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6.013/ESD.013J Electromagnetics and Applications, Fall 2005

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Massachusetts Institute of Technology
 Department of Electrical Engineering and Computer Science
 6.013 Electromagnetics and Applications
 Problem Set #2 SOLUTION
 Fall Term 2005

Problem 2.1

a. By the divergence theorem;

$$\int_V \nabla \cdot (\nabla \times \bar{A}) dV = \oint_S (\nabla \times \bar{A}) \cdot d\bar{a} \quad \text{where } S \text{ encloses } V.$$

By Stokes' theorem:

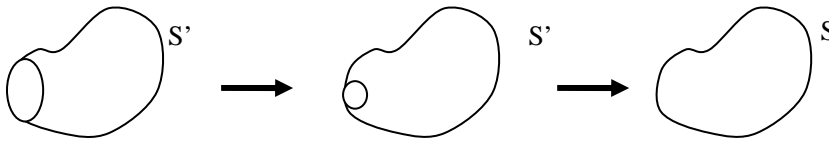
$$\int_{S'} (\nabla \times \bar{A}) \cdot d\bar{a} = \oint_C \bar{A} \cdot d\bar{l}$$

Suppose $S'=S$ is a closed surface



Or S' is an open surface with boundary contour C . i.e., S' is the same as S , except for the curve C , which makes S' not a closed surface.

Now consider the limit as $C \rightarrow 0$;



So that $S'=S$.

$$\text{If } C \text{ is } 0 \text{ then } \int_{S'} (\nabla \times \bar{A}) \cdot d\bar{a} = \oint_C \bar{A} \cdot d\bar{l} = 0 \text{ and } \int_V \nabla \cdot (\nabla \times \bar{A}) dV = \oint_S (\nabla \times \bar{A}) \cdot d\bar{a} = 0$$

Since V can be any volume, the argument of the integral must be identically 0:

$$\nabla \cdot (\nabla \times \bar{A}) = 0$$

b. $\bar{A} = A_r \bar{i}_r + A_\phi \bar{i}_\phi + A_z \bar{i}_z$ in the cylindrical geometry.

$$\nabla \times \bar{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \bar{i}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \bar{i}_\phi + \frac{1}{r} \left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right) \bar{i}_z$$

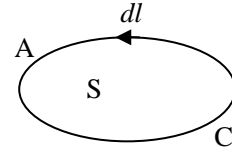
$$\begin{aligned} \nabla \cdot (\nabla \times \bar{A}) &= \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \right] + \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \frac{\partial}{\partial z} \left[\frac{1}{r} \left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right) \right] \\ &= \frac{1}{r} \frac{\partial^2 A_z}{\partial r \partial \phi} - \frac{1}{r} \frac{\partial A_\phi}{\partial z} - \frac{\partial^2 A_\phi}{\partial r \partial z} + \frac{1}{r} \frac{\partial^2 A_r}{\partial \phi \partial z} - \frac{1}{r} \frac{\partial^2 A_z}{\partial r \partial \phi} + \frac{1}{r} \frac{\partial A_\phi}{\partial z} + \frac{\partial^2 A_\phi}{\partial r \partial z} - \frac{1}{r} \frac{\partial^2 A_r}{\partial \phi \partial z} = 0 \end{aligned}$$

c. From stokes' theorem

$$\int_S (\nabla \times (\nabla f)) \cdot d\vec{a} = \oint_C \nabla f \cdot d\vec{l} \quad \text{Here} \quad \nabla f = \frac{\partial f}{\partial x} \vec{i}_x + \frac{\partial f}{\partial y} \vec{i}_y + \frac{\partial f}{\partial z} \vec{i}_z \quad ,$$

$$d\vec{l} = dx\vec{i}_x + dy\vec{i}_y + dz\vec{i}_z$$

$$\nabla f \cdot d\vec{l} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$



$$\text{So } \int_S (\nabla \times (\nabla f)) \cdot d\vec{a} = \oint_C \nabla f \cdot d\vec{l} = \oint_C df = f \Big|_A^A = 0$$

Since S can be any surface, the argument of the integral must be identically 0:

$$\nabla \times (\nabla f) = 0$$

d. $\nabla f = \frac{\partial f}{\partial r} \vec{i}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{i}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \vec{i}_\phi$ in spherical coordinate.

$$\begin{aligned} \nabla \times (\nabla f) &= \frac{1}{r \sin \theta} \left[\frac{\partial \left(\sin \theta \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \right)}{\partial \theta} - \frac{\partial \left(\frac{1}{r} \frac{\partial f}{\partial \theta} \right)}{\partial \phi} \right] \vec{i}_r + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial \left(\frac{\partial f}{\partial r} \right)}{\partial \phi} - \frac{\partial \left(r \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \right)}{\partial \phi} \right] \vec{i}_\theta \\ &\quad + \frac{1}{r} \left[\frac{\partial \left(r \frac{1}{r} \frac{\partial f}{\partial \theta} \right)}{\partial r} - \frac{\partial \left(\frac{\partial f}{\partial r} \right)}{\partial \theta} \right] \vec{i}_\phi \\ &= \frac{1}{r^2 \sin \theta} \left(\frac{\partial^2 f}{\partial \theta \partial \phi} - \frac{\partial^2 f}{\partial \theta \partial \phi} \right) \vec{i}_r + \frac{1}{r} \left(\frac{\partial^2 f}{\partial r \partial \phi} - \frac{\partial^2 f}{\partial r \partial \phi} \right) \vec{i}_\theta + \frac{1}{r} \left(\frac{\partial^2 f}{\partial r \partial \theta} - \frac{\partial^2 f}{\partial r \partial \theta} \right) \vec{i}_\phi = 0 \end{aligned}$$

Problem 2.2

a. The total z directed current on the cylinder is:

$$I_0 = \int_0^{R_1} J_z(r) 2\pi r dr = \int_0^{R_1} J_0 \left(\frac{r}{R_1} \right)^2 2\pi r dr = \frac{J_0 2\pi}{4R_1^2} r^4 \Big|_0^{R_1} = \frac{J_0 \pi R_1^2}{2}$$

b. By Ampere's integral law $\oint_C \vec{H} \cdot d\vec{s} = \int_S \vec{J} \cdot d\vec{a}$

$$H_\phi(r) 2\pi r = \begin{cases} \int_0^r J_z(r') 2\pi r' dr' = \frac{J_0 2\pi}{4R_1^2} r'^4 \Big|_0^r = \frac{J_0 2\pi}{4R_1^2} r^4, & r < R_1 \\ I_0, & R_1 < r < R_2 \end{cases}$$

$$H_\phi(r) = \begin{cases} \frac{J_0}{4R_1^2} r^3, & r < R_1 \\ \frac{J_0 R_1^2}{4r}, & R_1 < r < R_2 \end{cases}$$

c. The current density at $r = R_2$ is: $K_z = -H_\phi(r = R_2) = -\frac{J_0 R_1^2}{4R_2}$

d. The total current on cylinder of $r = R_2$ is: $I(r = R_2) = 2\pi R_2 K_z = -\frac{J_0 \pi R_1^2}{2} = -I_0$

Problem 2.3

a. The total charge on the sphere is: $q = \int_0^{R_1} \rho_0 \left(\frac{r}{R_1} \right)^4 4\pi r^2 dr = \frac{4\pi \rho_0 R_1^3}{7}$

b. By Gauss' law in free space $\oiint_s \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int_V \rho dV$

$$4\pi r^2 E_r = \begin{cases} \frac{1}{\epsilon_0} \int_0^r \rho_0 \left(\frac{r'}{R_1} \right)^4 4\pi r'^2 dr' = \frac{4\pi \rho_0 r^7}{7\epsilon_0 R_1^4}, & r < R_1 \\ \frac{1}{\epsilon_0} q = \frac{4\pi \rho_0 R_1^3}{7\epsilon_0}, & R_1 < r < R_2 \end{cases}$$

$$E_r = \begin{cases} \frac{\rho_0 r^5}{7\epsilon_0 R_1^4}, & r < R_1 \\ \frac{\rho_0 R_1^3}{7\epsilon_0 r^2}, & R_1 < r < R_2 \end{cases}$$

c. The surface charge on the sphere of radius R_2 is: $\sigma_s = -\epsilon_0 E_r(r = R_2) = -\frac{\rho_0 R_1^3}{7R_2^2}$

d. The total charge at $r = R_2$ is: $q(r = R_2) = \sigma_s 4\pi R_2^2 = -\frac{4\pi\rho_0 R_1^3}{7} = -q$

Problem 2.4

a. $H_\phi = \frac{i}{2\pi r}, \lambda_f = \mu_0 N \int_z^{z+l} dz \int_R^{R+h} H_\phi dr = \left[\frac{\mu_0 N l}{2\pi} \ln\left(1 + \frac{h}{R}\right) \right] i$

b. $v(t) = \frac{d\lambda_f}{dt} = \left[\frac{\mu_0 N l}{2\pi} \ln\left(1 + \frac{h}{R}\right) \right] \omega I_0 \cos(\omega t)$

with the values given in the problem statement $v(t) = \frac{d\lambda_f}{dt} = 1.35 \cos(120\pi t) \text{ mV}$

c. The coil should be placed vertically centered on the wire so that half the coil flux is positive and half is negative so that no net flux links the coil.

