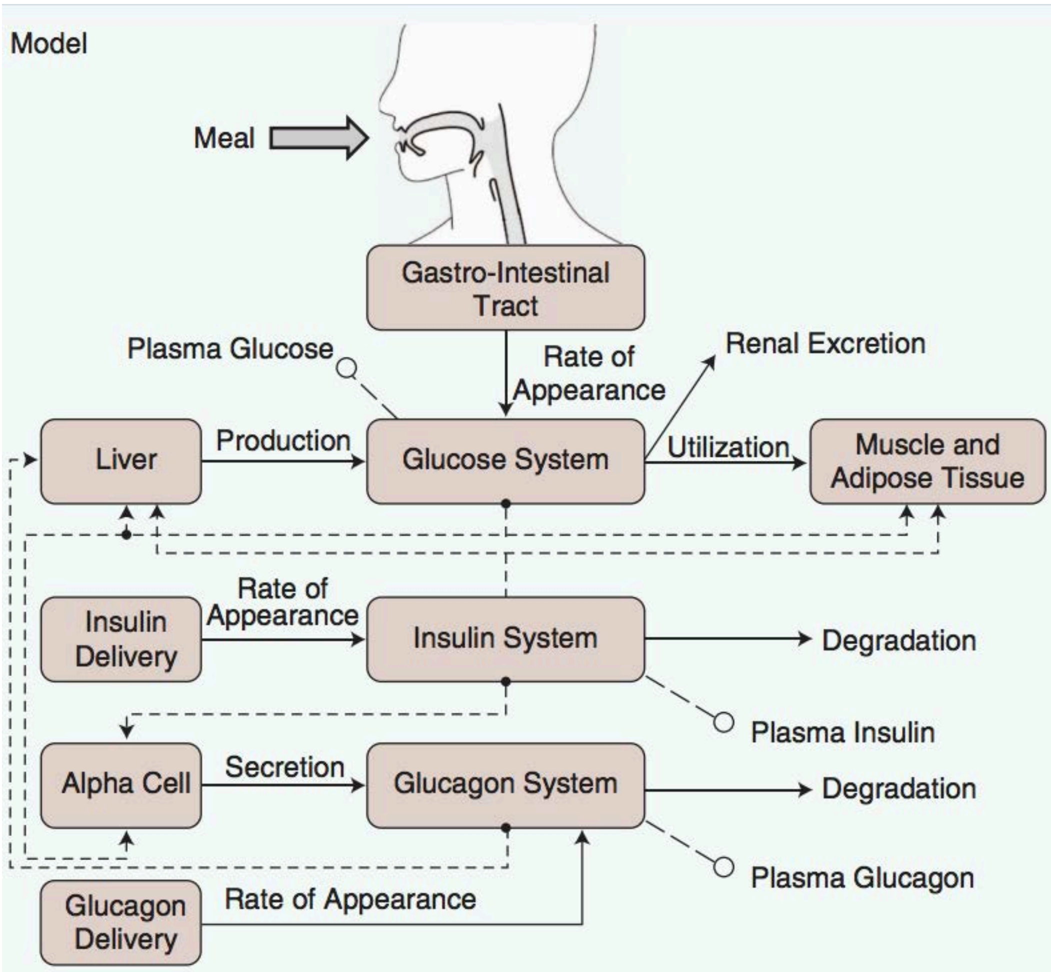
Modal solution of undriven CT LTI state-space models

6.011, Spring 2018

Lec 6

Glucose-insulin system



From Messori et al., *IEEE Control Systems Magazine* Feb 2018

UVA/Padova model (FDA approved!)

$$
\begin{cases}
\dot{x}_1(t) = -k_{gri}x_1(t) + d(t), \\
\dot{x}_2(t) = k_{gri}x_1(t) - k_{empt}(x_1(t) + x_2(t))x_2(t), \\
\dot{x}_3(t) = -k_{abs}x_3(t) + k_{empt}(x_1(t) + x_2(t))x_2(t), \\
\dot{x}_4(t) = EGP(t) + Ra(t) - U_{ii}(t) - E(t) - k_1x_4(t) + k_2x_5(t), \\
\dot{x}_5(t) = -U_{id}(t) + k_1x_4(t) - k_2x_5(t), \\
\dot{x}_6(t) = -(m_2 + m_4)x_6(t) + m_1x_{10}(t) + k_{a1}x_{11}(t) + k_{a2}x_{12}(t), \\
\dot{x}_7(t) = -p_{2U}x_7(t) + p_{2U}\left(\frac{x_6(t)}{V_I} - I_b\right), \\
\dot{x}_8(t) = -k_ix_8(t) + k_i\frac{x_6(t)}{V_I}, \\
\dot{x}_9(t) = -k_ix_9(t) + k_ix_8(t), \\
\dot{x}_{10}(t) = -(m_1 + m_3(t))x_{10}(t) + m_2x_6(t), \\
\dot{x}_{11}(t) = -(k_d + k_{a1})x_{11}(t) + i(t), \\
\dot{x}_{12}(t) = k_dx_{11}(t) - k_{a2}x_{12}(t), \\
\dot{x}_{13}(t) = -k_{sc}x_{13}(t) + k_{sc}x_4(t), \\
\dot{x}_{14}(t) = -n_Gx_{14}(t) + SR_H(t), \\
\dot{x}_{15}(t) = -k_Hx_{15}(t) + k_H\max\{x_{14}(t) - H_b, 0\}, \\
\dot{x}_{16}(t) = \dot{SR}_H^s(t),
\end{cases}
\quad (1)
$$

From Messori et al., *IEEE Control Systems Magazine* Feb 2018

Linearization at an equilibrium yields an LTI model

$$\text{CT case:} \quad \mathbf{q}(t) = \bar{\mathbf{q}} + \widetilde{\mathbf{q}}(t) \ , \quad x(t) = \bar{x} + \widetilde{x}(t) \ ,$$

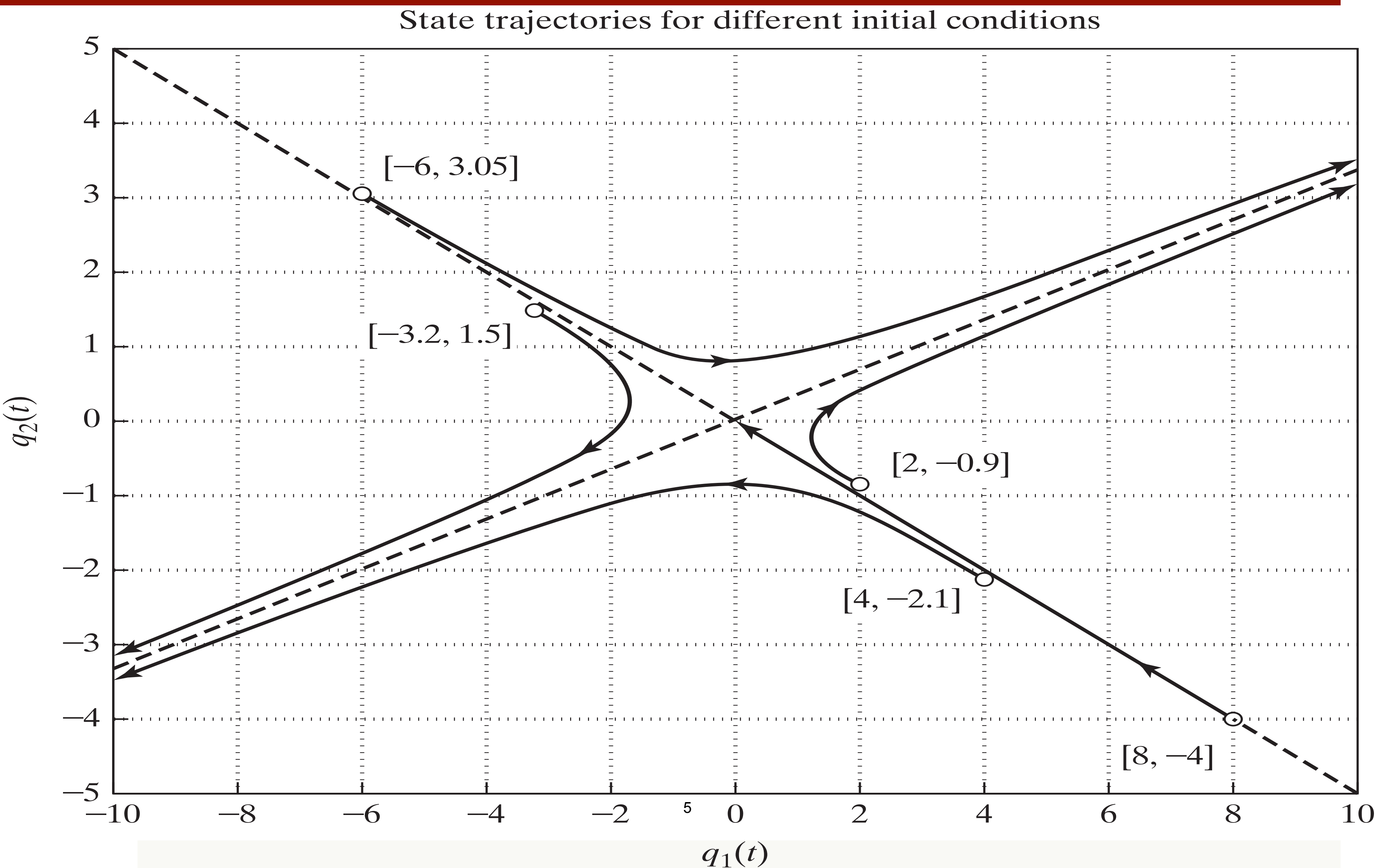
$$\dot{\mathbf{q}}(t) = \mathbf{f}(\mathbf{q}(t), x(t))$$

$$\downarrow$$

$$\dot{\widetilde{\mathbf{q}}}(t) \approx \left[\frac{\partial \mathbf{f}}{\partial \mathbf{q}}_{\ \bar{\mathbf{q}},\bar{x}}\right] \widetilde{\mathbf{q}}(t) + \left[\frac{\partial \mathbf{f}}{\partial x}_{\ \bar{\mathbf{q}},\bar{x}}\right] \widetilde{x}(t)$$

for small perturbations $\widetilde{\mathbf{q}}(t)$ and $\widetilde{x}(t)$ from equilibrium

Phase plane trajectories



Complex eigenvalue pairs (CT case)

If λ_i is a (complex) eigenvalue with eigenvector $\mathbf{v}_i$, then its complex conjugate λ_i^* is also an eigenvalue, with associated eigenvector $\mathbf{v}_i^*$.

Write $\ _i = \ _i + j\omega_i$, $\mathbf{v}_i = \mathbf{u}_i + j\mathbf{v}_i$. Then the contribution of the complex pair to the modal solution is

$$\alpha_i \mathbf{v}_i e^{\ _i t} + \alpha_i^* \mathbf{v}_i^* e^{\ _i^* t} =$$

$$K_i e^{\ _i t}\Big[\mathbf{u}_i \cos(\omega_i t + \theta_i) - \mathbf{w}_i \sin(\omega_i t + \theta_i)\Big]$$

Acoustics and Vibration Animations

Have fun exploring the animations created by

Prof. Dan Russell, Penn State