Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.011: Signals, Systems and Inference

Spring 2018

Problem Set 2
State-Space Models

Reading: Section 1.5.1 and 1.5.2; then move on to Chapter 4, also Sections 5.1–5.2.

The first two problems below still relate to the 6.003 review material we have been occupied with so far, and you should try to get them done before next week's classes. We will be moving on to state-space models, Chapters 4–6, for the next couple of weeks.

We haven't listed any optional problems, because all the unassigned ones in Chapter 4 could usefully serve for additional practice.

Problem 2.1

Consider a discrete-time LTI filter whose frequency response is

$$H(e^{j\Omega}) = 2\exp\{j(\Omega + \Omega^3)\}\ , \qquad |\Omega| < \pi\ .$$

(a) Determine the associated magnitude $\left|H(e^{j\Omega})\right|$ and phase $\angle H(e^{j\Omega})$, verifying that they have the required symmetry properties (magnitude even in Ω, phase odd in Ω).

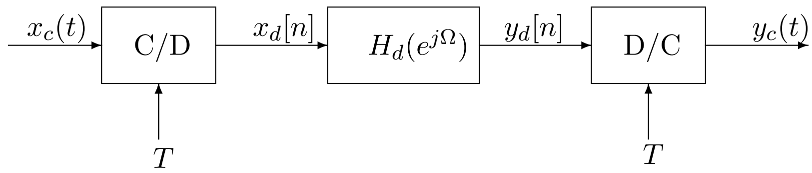
(b) If the input to the system is $x[n] = 4\cos(2n+1)$, what is the output $y[n]$?

(c) Determine the following quantities associated with the unit sample response $h[n]$ of the above system (none of these requires computing $h[n]$ itself):

- $\sum_n h[n]$;
- $\sum_n \left|h[n]\right|^2$;
- the deterministic autocorrelation function $R_{hh}[m] = \sum_{n=-\infty}^{\infty} h[n+m]h[n]$.

Problem 2.2

The figure below shows a standard configuration for DT processing of a CT signal $x_c(t)$ to produce a CT signal $y_c(t)$.



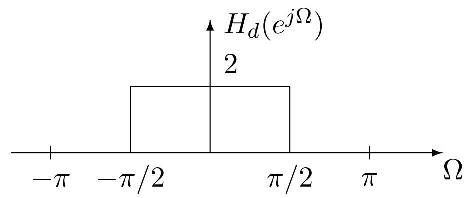
The C/D converter samples its input at integer multiples of T:

$$x_d[n] = x_c(nT) \ .$$

The DT processing here involves filtering by an LTI system with frequency response $H_d(e^{j\Omega})$. We assume an ideal D/C converter that performs bandlimited sinc interpolation of its input samples with reconstruction interval T, so

$$y_c(t) = \sum_{n=-\infty}^{\infty} y_d[n] \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T} \ .$$

(a) Assume that $x_c(t)$ is bandlimited to the frequency range $|\omega| < \pi/T$, and that $H_d(e^{j\Omega})$ is as given in the figure below, with a constant gain of 2 in its passband and 0 elsewhere in the region $|\Omega| < \pi$.



(i) If the energy $\int |x_c(t)|^2 \, dt$ of the CT signal $x_c(t)$ is $\mathcal{E}$, what is the energy $\sum_n |x_d[n]|^2$ of the DT signal $x_d[n]$? Be sure to explain how you arrive at your answer, and state whether the answer is approximate or exact.

(ii) Is it possible for the energy of $y_d[n]$ to exceed that of $x_d[n]$ for this particular system? Again, explain your answer.

(b) Suppose still that $x_c(t)$ is bandlimited to the frequency range $|\omega| < \pi/T$.

(i) Fully and carefully specify a frequency response $H_d(e^{j\Omega})$ that will result in $y_d[n] = x_c(nT + \frac{T}{2})$. (This $y_d[n]$ comprises samples of $x_c(t)$ at *odd* integer multiples of $T/2$, whereas $x_d[n]$ comprised samples at *even* integer multiples of $T/2$.) Be sure to explain your reasoning.

(ii) Find the unit sample response $h_d[n]$ corresponding to the filter you found in part (b)(i).

Problem 2.3

Part (a) of Problem 4.4, followed by **all** of Problem 4.5. Note that the state-space models here will be nonlinear, so of the form given in Eq. (4.43) rather than in the matrix form given in Eq. (4.41). (These are Problems 4.1(a) and 4.2 in softcover SSI.)

Problem 2.4

The following *continuous-time* state-space model has been used to represent the time-evolution of glucose concentration $y(t)$ in blood plasma, in response to plasma insulin concentration $x(t)$:

$$\begin{aligned}
\frac{dq_1(t)}{dt} &= -k_1\Big(q_1(t) - 90\Big) - q_1(t)q_2(t) \\
\frac{dq_2(t)}{dt} &= -k_2 q_2(t) + k_3\Big(x(t) - 11\Big) \\
y(t) &= q_1(t) \ .
\end{aligned}$$

Here $q_1(t) = y(t)$ is the plasma concentration of glucose, while $q_2(t)$ represents the effective insulin activity, and the parameters k_1, k_2 and k_3 are all positive constants (assume k_1 is different from k_2).

(a) Determine the equilibrium state of this model when $x(t)$ is fixed at the value $\overline{x} = 11$, i.e., determine the equilibrium values $\overline{q}_1$ and $\overline{q}_2$ of the two state variables.

(b) Write down the LTI linearized state-space model that governs small perturbations $\widetilde{q}_1(t)$, $\widetilde{q}_2(t)$, $\widetilde{x}(t)$ and $\widetilde{y}(t)$ of the state variables, input and output away from their equilibrium values.

(c) With $\widetilde{x}(t)$ set identically to 0 (i.e., with $x(t)$ fixed at 11), is it possible for $\widetilde{q}_1(t)$ to vary without $\widetilde{q}_2(t)$ varying? And is it possible for $\widetilde{q}_2(t)$ to vary without $\widetilde{q}_1(t)$ varying?

Problem 2.5

Problem 4.17 (a) and (b) (in the hardcover "North American" edition of SSI, which is Problem 4.15 (a) and (b) in the softcover "Global Edition" of SSI).

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