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## 6.013 Final Exam Formula Sheets

December 21, 2005

### Cartesian Coordinates (x,y,z):

$$\nabla\Psi = \hat{x}\frac{\partial\Psi}{\partial x} + \hat{y}\frac{\partial\Psi}{\partial y} + \hat{z}\frac{\partial\Psi}{\partial z}$$

$$\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\bar{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}$$

### Cylindrical coordinates (r,φ,z):

$$\nabla\Psi = \hat{r}\frac{\partial\Psi}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial\Psi}{\partial\phi} + \hat{z}\frac{\partial\Psi}{\partial z}$$

$$\nabla\cdot\bar{A} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\bar{A} = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left(\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right) = \frac{1}{r}\det\begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{\partial^2\Psi}{\partial z^2}$$

### Spherical coordinates (r,θ,φ):

$$\nabla\Psi = \hat{r}\frac{\partial\Psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\Psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}$$

$$\nabla\cdot\bar{A} = \frac{1}{r^2}\frac{\partial(r^2A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\bar{A} = \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right) + \hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)$$

$$= \frac{1}{r^2\sin\theta}\det\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$$

$$\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$$

### Gauss' Divergence Theorem:

$$\int_V \nabla\cdot\bar{G} \, dv = \oint_A \bar{G}\cdot\hat{n} \, da$$

### Stokes' Theorem:

$$\int_A (\nabla\times\bar{G})\cdot\hat{n} \, da = \oint_C \bar{G}\cdot d\bar{\ell}$$

### Vector Algebra:

$$\nabla = \hat{x}\partial/\partial x + \hat{y}\partial/\partial y + \hat{z}\partial/\partial z$$

$$\bar{A}\cdot\bar{B} = A_x B_x + A_y B_y + A_z B_z$$

$$\nabla\cdot(\nabla\times\bar{A}) = 0$$

$$\nabla\times(\nabla\times\bar{A}) = \nabla(\nabla\cdot\bar{A}) - \nabla^2\bar{A}$$

# Basic Equations for Electromagnetics and Applications

## Fundamentals

$$\vec{f} = q(\vec{E} + \vec{v} \times \mu_0 \vec{H}) [\text{N}] \quad (\text{Force on point charge})$$

$$\nabla \times \vec{E} = -\partial \vec{B} / \partial t$$

$$\oint_c \vec{E} \cdot d\vec{s} = -\frac{d}{dt} \int_A \vec{B} \cdot d\vec{a}$$

$$\nabla \times \vec{H} = \vec{J} + \partial \vec{D} / \partial t$$

$$\oint_c \vec{H} \cdot d\vec{s} = \int_A \vec{J} \cdot d\vec{a} + \frac{d}{dt} \int_A \vec{D} \cdot d\vec{a}$$

$$\nabla \cdot \vec{D} = \rho \rightarrow \int_A \vec{D} \cdot d\vec{a} = \int_V \rho dv$$

$$\nabla \cdot \vec{B} = 0 \rightarrow \int_A \vec{B} \cdot d\vec{a} = 0$$

$$\nabla \cdot \vec{J} = -\partial \rho / \partial t$$

$$\vec{E} = \text{electric field (Vm}^{-1}\text{)}$$

$$\vec{H} = \text{magnetic field (Am}^{-1}\text{)}$$

$$\vec{D} = \text{electric displacement (Cm}^{-2}\text{)}$$

$$\vec{B} = \text{magnetic flux density (T)}$$

$$\text{Tesla (T)} = \text{Weber m}^{-2} = 10,000 \text{ gauss}$$

$$\rho = \text{charge density (Cm}^{-3}\text{)}$$

$$\vec{J} = \text{current density (Am}^{-2}\text{)}$$

$$\sigma = \text{conductivity (Siemens m}^{-1}\text{)}$$

$$\vec{J}_s = \text{surface current density (Am}^{-1}\text{)}$$

$$\rho_s = \text{surface charge density (Cm}^{-2}\text{)}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ Fm}^{-1}$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ Hm}^{-1}$$

$$c = (\epsilon_0 \mu_0)^{-0.5} \cong 3 \times 10^8 \text{ ms}^{-1}$$

$$e = -1.60 \times 10^{-19} \text{ C}$$

$$\eta_0 \cong 377 \text{ ohms} = (\mu_0 / \epsilon_0)^{0.5}$$

$$(\nabla^2 - \mu\epsilon \partial^2 / \partial t^2) \vec{E} = 0 \quad [\text{Wave Eqn.}]$$

$$E_y(z,t) = E_+(z-ct) + E_-(z+ct) = \text{Re} \{ \underline{E}_y(z) e^{j\omega t} \}$$

$$H_x(z,t) = \eta_0^{-1} [E_+(z-ct) - E_-(z+ct)] \quad [\text{or } (\omega t - kz) \text{ or } (t - z/c)]$$

$$\int_A (\vec{E} \times \vec{H}) \cdot d\vec{a} + (d/dt) \int_V (\epsilon |\vec{E}|^2 / 2 + \mu |\vec{H}|^2 / 2) dv$$

$$= - \int_V \vec{E} \cdot \vec{J} dv \quad (\text{Poynting Theorem})$$

## Media and Boundaries

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\nabla \cdot \vec{D} = \rho_f, \quad \tau = \epsilon / \sigma$$

$$\nabla \cdot \epsilon_0 \vec{E} = \rho_f + \rho_p$$

$$\nabla \cdot \vec{P} = -\rho_p, \quad \vec{J} = \sigma \vec{E}$$

$$\vec{B} = \mu \vec{H} = \mu_0 (\vec{H} + \vec{M})$$

$$\epsilon(\omega) = \epsilon (1 - \omega_p^2 / \omega^2), \quad \omega_p = (Ne^2 / m\epsilon)^{0.5} \quad (\text{plasma})$$

$$\epsilon_{eff} = \epsilon (1 - j\sigma / \omega\epsilon)$$

$$\text{Skin depth } \delta = (2 / \omega\mu\sigma)^{0.5} [m]$$

$$\vec{E}_{1//} - \vec{E}_{2//} = 0$$

$$\vec{H}_{1//} - \vec{H}_{2//} = \vec{J}_s \times \hat{n}$$

$$B_{1\perp} - B_{2\perp} = 0$$

$$(D_{1\perp} - D_{2\perp}) = \rho_s$$

$$\downarrow 0 = \text{if } \sigma = \infty$$



## Electromagnetic Quasistatics

$$\vec{E} = -\nabla \Phi(r), \quad \Phi(r) = \int_{V'} (\rho(\vec{r}') / 4\pi\epsilon |\vec{r}' - \vec{r}|) dv'$$

$$\nabla^2 \Phi = -\rho_f / \epsilon$$

$$C = Q/V = A\epsilon/d [F]$$

$$L = \Lambda/I$$

$$i(t) = C dv(t)/dt$$

$$v(t) = L di(t)/dt = d\Lambda/dt$$

$$w_e = Cv^2(t)/2; \quad w_m = Li^2(t)/2$$

$$L_{\text{solenoid}} = N^2 \mu A / W$$

$$\tau = RC, \quad \tau = L/R$$

$$\Lambda = \int_A \vec{B} \cdot d\vec{a} \quad (\text{per turn})$$

$$\text{KCL: } \sum_i I_i(t) = 0 \quad \text{at node}$$

$$\text{KVL: } \sum_i V_i(t) = 0 \quad \text{around loop}$$

$$Q = \omega_0 w_T / P_{diss} = \omega_0 / \Delta\omega$$

$$\omega_0 = (LC)^{-0.5}$$

$$\langle V^2(t) \rangle / R = kT$$

## Electromagnetic Waves

$$(\nabla^2 - \mu\epsilon \partial^2 / \partial t^2) \vec{E} = 0 \quad [\text{Wave Eqn.}]$$

$$(\nabla^2 + k^2) \hat{E} = 0, \quad \hat{E} = \hat{E}_0 e^{-j\vec{k} \cdot \vec{r}}$$

$$k = \omega(\mu\epsilon)^{0.5} = \omega/c = 2\pi/\lambda$$

$$k_x^2 + k_y^2 + k_z^2 = k_0^2 = \omega^2 \mu\epsilon$$

$$v_p = \omega/k, \quad v_g = (\partial k / \partial \omega)^{-1}$$

$$\theta_r = \theta_i$$

$$\sin \theta_t / \sin \theta_i = k_i / k_t = n_i / n_t$$

$$\theta_c = \sin^{-1} (n_t / n_i)$$

$$\theta_B = \tan^{-1} (\epsilon_t / \epsilon_i)^{0.5} \quad \text{for TM}$$

$$\theta > \theta_c \Rightarrow \hat{E}_t = \hat{E}_i T e^{+\alpha x - jk_z z}$$

$$\vec{k} = \vec{k}' - j\vec{k}''$$

$$\Gamma = T - 1$$

$$T_{TE} = 2 / (1 + [\eta_i \cos \theta_t / \eta_t \cos \theta_i])$$

$$T_{TM} = 2 / (1 + [\eta_t \cos \theta_t / \eta_i \cos \theta_i])$$

## Radiating Waves

$$\nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \mathbf{J}_f$$

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho_f}{\epsilon}$$

$$\bar{A} = \int_{V'} \frac{\mu \mathbf{J}_f(t - r_{QP}/c) dV'}{4\pi r_{QP}}$$

$$\Phi = \int_{V'} \frac{\rho_f(t - r_{QP}/c) dV'}{4\pi \epsilon r_{QP}}$$

$$\bar{E} = -\nabla \Phi - \frac{\partial \bar{A}}{\partial t}, \quad \bar{B} = \nabla \times \bar{A}$$

$$\nabla^2 \hat{\Phi} + \omega^2 \mu \epsilon \hat{\Phi} = -\hat{\rho}/\epsilon,$$

$$\Phi(x, y, z, t) = \text{Re} \left[ \hat{\Phi}(x, y, z) e^{j\omega t} \right]$$

$$\nabla^2 \hat{A} + \omega^2 \mu \epsilon \hat{A} = -\mu \hat{J},$$

$$\bar{A}(x, y, z, t) = \text{Re} \left[ \hat{A}(x, y, z) e^{j\omega t} \right]$$

$$\hat{\Phi}(\mathbf{r}) = \int_{V'} \hat{\rho}(\bar{r}) e^{-jk|\bar{r}-\bar{r}'|} / (4\pi \epsilon |\bar{r}-\bar{r}'|) dV'$$

$$\hat{A}(\mathbf{r}) = \int_{V'} (\mu \hat{J}(\bar{r}) e^{-jk|\bar{r}-\bar{r}'|} / 4\pi |\bar{r}-\bar{r}'|) dV'$$

$$\hat{E}_{\text{m0}} = \sqrt{\frac{\mu}{\epsilon}} \hat{H}_{\text{m0}} = (j\eta k \hat{d} / 4\pi r) e^{-jkr} \sin \theta$$

## Forces, Motors, and Generators

$$\bar{J} = \sigma(\bar{E} + \bar{v} \times \bar{B})$$

$$\bar{F} = \bar{I} \times \bar{B} \quad [\text{Nm}^{-1}] \quad (\text{force per unit length})$$

$$\bar{E} = -\bar{v} \times \bar{B} \quad \text{inside perfectly conducting wire} \quad (\sigma \rightarrow \infty)$$

$$\text{Max } f/A = B^2/2\mu, \quad D^2/2\epsilon \quad [\text{Nm}^{-2}]$$

$$v_i = \frac{dw_T}{dt} + f \frac{dz}{dt}$$

$$f = ma = d(mv)/dt$$

$$P = fv = T\omega \quad (\text{Watts})$$

$$T = I d\omega/dt$$

$$I = \sum_i m_i r_i^2$$

$$\bar{F}_E = \lambda \bar{E} \quad [\text{Nm}^{-1}] \quad \text{Force per unit length on line charge } \lambda$$

$$W_M(\lambda, x) = \frac{1}{2} \frac{\lambda^2}{L(x)}; \quad W_E(q, x) = \frac{1}{2} \frac{q^2}{C(x)}$$

$$f_M(\lambda, x) = -\frac{\partial W_M}{\partial x} \Big|_{\lambda} = -\frac{1}{2} \lambda^2 \frac{d}{dx} (1/L(x)) = \frac{1}{2} I^2 \frac{dL(x)}{dx}$$

$$f_E(q, x) = -\frac{\partial W_E}{\partial x} \Big|_q = -\frac{1}{2} q^2 \frac{d}{dx} (1/C(x)) = \frac{1}{2} v^2 \frac{dC(x)}{dx}$$

## Wireless Communications and Radar

$$G(\theta, \phi) = P_r / (P_R / 4\pi r^2)$$

$$P_R = \int_{4\pi} P_r(\theta, \phi, r) r^2 \sin \theta \, d\theta d\phi$$

$$P_{\text{rec}} = P_r(\theta, \phi) A_e(\theta, \phi)$$

$$A_e(\theta, \phi) = G(\theta, \phi) \lambda^2 / 4\pi$$

$$G(\theta, \phi) = 1.5 \sin^2 \theta \quad (\text{Hertzian Dipole})$$

$$R_r = P_R / \langle i^2(t) \rangle = \frac{2\pi \sqrt{\mu/\epsilon}}{3} \left( \frac{dl_{\text{eff}}}{\lambda} \right)^2$$

$$E_{\text{ff}}(\theta \cong 0) = (j e^{jkr} / \lambda r) \int_A E_t(x, y) e^{jk_x x + jk_y y} dx dy$$

$$\hat{E}_z = \sum_i a_i \bar{E} e^{-jk r_i} = (\text{element factor})(\text{array } f)$$

$$E_{\text{bit}} \geq \sim 4 \times 10^{-20} \quad [\text{J}]$$

$$Z_{12} = Z_{21} \quad \text{if reciprocity}$$

$$\text{At } \omega_0, \quad \langle w_e \rangle = \langle w_m \rangle$$

$$\langle w_e \rangle = \int_V \left( \epsilon |\hat{E}|^2 / 4 \right) dv$$

$$\langle w_m \rangle = \int_V \left( \mu |\hat{H}|^2 / 4 \right) dv$$

$$Q_n = \omega_n w_{Tn} / P_n = \omega_n / 2\alpha_n$$

$$f_{\text{mnp}} = (c/2) \left( [m/a]^2 + [n/b]^2 + [p/d]^2 \right)^{0.5}$$

$$S_n = j\omega_n - \alpha_n$$

## Acoustics

$$P = P_0 + p, \quad \bar{U} = 0 + \bar{u}$$

$$\nabla p = -\rho_0 \partial \bar{u} / \partial t$$

$$\nabla \cdot \bar{u} = -(1/\rho_0 c_s^2) \partial p / \partial t$$

$$(\nabla^2 - k^2 \partial^2 / \partial t^2) p = 0$$

$$k^2 = \omega^2 / c_s^2 = \omega^2 \rho_0 / \gamma P_0$$

$$c_s = v_p = v_g = (\gamma P_0 / \rho_0)^{0.5} \quad \text{or} \quad (K / \rho_0)^{0.5} \quad \text{or} \quad \sqrt{RT}$$

$$\eta_s = p/u = \rho_0 c_s = (\rho_0 \gamma P_0)^{0.5} \quad \text{or} \quad \rho_0 \sqrt{RT} \quad \text{gases}$$

$$\eta_s = (\rho_0 K)^{0.5} \quad \text{solids, liquids}$$

$$p, \bar{u}_\perp \quad \text{continuous at boundaries}$$

$$\hat{p}(z) = \hat{p}_+ e^{-jkz} + \hat{p}_- e^{+jkz}, \quad p(z, t) = \text{Re} \left[ \hat{p}(z) e^{j\omega t} \right]$$

$$\hat{u}_z = \eta_s^{-1} (\hat{p}_+ e^{-jkz} - \hat{p}_- e^{+jkz}), \quad u_z(z, t) = \text{Re} \left[ \hat{u}_z(z) e^{j\omega t} \right]$$

$$\int_A \bar{u} p \cdot \bar{d}\bar{a} + (d/dt) \int_V (\rho_0 |\bar{u}|^2 / 2 + p^2 / 2\gamma P_0) dV$$

## Optical Communications

$E = hf$ , photons or phonons

$hf/c = \text{momentum [kg ms}^{-1}\text{]}$

$$dn_2/dt = -[An_2 + B(n_2 - n_1)]$$

## Transmission Lines

### Time Domain

$$\partial v(z,t)/\partial z = -L\partial i(z,t)/\partial t$$

$$\partial i(z,t)/\partial z = -C\partial v(z,t)/\partial t$$

$$\partial^2 v/\partial z^2 = LC \partial^2 v/\partial t^2$$

$$v(z,t) = V_+(t - z/c) + V_-(t + z/c)$$

$$i(z,t) = Y_0[V_+(t - z/c) - V_-(t + z/c)]$$

$$c = (LC)^{-0.5} = (\mu\epsilon)^{-0.5}$$

$$Z_0 = Y_0^{-1} = (L/C)^{0.5}$$

$$\Gamma_L = V_-/V_+ = (R_L - Z_0)/(R_L + Z_0)$$

### Frequency Domain

$$(d^2/dz^2 + \omega^2 LC)\hat{V}(z) = 0$$

$$\hat{V}(z) = \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz}, \quad v(z,t) = \text{Re}[\hat{V}(z)e^{j\omega t}]$$

$$\hat{I}(z) = Y_0[\hat{V}_+ e^{-jkz} - \hat{V}_- e^{+jkz}], \quad i(z,t) = \text{Re}[\hat{I}(z)e^{j\omega t}]$$

$$k = 2\pi/\lambda = \omega/c = \omega(\mu\epsilon)^{0.5}$$

$$Z(z) = \hat{V}(z)/\hat{I}(z) = Z_0 Z_n(z)$$

$$Z_n(z) = [1 + \Gamma(z)]/[1 - \Gamma(z)] = R_n + jX_n$$

$$\Gamma(z) = (V_-/V_+)e^{2jkz} = [Z_n(z) - 1]/[Z_n(z) + 1]$$

$$Z(z) = Z_0 (Z_L - jZ_0 \tan kz)/(Z_0 - jZ_L \tan kz)$$

$$\text{VSWR} = |V_{\max}|/|V_{\min}|$$