{\partial t}\right] \bullet \hat{n}\ da \Rightarrow \iint_A \bar{J} \bullet \hat{n}_a\ da = \bar{J}_s W \tag{2.6.10}$$

where we note that the area integral of $\partial \bar{D}/\partial t$ approaches zero as $\delta \to 0$, but not the integral over the surface current $\bar{J}_s$, which occupies a surface layer thin compared to δ. Thus $\bar{H}_{1//} - \bar{H}_{2//} = \bar{J}_s$, or more generally:

$$\hat{n} \times \left(\bar{H}_1 - \bar{H}_2\right) = \bar{J}_s \qquad \text{(boundary condition for } \bar{H}_{//}\text{)} \tag{2.6.11}$$

where $\hat{n}$ is defined as pointing from medium 2 into medium 1. If the medium is non-conducting, $\bar{J}_s = 0$.

A simple static example illustrates how these boundary conditions generally result in fields on two sides of a boundary pointing in different directions. Consider the magnetic fields $\bar{H}_1$ and $\bar{H}_2$ illustrated in Figure 2.6.3, where $\mu_2 \neq \mu_1$, and both media are insulators so the surface current must be zero. If we are given $\bar{H}_1$, then the magnitude and angle of $\bar{H}_2$ are determined because $\bar{H}_{//}$ and $\bar{B}_\perp$ are continuous across the boundary, where $\bar{B}_i = \mu_i \bar{H}_i$. More specifically, $\bar{H}_{2//} = \bar{H}_{1//}$, and:

$$H_{2\perp} = B_{2\perp}/\mu_2 = B_{1\perp}/\mu_2 = \mu_1 H_{1\perp}/\mu_2 \tag{2.6.12}$$

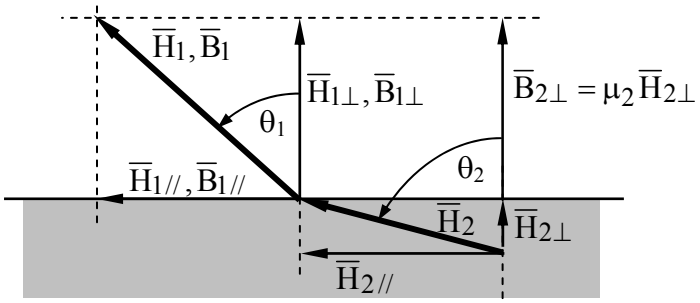

Figure 2.6.3 Static magnetic field directions at a boundary.

It follows that:

$$\theta_2 = \tan^{-1}\left(|\bar{H}_{2//}|/H_{2\perp}\right) = \tan^{-1}\left(\mu_2 |\bar{H}_{1//}|/\mu_1 H_{1\perp}\right) = \tan^{-1}\left[(\mu_2/\mu_1)\tan\theta_1\right] \tag{2.6.13}$$

Thus θ_2 approaches 90 degrees when $\mu_2 >> \mu_1$, almost regardless of θ_1, so the magnetic flux inside high permeability materials is nearly parallel to the walls and trapped inside, even when the field orientation outside the medium is nearly perpendicular to the interface. The flux escapes high-μ material best when $\theta_1 \cong 90°$. This phenomenon is critical to the design of motors or other systems incorporating iron or nickel.

If a static surface current $\bar{J}_s$ flows at the boundary, then the relations between $\bar{B}_1$ and $\bar{B}_2$ are altered along with those for $\bar{H}_1$ and $\bar{H}_2$. Similar considerations and methods apply to static electric fields at a boundary, where any static surface charge on the boundary alters the relationship between $\bar{D}_1$ and $\bar{D}_2$. Surface currents normally arise only in non-static or "dynamic" cases.

Example 2.6A

Two insulating planar dielectric slabs having ε_1 and ε_2 are bonded together. Slab 1 has $\bar{E}_1$ at angle θ_1 to the surface normal. What are $\bar{E}_2$ and θ_2 if we assume the surface charge at the boundary $\rho_s = 0$? What are the components of $\bar{E}_2$ if $\rho_s \neq 0$?

Solution: $\bar{E}_{//}$ is continuous across any boundary, and if $\rho_s = 0$, then $\bar{D}_\perp = \varepsilon_i \bar{E}_\perp$ is continuous too, which implies $\bar{E}_{2\perp} = (\varepsilon_1/\varepsilon_2)\bar{E}_{1\perp}$. Also, $\theta_1 = \tan^{-1}(E_{//}/E_{1\perp})$, and $\theta_2 = \tan^{-1}(E_{//}/E_{2\perp})$. It follows that $\theta_2 = \tan^{-1}\left[(\varepsilon_2/\varepsilon_1)\tan\theta_1\right]$. If $\rho_s \neq 0$ then $\bar{E}_{//}$ is unaffected and $\bar{D}_{2\perp} = \bar{D}_{1\perp} + \hat{n}\rho_s$ so that $\bar{E}_{2\perp} = \bar{D}_{2\perp}/\varepsilon_2 = (\varepsilon_1/\varepsilon_2)\bar{E}_{1\perp} + \hat{n}\rho_s/\varepsilon_2$.

2.6.4 Boundary conditions adjacent to perfect conductors

The four boundary conditions (2.6.4), (2.6.5), (2.6.9), and (2.6.11) are simplified when one medium is a perfect conductor ($\sigma = \infty$) because electric and magnetic fields must be zero inside it. The electric field is zero because otherwise it would produce enormous $\bar{J} = \sigma\bar{E}$ so as to redistribute the charges and to neutralize that $\bar{E}$ almost instantly, with a time constant $\tau = \varepsilon/\sigma$ seconds, as shown in Equation (4.3.3).

It can also be easily shown that $\bar{B}$ is zero inside perfect conductors. Faraday's law says $\nabla \times \bar{E} = -\partial\bar{B}/\partial t$, so if $\bar{E} = 0$, then $\partial\bar{B}/\partial t = 0$. If the perfect conductor were created in the absence of $\bar{B}$ then $\bar{B}$ would always remain zero inside. It has further been observed that when certain special materials become superconducting at low temperatures, as discussed in Section 2.5.2, any pre-existing $\bar{B}$ is thrust outside.

The boundary conditions for perfect conductors are also relevant for normal conductors because most metals have sufficient conductivity σ to enable $\bar{J}$ and ρ_s to cancel the incident electric field, although not instantly. As discussed in Section 4.3.1, this relaxation process by which charges move to cancel $\bar{E}$ is sufficiently fast for most metallic conductors that they

largely obey the perfect-conductor boundary conditions for most wavelengths of interest, from DC to beyond the infrared region. This relaxation time constant is $\tau = \varepsilon/\sigma$ seconds. One consequence of finite conductivity is that any surface current penetrates metals to some depth $\delta = \sqrt{2/\omega\mu\sigma}$, called the skin depth, as discussed in Section 9.2. At sufficiently low frequencies, even sea water with its limited conductivity largely obeys the perfect-conductor boundary condition.

The four boundary conditions for fields adjacent to perfect conductors are presented below together with the more general boundary condition from which they follow when all fields in medium 2 are zero:

$$\hat{n} \bullet \overline{B} = 0 \qquad \left[\text{from } \hat{n} \bullet \left(\overline{B}_1 - \overline{B}_2 \right) = 0 \right] \qquad (2.6.14)$$

$$\hat{n} \bullet \overline{D} = \rho_s \qquad \left[\text{from } \hat{n} \bullet \left(\overline{D}_1 - \overline{D}_2 \right) = \rho_s \right] \qquad (2.6.15)$$

$$\hat{n} \times \overline{E} = 0 \qquad \left[\text{from } \hat{n} \times \left(\overline{E}_1 - \overline{E}_2 \right) = 0 \right] \qquad (2.6.16)$$

$$\hat{n} \times \overline{H} = \overline{J}_s \qquad \left[\text{from } \hat{n} \times \left(\overline{H}_1 - \overline{H}_2 \right) = \overline{J}_s \right] \qquad (2.6.17)$$

These four boundary conditions state that magnetic fields can only be parallel to perfect conductors, while electric fields can only be perpendicular. Moreover, the magnetic fields are always associated with surface currents flowing in an orthogonal direction; these currents have a numerical value equal to $\overline{H}$. The perpendicular electric fields are always associated with a surface charge ρ_s numerically equal to $\overline{D}$; the sign of σ is positive when $\overline{D}$ points away from the conductor.

Example 2.6B

What boundary conditions apply when $\mu \to \infty$, $\sigma = 0$, and $\varepsilon = \varepsilon_o$?

Solution: Inside this medium $\overline{H} = 0$ and $\overline{J} = 0$ because otherwise infinite energy densities, $\mu|H|^2/2$, are required; static $\overline{E}$ and $\overline{B}$ are unconstrained, however. Since $\nabla \times \overline{H} = 0 = \overline{J} + \partial \overline{D}/\partial t$ inside, dynamic $\overline{E}$ and $\overline{D} = 0$ there too. Since $\overline{H}_{//}$ and $\overline{B}_{\perp}$ are continuous across the boundary, $\overline{H}_{//} = 0$ and $\overline{H}_{\perp}$ can be anything at the boundary. Since $\overline{E}_{//}$ and $\overline{D}_{\perp}$ are continuous (let's assume $\rho_s = 0$ if $\overline{J} = 0$), static $\overline{E}$ and $\overline{D}$ are unconstrained at the boundary while dynamic $\overline{E} = \overline{D} = 0$ there because there is no dynamic electric field inside and no dynamic surface charge. Since only $\overline{H}_{\perp} \neq 0$ at the boundary, this is non-physical and such media don't exist. For example, there is no way to match boundary conditions for an incoming plane wave. This impasse would be avoided if $\sigma \neq 0$, for then dynamic $\overline{H}_{//}$ and $\overline{E}_{\perp}$ could be non-zero.

2.7 Power and energy in the time and frequency domains, Poynting theorem

2.7.1 Poynting theorem and definition of power and energy in the time domain

To derive the Poynting theorem we can manipulate Maxwell's equations to produce products of variables that have the dimensions and character of power or energy. For example, the power dissipated in a resistor is the product of its voltage and current, so the product $\overline{E} \bullet \overline{J}$ [W m^{-3}] would be of interest. The dimensions of $\overline{E}$ and $\overline{J}$ are volts per meter and amperes per square meter, respectively. Faraday's and Ampere's laws are:

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \text{(Faraday's law)} \qquad (2.7.1)$$

$$\nabla \times \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t} \qquad \text{(Ampere's law)} \qquad (2.7.2)$$

We can produce the product $\overline{E} \bullet \overline{J}$ and preserve symmetry in the resulting equation by taking the dot product of $\overline{E}$ with Ampere's law and subtracting from it the dot product of $\overline{H}$ with Faraday's law, yielding:

$$\overline{E} \bullet (\nabla \times \overline{H}) - \overline{H} \bullet (\nabla \times \overline{E}) = \overline{E} \bullet \overline{J} + \overline{E} \bullet \frac{\partial \overline{D}}{\partial t} + \overline{H} \bullet \frac{\partial \overline{B}}{\partial t} \qquad (2.7.3)$$

$$= -\nabla \bullet (\overline{E} \times \overline{H}) \qquad (2.7.4)$$

where (2.7.4) is a vector identity. Equations (2.7.3) and (2.7.4) can be combined to form the *Poynting theorem*:

$$\nabla \bullet (\overline{E} \times \overline{H}) + \overline{E} \bullet \overline{J} + \overline{E} \bullet \frac{\partial \overline{D}}{\partial t} + \overline{H} \bullet \frac{\partial \overline{B}}{\partial t} = 0 \ \left[\mathrm{Wm^{-3}} \right] \qquad (2.7.5)$$

The dimension of $\overline{E} \bullet \overline{J}$ and every other term in this equation is W m^{-3}. If $\overline{D} = \varepsilon \overline{E}$ and $\overline{B} = \mu \overline{H}$, then $\overline{E} \bullet \partial \overline{D} / \partial t = \partial \left[\varepsilon |\overline{E}|^2 / 2 \right] / \partial t$ and $\overline{H} \bullet \partial \overline{B} / \partial t = \partial \left[\mu |\overline{H}|^2 / 2 \right] / \partial t$. The factor of one-half arises because we are now differentiating with respect to a squared quantity rather than a single quantity, as in (2.7.5). It follows that $\varepsilon |\overline{E}|^2 / 2$ and $\mu |\overline{H}|^2 / 2$ have the dimension of J m^{-3} and represent electric and magnetic energy density, respectively, denoted by W_e and W_m. The product $\overline{E} \bullet \overline{J}$ can represent either power dissipation or a power source, both denoted by P_d. If $\overline{J} = \sigma \overline{E}$, then $P_d = \sigma |\overline{E}|^2$ [W m^{-3}], where σ is the conductivity of the medium, as discussed later in Section 3.1.2. To summarize:

$$P_d = \overline{J} \bullet \overline{E} \ \left[\mathrm{W\ m^{-3}} \right] \qquad \text{(\textit{power dissipation density})} \qquad (2.7.6)$$

$$W_e = \frac{1}{2}\varepsilon|\overline{E}|^2 \quad \left[\text{J m}^{-3}\right] \qquad \text{(}\textit{electric energy density}\text{)} \qquad (2.7.7)$$

$$W_m = \frac{1}{2}\mu|\overline{H}|^2 \quad \left[\text{J m}^{-3}\right] \qquad \text{(}\textit{magnetic energy density}\text{)} \qquad (2.7.8)$$

Thus we can write Poynting's theorem in a simpler form:

$$\nabla \bullet (\overline{E} \times \overline{H}) + \overline{E} \bullet \overline{J} + \frac{\partial}{\partial t}(W_e + W_m) = 0 \quad \left[\text{Wm}^{-3}\right] \qquad \text{(Poynting theorem)} \qquad (2.7.9)$$

suggesting that the sum of the divergence of electromagnetic power associated with $\overline{E} \times \overline{H}$, the density of power dissipated, and the rate of increase of energy storage density must equal zero.

The physical interpretation of $\nabla \bullet (\overline{E} \times \overline{H})$ is best seen by applying Gauss's divergence theorem to yield the integral form of the Poynting theorem:

$$\oiint_A (\overline{E} \times \overline{H}) \bullet \hat{n}\, da + \iiint_V \overline{E} \bullet \overline{J}\, dv + (\partial/\partial t) \iiint_V \frac{1}{2}(\overline{E} \bullet \overline{D} + \overline{H} \bullet \overline{B})\, dv = 0 \text{ [W]} \qquad (2.7.10)$$

which can also be represented as:

$$\oiint_A (\overline{E} \times \overline{H}) \bullet \hat{n}\, da + p_d + \frac{\partial}{\partial t}(w_e + w_m) = 0 \text{ [W]} \qquad \text{(Poynting theorem)} \qquad (2.7.11)$$

Based on (2.7.8) and conservation of power (1.1.6) it is natural to associate $\overline{E} \times \overline{H}$ [W m^{-2}] with the instantaneous power density of an electromagnetic wave characterized by $\overline{E}$ [V m^{-1}] and $\overline{H}$ [A m^{-1}]. This product is defined as the *Poynting vector*:

$$\overline{S} \equiv \overline{E} \times \overline{H} \quad \left[\text{Wm}^{-2}\right] \qquad \text{(Poynting vector)} \qquad (2.7.12)$$

The instantaneous *electromagnetic wave intensity* of a uniform plane wave. Thus Poynting's theorem says that the integral of the inward component of the Poynting vector over the surface of any volume V equals the sum of the power dissipated and the rate of energy storage increase inside that volume.

Example 2.7A

Find $\overline{S}(t)$ and $\langle \overline{S}(t) \rangle$ for a uniform plane wave having $\overline{E} = \hat{x}E_o \cos(\omega t - kz)$. Find the electric and magnetic energy densities $W_e(t,z)$ and $W_m(t,z)$ for the same wave; how are they related?

Solution: $\overline{H} = \hat{z} \times \overline{E}/\eta_o = \hat{y}E_o \cos(\omega t - kz)/\eta_o$ where $\eta_o = (\mu_o/\varepsilon_o)^{0.5}$. $\overline{S} = \overline{E} \times \overline{H} = \hat{z}\, E_o^2 \cos^2(\omega t - kz)/\eta_o$ and $\langle \overline{S}(t) \rangle = \hat{z}\, E_o^2/2\eta_o$ [Wm^{-2}]. $W_e(t,z) = \varepsilon_o E_o^2 \cos^2(\omega t\text{-}kz)/2$ [Jm^{-3}] and $W_m(t,z) = \mu_o E_o^2 \cos^2(\omega t - kz)/2\eta_o^2 = \varepsilon_o E_o^2 \cos^2(\omega t - kz)/2 = W_e(t,z)$; the electric and

magnetic energy densities vary together in space and time and are equal for a single uniform plane wave.

2.7.2 Complex Poynting theorem and definition of complex power and energy

Unfortunately we cannot blindly apply to power and energy our standard conversion protocol between frequency-domain and time-domain representations because we no longer have only a single frequency present. Time-harmonic power and energy involve the products of sinusoids and therefore exhibit sum and difference frequencies. More explicitly, we cannot simply represent the Poynting vector $\overline{S}(t)$ for a field at frequency f [Hz] by $\mathrm{Re}\{\underline{\overline{S}}\, e^{j\omega t}\}$ because power has components at both f = 0 and 2f Hz, where $\omega = 2\pi f$. Nonetheless we can use the convenience of the time-harmonic notation by restricting it to fields, voltages, and currents while representing their products, i.e. power and energy, only by their complex averages.

To understand the definitions of complex power and energy, consider the product of two sinusoids, a(t) and b(t), where:

$$a(t) = R_e\{\underline{A}e^{j\omega t}\} = R_e\{Ae^{j\alpha}e^{j\omega t}\} = A\cos(\omega t+\alpha), \quad b(t) = B\cos(\omega t+\beta) \tag{2.7.13}$$

$$a(t)b(t) = AB\cos(\omega t+\alpha)\cos(\omega t+\beta) = (AB/2)\left[\cos(\alpha-\beta)+\cos(2\omega t+\alpha+\beta)\right] \tag{2.7.14}$$

where we used the identity $\cos\gamma\cos\theta = \left[\cos(\gamma-\theta)+\cos(\gamma+\theta)\right]/2$. If we time average (2.7.14) over a full cycle, represented by the operator $\langle\bullet\rangle$, then the last term becomes zero, leaving:

$$\langle a(t)b(t)\rangle = \frac{1}{2}AB\cos(\alpha-\beta) = \frac{1}{2}R_e\{Ae^{j\alpha}Be^{-j\beta}\} = \frac{1}{2}R_e\{\underline{A}\underline{B}^*\} \tag{2.7.15}$$

By treating each of the x, y, and z components separately, we can readily show that (2.7.15) can be extended to vectors:

$$\langle\overline{A}(t)\bullet\overline{B}(t)\rangle = \frac{1}{2}R_e\left\{\underline{\overline{A}}\bullet\underline{\overline{B}}^*\right\} \tag{2.7.16}$$

The time average of the Poynting vector $\overline{S}(t) = \overline{E}(t)\times\overline{H}(t)$ is:

$$\langle\overline{S}(t)\rangle = \frac{1}{2}R_e\{\underline{\overline{E}}\times\underline{\overline{H}}^*\} = \frac{1}{2}\mathrm{Re}\{\underline{\overline{S}}\}\ \left[\mathrm{W/m^2}\right] \quad \text{(Poynting average power density)} \tag{2.7.17}$$

where we define the complex Poynting vector as:

$$\underline{\overline{S}} = \underline{\overline{E}}\times\underline{\overline{H}}^*\ \left[\mathrm{W\ m^{-2}}\right] \qquad (\textit{complex Poynting vector}) \tag{2.7.18}$$

Note that $\underline{\bar{S}}$ is complex and can be purely imaginary. Its average power density is given by (2.7.17).

We can re-derive Poynting's theorem to infer the physical significance of this complex vector, starting from the complex Maxwell equations:

$$\nabla \times \underline{\bar{E}} = -j\omega \underline{\bar{B}} \qquad \text{(Faraday's law)} \qquad (2.7.19)$$

$$\nabla \times \underline{\bar{H}} = \underline{\bar{J}} + j\omega \underline{\bar{D}} \qquad \text{(Ampere's law)} \qquad (2.7.20)$$

To see how time-average dissipated power, $\underline{\bar{E}} \bullet \underline{\bar{J}}^* / 2$ is related to other terms, we compute the dot product of $\underline{\bar{E}}$ and the complex conjugate of Ampere's law, and subtract it from the dot product of $\underline{\bar{H}}^*$ and Faraday's law to yield:

$$\underline{\bar{H}}^* \bullet \left(\nabla \times \underline{\bar{E}}\right) - \underline{\bar{E}} \bullet \left(\nabla \times \underline{\bar{H}}^*\right) = -j\omega \underline{\bar{H}}^* \bullet \underline{\bar{B}} - \underline{\bar{E}} \bullet \underline{\bar{J}}^* + j\omega \underline{\bar{E}} \bullet \underline{\bar{D}}^* \qquad (2.7.21)$$

Using the vector identity in (2.7.4), we obtain from (2.7.21) the differential form of the complex Poynting theorem:

$$\nabla \bullet \left(\underline{\bar{E}} \times \underline{\bar{H}}^*\right) = -\underline{\bar{E}} \bullet \underline{\bar{J}}^* - j\omega \left(\underline{\bar{H}}^* \bullet \underline{\bar{B}} - \underline{\bar{E}} \bullet \underline{\bar{D}}^*\right) \qquad (\textit{complex Poynting theorem}) \qquad (2.7.22)$$

The integral form of the complex Poynting theorem follows from the complex differential form as it did in the time domain, by using Gauss's divergence theorem. The integral form of (2.7.22), analogous to (2.7.10), therefore is:

$$\oiint_A \left(\underline{\bar{E}} \times \underline{\bar{H}}^*\right) \bullet \hat{n}\, da + \iiint_V \left\{\underline{\bar{E}} \bullet \underline{\bar{J}}^* + j\omega \left(\underline{\bar{H}}^* \bullet \underline{\bar{B}} - \underline{\bar{E}} \bullet \underline{\bar{D}}^*\right)\right\} dv = 0 \qquad (2.7.23)$$

We can interpret the complex Poynting theorem in terms of physical quantities using (2.7.17) and by expressing the integral form of the complex Poynting theorem as:

$$\frac{1}{2} \oiint_A \underline{\bar{S}} \bullet \hat{n}\, da + \frac{1}{2} \iiint_V \left[\underline{\bar{E}} \bullet \underline{\bar{J}}^* + 2j\omega \left(\underline{W}_m - \underline{W}_e\right)\right] dv = 0 \qquad (2.7.24)$$

where the complex energy densities and time-average power density dissipated P_d are:

$$\underline{W}_m = \frac{1}{2} \underline{\bar{H}}^* \bullet \underline{\bar{B}} = \frac{1}{2} \underline{\mu} \left|\underline{\bar{H}}\right|^2 \quad \left[\text{J/m}^3\right] \qquad (2.7.25)$$

$$\underline{W}_e = \frac{1}{2} \underline{\bar{E}} \bullet \underline{\bar{D}}^* = \frac{1}{2} \underline{\varepsilon} \left|\underline{\bar{E}}\right|^2 \quad \left[\text{J/m}^3\right] \qquad (2.7.26)$$

$$P_d = \frac{1}{2}\sigma|\underline{\overline{E}}|^2 + \iiint_V 2\omega I_m\left[\underline{W}_e - \underline{W}_m\right]dv \equiv \quad \left[W/m^3\right] \tag{2.7.27}$$

We recall that the instantaneous *magnetic energy density* $W_m(t)$ is $\mu|\overline{H}|^2/2$ [J/m^3] from (2.7.8), and that its time average is $\mu|\underline{\overline{H}}|^2/4$ because its peak density is $\mu|\overline{H}|^2/2$ and it varies sinusoidally at 2f Hz. If $\underline{\overline{B}} = \underline{\mu}\,\underline{\overline{H}} = (\mu_r + j\mu_i)\underline{\overline{H}}$ and $\underline{\overline{D}} = \underline{\varepsilon}\,\underline{\overline{E}} = (\varepsilon_r + j\varepsilon_i)\underline{\overline{E}}$, then (2.7.25)-(2.5.27) become:

$$\left\langle W_m(t)\right\rangle = \frac{1}{2}R_e\left\{\underline{W}_m\right\} = \frac{1}{4}\mu_r|\underline{\overline{H}}|^2 \quad \left[J/m^3\right] \qquad \text{(magnetic energy density)} \tag{2.7.28}$$

$$\left\langle W_e(t)\right\rangle = \frac{1}{2}R_e\left\{\underline{W}_e\right\} = \frac{1}{4}\varepsilon_r|\underline{\overline{E}}|^2 \quad \left[J/m^3\right] \qquad \text{(electric energy density)} \tag{2.7.29}$$

$$P_d = \frac{1}{2}\sigma|\underline{\overline{E}}|^2 + \iiint_V 2\omega I_m\left[\underline{W}_e - \underline{W}_m\right]dv \equiv \quad \left[W/m^3\right] \text{(power dissipation density)} \tag{2.7.30}$$

We can now interpret the physical significance of the complex Poynting vector $\underline{\overline{S}}$ by restating the real part of (2.7.24) as the time-average quantity:

$$p_r + p_d = 0 \ \ [W] \tag{2.7.31}$$

where the time-average total *power radiated* outward across the surface area A is:

$$p_r = \frac{1}{2}R_e\oiint_A\left\{\underline{\overline{S}}\bullet\hat{n}\right\}da \quad [W] \tag{2.7.32}$$

as also given by (2.7.17), and p_d is the time-average power dissipated [W] within the volume V, as given by (2.7.27). The *flux density* or time-average radiated power intensity [W/m^2] is therefore $P_r = 0.5R_e\left[\underline{\overline{S}}\right]$. Note that the dissipated power p_d can be negative if there is an external or internal source (e.g., a battery) supplying power to the volume; it is represented by negative contribution to $\underline{\overline{E}}\bullet\underline{\overline{J}}^*$. The imaginary part of the radiation (2.7.24) becomes:

$$\oiint_A I_m\left[\underline{\overline{S}}\bullet\hat{n}\right]da + \iiint_V\left(I_m\left\{\underline{\overline{E}}\bullet\underline{\overline{J}}^*\right\} + 4\omega\left[\left\langle w_m(t)\right\rangle - \left\langle w_e(t)\right\rangle\right]\right)dv = 0 \ \ [W] \tag{2.7.33}$$

The surface integral over A of the imaginary part of the Poynting vector is the reactive power, which is simply related by (2.7.33) to the difference between the average magnetic and electric energies stored in the volume V and to any reactance associated with $\overline{J}$.

2.7.3 Power and energy in uniform plane waves

Consider the +z-propagating uniform time-harmonic plane wave of (2.3.1–2), where:

$$\overline{E}(\bar{r},t) = \hat{x}E_o \cos(z-ct) \quad [V/m] \tag{2.7.34}$$

$$\overline{H}(\bar{r},t) = \hat{y}\sqrt{\frac{\varepsilon_o}{\mu_o}}E_o \cos(z-ct) \quad [A/m^2] \tag{2.7.35}$$

The flux density for this wave is given by the Poynting vector $\overline{S}(t)$ (2.7.12):

$$\overline{S}(t) = \overline{E}\times\overline{H} = \hat{z}\frac{E_o^2}{\eta_o}\cos^2(z-ct) \quad [W/m^2] \tag{2.7.36}$$

where the characteristic impedance of free space is $\eta_o = \sqrt{\mu_o/\varepsilon_o}$ ohms (2.2.19). The time average of $\overline{S}(t)$ in (2.7.36) is $\hat{z}E_o^2/2\eta_o$ [W/m^2].

The electric and magnetic energy densities for this wave can be found from (2.7.7–8):

$$W_e = \frac{1}{2}\varepsilon|\overline{E}|^2 = \frac{1}{2}\varepsilon_o E_o^2\cos^2(z-ct) \quad [J/m^3] \qquad \text{(electric energy density)} \tag{2.7.37}$$

$$W_m = \frac{1}{2}\mu|\overline{H}|^2 = \frac{1}{2}\varepsilon_o E_o^2\cos^2(z-ct) \quad [J/m^3] \qquad \text{(magnetic energy density)} \tag{2.7.38}$$

Note that these two energy densities, W_e and W_m, are equal, non-negative, and sinusoidal in behavior at twice the spatial frequency $k = 2\pi/\lambda$ [radians/m] of the underlying wave, where $\cos^2(z - ct) = 0.5[1 + \cos 2(z - ct)]$, as illustrated in Figure 2.7.1; their frequency f[Hz] is also double that of the underlying wave. They have the same space/time form as the flux density, except with a different magnitude.

The complex electric and magnetic fields corresponding to (2.7.34–5) are:

$$\underline{\overline{E}}(\bar{r}) = \hat{x}\underline{E}_o \quad [V/m] \tag{2.7.39}$$

$$\underline{\overline{H}}(\bar{r}) = \hat{y}\sqrt{\varepsilon_o/\mu_o}\,\underline{E}_o \quad [A/m^2] \tag{2.7.40}$$

The real and imaginary parts of the complex power for this uniform plane wave indicate the time average and reactive powers, respectively:

$$\langle\overline{S}(t)\rangle = \frac{1}{2}R_e[\underline{\overline{S}}] = \frac{1}{2}R_e\left[\underline{\overline{E}}\times\underline{\overline{H}}^*\right] = \hat{z}\frac{1}{2\eta_o}|\underline{\overline{E}}_o|^2 = \hat{z}\frac{1}{2}\eta_o|\underline{\overline{H}}_o|^2 \quad [W/m^2] \tag{2.7.41}$$

$$I_m\left\{\underline{\bar{S}}(\bar{r})\right\} = I_m\left\{\underline{\bar{E}}\times\underline{\bar{H}}^*\right\} = 0 \tag{2.7.42}$$

If two superimposed plane waves are propagating in opposite directions with the same polarization, then the imaginary part of the Poynting vector is usually non-zero. Positive reactive power flowing into a volume is generally associated with an excess of time-average magnetic energy storage over electric energy storage in that volume, and vice-versa, with negative reactive power input corresponding to excess electric energy storage.

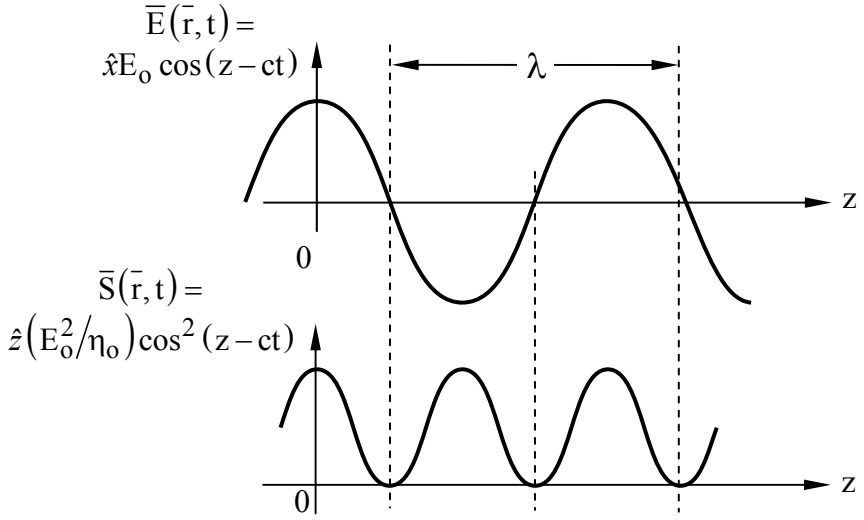

Figure 2.71 Electric field, electric and magnetic storage, and wave intensity for a uniform plane wave.

Example 2.7B

Two equal x-polarized plane waves are propagating along the z axis in opposite directions such that $\bar{E}(t) = 0$ at z = 0 for all time t. What is $\underline{\bar{S}}(z)$?

Solution: $\underline{\bar{E}}(z) = \hat{x}E_o\left(e^{-jkz} - e^{+jkz}\right) = 0$ at z = 0; and $\underline{\bar{H}}(z) = \hat{y}\left(E_o/\eta_o\right)\left(e^{-jkz} + e^{+jkz}\right)$, so $\underline{\bar{S}} = \underline{\bar{E}}\times\underline{\bar{H}}^* = \hat{z}\left(E_o{}^2/\eta_o\right)\left(e^{-j2kz} - e^{+j2kz}\right) = -\hat{z}2j\left(E_o{}^2/\eta_o\right)\sin(2kz)$. $\underline{\bar{S}}$ is pure imaginary and varies sinusoidally along z between inductive and capacitive reactive power, with nulls at intervals of $\lambda/2$.

2.8 Uniqueness theorem

Throughout this text we often implicitly assume uniqueness when we first guess the solution to Maxwell's equations for a given set of boundary conditions and then test that solution against those equations. This process does not guarantee that the resulting solution is unique, and often there are an infinite number of possible solutions, of which we might guess only one. The

uniqueness theorem is quite useful for it sets forth constraints on the boundary conditions that guarantee there is only one solution to Maxwell's equations, which we can find as usual.

To prove the uniqueness theorem we begin by considering a volume V enclosed by surface S and governed by Maxwell's equations:

$$\nabla \bullet \overline{D}_i = \rho \tag{2.8.1}$$

$$\nabla \bullet \overline{B}_i = 0 \qquad \nabla \times \overline{E}_i = -\frac{\partial \overline{B}_i}{\partial t} \qquad \nabla \times \overline{H}_i = \overline{J} + \frac{\partial \overline{D}_i}{\partial t} \tag{2.8.2}$$

where i = 1, 2 correspond to two possible solutions consistent with the given source distributions ρ and $\overline{J}$. We can now show that the difference $\overline{A}_d = \overline{A}_1 - \overline{A}_2$ between these two solutions must be zero under certain conditions, and therefore there can then be no more than one solution: $\overline{A}$ represents $\overline{D}$, $\overline{B}$, $\overline{E}$, $\overline{H}$, or $\overline{J}$.

If we subtract (2.8.1) for i = 2 from (2.8.1) for i = 1 we obtain:

$$\nabla \bullet \left(\overline{D}_1 - \overline{D}_2\right) = \nabla \bullet \overline{D}_d = 0 \tag{2.8.3}$$

Similar subtraction of corresponding equations for (2.8.2) yield three more Maxwell's equations that the difference fields $\overline{B}_d$ and $\overline{D}_d$ must satisfy:

$$\nabla \bullet \overline{B}_d = 0 \qquad \nabla \times \overline{E}_d = -\frac{\partial \overline{B}_d}{\partial t} \qquad \nabla \times \overline{H}_d = \frac{\partial \overline{D}_d}{\partial t} \tag{2.8.4}$$

where we note that the source terms ρ and $\overline{J}$ have vanished from (2.8.3) and (2.8.4) because they are given and fixed.

The boundary constraints that ensure uniqueness are:

(1) At some time t = 0 the fields are known everywhere so that at that instant $\overline{E}_d = \overline{D}_d = \overline{H}_d = \overline{B}_d = 0$.

(2) At all times and at each point on the surface S either the tangential $\overline{E}$ or the tangential $\overline{H}$ is known.

Applying Poynting's theorem (2.7.10) to the difference fields at time t proves uniqueness subject to these constraints:

$$\iiint_V \left[\overline{H}_d \bullet \frac{\partial \overline{B}_d}{\partial t} + \overline{E}_d \bullet \frac{\partial \overline{D}_d}{\partial t} \right] dv + \oiint_S \left(\overline{E}_d \times \overline{H}_d\right) \bullet d\overline{a} = 0 \tag{2.8.5}$$

Boundary constraint (2) ensures that the tangential component of either $\overline{E}_d$ or $\overline{H}_d$ is always zero, thus forcing the cross product in the second integral of (2.8.5) to zero everywhere on the enclosing surface S. The first integral can be simplified if $\overline{D} = \varepsilon\overline{E}$ and $\overline{B} = \mu\overline{H}$, where both ε and μ can be functions of position. Because this volume integral then involves only the time derivative of the squares of the difference fields $\left(|\overline{H}_d|^2 \text{ and } |\overline{E}_d|^2\right)$, and because these fields are zero at t = 0 by virtue of constraint (1), the difference fields can never depart from zero while satisfying (2.8.5). Since (2.8.5) holds for all time, the difference fields must therefore always be zero everywhere, meaning there can be no more than one solution to Maxwell's equations subject to the two constraints listed above.

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