Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.013 Electromagnetics and Applications
Problem Set #2 SOLUTION
Fall Term 2005

Problem 2.1

a. By the divergence theorem;

$$\int_V \nabla\bullet\left(\nabla\times\overline{A}\right)dV = \oint_S \left(\nabla\times\overline{A}\right)\bullet d\overline{a} \quad \text{where S encloses V.}$$

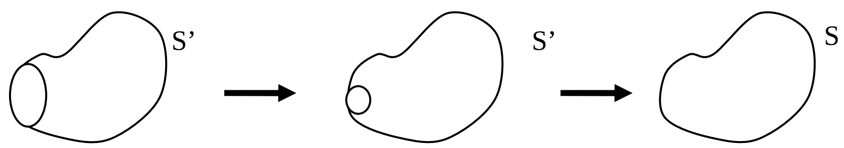
By Stokes' theorem:

$$\int_{S'} \left(\nabla\times\overline{A}\right)\bullet d\overline{a} = \oint_C \overline{A}\bullet d\overline{l}$$

Suppose S'=S is a closed surface

Or S' is an open surface with boundary contour C. i.e., S' is the same as S, except for the curve C, which makes S' not a closed surface.

Now consider the limit as C→0;



So that S'=S.

If C is 0 then $\int_{S'} \left(\nabla\times\overline{A}\right)\bullet d\overline{a} = \oint_C \overline{A}\bullet d\overline{l} = 0$ and $\int_V \nabla\bullet\left(\nabla\times\overline{A}\right)dV = \oint_S \left(\nabla\times\overline{A}\right)\bullet d\overline{a} = 0$

Since V can be any volume, the argument of the integral must be identically 0:

$$\nabla\bullet\left(\nabla\times\overline{A}\right) = 0$$

b. $\overline{A} = A_r\overline{i}_r + A_\phi\overline{i}_\phi + A_z\overline{i}_z$ in the cylindrical geometry.

$$\nabla\times\overline{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\overline{i}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\overline{i}_\phi + \frac{1}{r}\left(\frac{\partial\left(rA_\phi\right)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right)\overline{i}_z$$

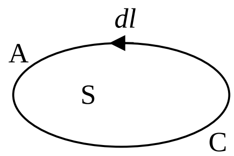
$$\nabla\bullet\left(\nabla\times\overline{A}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\right] + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \frac{\partial}{\partial z}\left[\frac{1}{r}\left(\frac{\partial\left(rA_\phi\right)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right)\right]$$

$$= \frac{1}{r}\frac{\partial^2 A_z}{\partial r\partial\phi} - \frac{1}{r}\frac{\partial A_\phi}{\partial z} - \frac{\partial^2 A_\phi}{\partial r\partial z} + \frac{1}{r}\frac{\partial^2 A_r}{\partial\phi\partial z} - \frac{1}{r}\frac{\partial^2 A_z}{\partial r\partial\phi} + \frac{1}{r}\frac{\partial A_\phi}{\partial z} + \frac{\partial^2 A_\phi}{\partial r\partial z} - \frac{1}{r}\frac{\partial^2 A_r}{\partial\phi\partial z} = 0$$

c. From stokes' theorem

$$\int_S \left(\nabla\times\left(\nabla f\right)\right)\bullet d\overline{a} = \oint_C \nabla f \bullet d\overline{l}$$ Here $$\nabla f = \frac{\partial f}{\partial x}\overline{i}_x + \frac{\partial f}{\partial y}\overline{i}_y + \frac{\partial f}{\partial z}\overline{i}_z$$,

$$d\overline{l} = dx\overline{i}_x + dy\overline{i}_y + dz\overline{i}_z$$



$$\nabla f \bullet d\overline{l} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

So $$\int_S \left(\nabla\times\left(\nabla f\right)\right)\bullet d\overline{a} = \oint_C \nabla f \bullet d\overline{l} = \oint_C df = f\Big|_A^A = 0$$

Since S can be any surface, the argument of the integral must be identically 0:

$$\nabla\times\left(\nabla f\right) = 0$$

d. $\nabla f = \frac{\partial f}{\partial r}\overline{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\overline{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\overline{i}_\phi$ in spherical coordinate.

$$\nabla\times\left(\nabla f\right) = \frac{1}{r\sin\theta}\left[\frac{\partial\left(\sin\theta\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)}{\partial\theta} - \frac{\partial\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right)}{\partial\phi}\right]\overline{i}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial\left(\frac{\partial f}{\partial r}\right)}{\partial\phi} - \frac{\partial\left(r\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\right)}{\partial\phi}\right]\overline{i}_\theta$$

$$+\frac{1}{r}\left[\frac{\partial\left(r\frac{1}{r}\frac{\partial f}{\partial\theta}\right)}{\partial r} - \frac{\partial\left(\frac{\partial f}{\partial r}\right)}{\partial\theta}\right]\overline{i}_\phi$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial^2 f}{\partial\theta\partial\phi} - \frac{\partial^2 f}{\partial\theta\partial\phi}\right)\overline{i}_r + \frac{1}{r}\left(\frac{\partial^2 f}{\partial r\partial\phi} - \frac{\partial^2 f}{\partial r\partial\phi}\right)\overline{i}_\theta + \frac{1}{r}\left(\frac{\partial^2 f}{\partial r\partial\theta} - \frac{\partial^2 f}{\partial r\partial\theta}\right)\overline{i}_\phi = 0$$

Problem 2.2

a. The total z directed current on the cylinder is:

$$I_0 = \int_0^{R_1} J_z(r) 2\pi r dr = \int_0^{R_1} J_0 \left(\frac{r}{R_1}\right)^2 2\pi r dr = \frac{J_0 2\pi}{4R_1^2} r^4 \Big|_0^{R_1} = \frac{J_0 \pi R_1^2}{2}$$

b. By Ampere's integral law $\oint_C \overline{H} \bullet d\overline{s} = \int_S \overline{J} \bullet d\overline{a}$

$$H_\phi(r) 2\pi r = \begin{cases} \int_0^r J_z(r') 2\pi r' dr' = \frac{J_0 2\pi}{4R_1^2} r'^4 \Big|_0^r = \frac{J_0 2\pi}{4R_1^2} r^4, & r < R_1 \\ I_0, & R_1 < r < R_2 \end{cases}$$

$$H_\phi(r) = \begin{cases} \frac{J_0}{4R_1^2} r^3, r < R_1 \\ \frac{J_0 R_1^2}{4r}, R_1 < r < R_2 \end{cases}$$

c. The current density at $r = R_2$ is: $K_z = -H_\phi(r = R_2) = -\frac{J_0 R_1^2}{4R_2}$

d. The total current on cylinder of $r = R_2$ is: $I(r = R_2) = 2\pi R_2 K_z = -\frac{J_0 \pi R_1^2}{2} = -I_0$

Problem 2.3

a. The total charge on the sphere is: $q = \int_0^{R_1} \rho_0 \left(\frac{r}{R_1}\right)^4 4\pi r^2 dr = \frac{4\pi \rho_0 R_1^3}{7}$

b. By Gauss' law in free space $\oiint_s \overline{E} \bullet d\overline{a} = \frac{1}{\varepsilon_0} \int_V \rho dV$

$$4\pi r^2 E_r = \begin{cases} \frac{1}{\varepsilon_0} \int_0^r \rho_0 \left(\frac{r'}{R_1}\right)^4 4\pi r'^2 dr' = \frac{4\pi \rho_0 r^7}{7\varepsilon_0 R_1^4}, r < R_1 \\ \frac{1}{\varepsilon_0} q = \frac{4\pi \rho_0 R_1^3}{7\varepsilon_0}, & R_1 < r < R_2 \end{cases}$$

$$E_r = \begin{cases} \frac{\rho_0 r^5}{7\varepsilon_0 R_1^4}, r < R_1 \\ \frac{\rho_0 R_1^3}{7\varepsilon_0 r^2}, R_1 < r < R_2 \end{cases}$$

c. The surface charge on the sphere of radius R_2 is: $\sigma_s = -\varepsilon_0 E_r(r = R_2) = -\frac{P_0 R_1^3}{7R_2^2}$

d. The total charge at $r = R_2$ is: $q(r = R_2) = \sigma_s 4\pi R_2^{\,2} = -\dfrac{4\pi\rho_0 R_1^{\,3}}{7} = -q$

<u>Problem 2.4</u>

a. $H_\phi = \dfrac{i}{2\pi r}, \lambda_f = \mu_0 N \int_z^{z+l} dz \int_R^{R+h} H_\phi dr = \left[\dfrac{\mu_0 N l}{2\pi} \ln\left(1 + \dfrac{h}{R}\right)\right] i$

b. $v(t) = \dfrac{d\lambda_f}{dt} = \left[\dfrac{\mu_0 N l}{2\pi} \ln\left(1 + \dfrac{h}{R}\right)\right] \omega I_0 \cos(\omega t)$

with the values given in the problem statement $v(t) = \dfrac{d\lambda_f}{dt} = 1.35 \cos(120\pi t)\ \ mV$

c. The coil should be placed vertically centered on the wire so that half the coil flux is positive and half is negative so that no net flux links the coil.