

Massachusetts Institute of Technology
 Department of Electrical Engineering and Computer Science
 6.011: Introduction to Communication, Control and Signal Processing
 QUIZ 1, March 16, 2010
ANSWER BOOKLET

Your Full Name:	SOLUTIONS
Recitation Time :	o'clock

- This quiz is **closed book**, but **two** sheet of notes are allowed. Calculators will not be necessary and are not allowed.
- Check that this **ANSWER BOOKLET** has pages numbered up to 14. The booklet contains spaces for **all** relevant reasoning and answers.
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to this booklet the work you would like us to look at. Let us know if you need additional scratch paper. **Only** this booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- There are **4 problems, weighted as shown.** (The points indicated on the following pages for the various subparts of the problems are our best guess for now, but may be modified slightly when we get to grading.)

Problem	Your Score
1 (5 points)	
2 (15 points)	
3 (10 points)	
4 (20 points)	
Total (50 points)	

Problem 1 (5 points)

$H(e^{j\Omega}) = \exp\{-j(60\Omega + 25\Omega^3)\}$ for $|\Omega| < \pi$. In this frequency range,

1(a) (1 point) the magnitude $|H(e^{j\Omega})|$ is:

1

1(b) (1 point) the phase $\angle H(e^{j\Omega})$ is:

$-(60\Omega + 25\Omega^3)$

1(c) (1 point) the group delay $\tau_g(\Omega)$ is:

**... the negative of the slope of the phase with respect to Ω , therefore $60 + 75\Omega^2$.
This is 60 samples at $\Omega = 0$ and rises to around 800 samples at $\Omega = \pi$.**

1(d) (2 points) The impulse response $h[n]$ of this filter is given by the plot in (choose A or B or C or D or E):

C

Two different features of this response that support choosing it as $h[n]$ are:

The bulk of the response starts around time 60, with the lowest frequencies emerging first, and increasingly higher frequencies emerging later, which is consistent with the group delay characteristic. None of the other waveforms has these features.

Problem 2 (15 points)

The deterministic autocorrelation function of $x(t)$ is

$$\overline{R}_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau) dt = 9 \frac{\sin(2\tau)}{\pi\tau} .$$

2(a) (3 points) The energy spectral density $\overline{S}_{xx}(j\omega)$ of this signal is given by the following — careful and fully labeled! — sketch:

What we want is the Fourier transform of the deterministic autocorrelation function, which is $\overline{S}_{xx}(j\omega) = 9$ for $|\omega| < 2$, and 0 elsewhere. (You can draw the figure!)

The energy of the signal, $\mathcal{E}_x = \int_{-\infty}^{\infty} x^2(t) dt$, is:

$$\mathcal{E}_x = \overline{R}_{xx}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{S}_{xx}(j\omega) d\omega = \frac{18}{\pi} .$$

The time-domain and frequency-domain expressions yield the same result.

- 2(b) (2 points) The magnitude of the Fourier transform of $x(t)$, i.e., $|X(j\omega)|$, is given by the following — careful and fully labeled! — sketch (be sure to explain your answer):

We know $\overline{S}_{xx}(j\omega) = |X(j\omega)|^2$, so $|X(j\omega)| = 3$ for $|\omega| < 2$, and 0 elsewhere. (You can draw the figure!)

- 2(c) (2 points) One possible signal $x(t)$ that has the specified deterministic autocorrelation function $\overline{R}_{xx}(\tau)$ is:

Assuming a signal that has zero phase, i.e., $\angle X(j\omega) = 0$, we have that $X(j\omega) = |X(j\omega)|$, so an appropriate signal would be

$$x(t) = 3 \frac{\sin(2t)}{\pi t} .$$

- 2(d) (2 points) The relation — in either the time domain or the frequency domain — between any other (correct but otherwise arbitrary) answer to 2(c) and the specific one you gave in 2(c):

A solution can have an arbitrary phase characteristic (as long as the phase is an odd function of ω). All solutions have the same magnitude characteristic, determined as in 2(b). Stated another way, one solution can be obtained from another by all-pass filtering.

- 2(e) (3 points) The $x(t)$ above is the input to an ideal lowpass filter that has gain 1 for frequencies ω satisfying $|\omega| < 1$, and gain 0 elsewhere. The corresponding output of the filter is $y(t)$, with energy spectral density $\overline{S}_{yy}(j\omega)$ given by the following — careful and fully labeled! — sketch (be sure to explain your answer):

We know

$$\overline{S}_{yy}(j\omega) = |H(j\omega)|^2 \overline{S}_{xx}(j\omega)$$

where $H(j\omega)$ is the frequency response of the lowpass filter, so $\bar{S}_{yy}(j\omega) = 9$ for $|\omega| < 1$, and 0 elsewhere. The energy of $y(t)$ is thus

$$\mathcal{E}_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{S}_{yy}(j\omega) d\omega = \frac{9}{\pi} .$$

2(f) (3 points) Suppose another signal $f(t)$ has deterministic autocorrelation function

$$\bar{R}_{ff}(\tau) = \int_{-\infty}^{\infty} f(t)f(t-\tau) dt = \cos(10\tau) \frac{\sin(2\tau)}{\tau} .$$

The magnitude of the Fourier transform of the signal, i.e., $|F(j\omega)|$, is derived below, and given by the following — careful and fully labeled! — sketch:

Taking the Fourier transform of the given deterministic autocorrelation function, we find $\bar{S}_{ff}(j\omega) = 0.5\pi$ for $8 < |\omega| < 12$, and 0 elsewhere. Thus $|F(j\omega)| = \sqrt{0.5\pi}$ for $8 < |\omega| < 12$, and 0 elsewhere. (You can draw the figure!)

We know the deterministic correlation $\int_{-\infty}^{\infty} x(t)f(t-\tau) dt$ has Fourier transform $X(j\omega)F^*(j\omega)$. But since the $|X(j\omega)|$ is 0 where $|F(j\omega)|$ is nonzero, this product of transforms is 0, hence the deterministic correlation of these two signals is 0.

Problem 3 (10 points)

Note $x_c(t) = \sin(2\pi f_1 t)$, with $f_1 = 1300$ Hz.

3(a) (1 point) The minimum value that the sampling frequency $1/T$ has to exceed, in order to avoid aliasing at the C/D converter:

This is the Nyquist frequency, namely twice the highest frequency in the signal, hence 2600 Hz.

For each of the following parts, fully specify what the output $y_c(t)$ is for the indicated choice of the sampling/reconstruction frequency $1/T$. Show how you obtain your answers!

In the absence of aliasing, the filter behaves as a CT filter whose frequency response is obtained by the substitution $\Omega = \omega T$ in the DT filter characteristic, i.e., as an ideal lowpass filter with unit gain and cut-off determined as $f_c = \Omega_c/(2\pi T) = 1/(4T)$ Hz.

3(b) (3 points) When $1/T = 8000$ Hz, the output $y_c(t)$ is:

For this sampling frequency, there is no aliasing, and the effective filter cut-off frequency is 2000 Hz, which allows the input to go through intact. Hence $y_c(t) = x_c(t) = \sin(2600\pi t)$.

3(c) (3 points) When $1/T = 4000$ Hz, the output $y_c(t)$ is:

For this sampling frequency, there is no aliasing, and the effective filter cut-off frequency is 1000 Hz, which blocks the input. Hence $y_c(t) = 0$.

3(d) (3 points) When $1/T = 1600$ Hz, the output $y_c(t)$ is:

For this sampling frequency of 1600 Hz, aliasing does occur, so we need to work more carefully. The principal frequency band is $[-800 \text{ Hz}, 800 \text{ Hz}]$ and the filter cut-off frequency is 400 Hz. The 1300 Hz input component is outside these. However, the *lower* sideband of the alias centered at the sampling frequency of 1600 Hz appears at a frequency that is 1300 Hz lower than 1600 Hz, i.e., appears at *positive* frequency 300 Hz, therefore as an *upper* sideband in the principal frequency range, and within the passband of the filter. The output will therefore be $y_c(t) = -\sin(2\pi 300t) = -\sin(600\pi t)$.

Alternatively, note that for this input the DT signal $x_d[n]$ is given by

$$\sin(2\pi 1300nT) = \sin(2\pi 1300n/1600) = \sin((2\pi n) - (2\pi 300n/1600)) = -\sin(2\pi 300n/1600)$$

which is precisely what one would get from an input signal of $-\sin(2\pi 300t)$. The latter signal *does* lie in the principal frequency band and in the passband of the effective CT filter, thus yielding an output $y_c(t) = -\sin(2\pi 300t)$.

Problem 4 (20 points)

$$\dot{\mathbf{q}}(t) = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = [1 \quad 1] \mathbf{q}(t).$$

4(a) (3 points) The eigenvalues and associated eigenvectors are:

... determined by starting with the characteristic polynomial, which is

$$s(s - 1) - 6 = s^2 - s - 6 = (s + 2)(s - 3).$$

The roots of this are $\lambda_1 = -2$ and $\lambda_2 = 3$, which are the desired eigenvalues. The corresponding eigenvectors are then directly obtained from the defining equation,

$$(\lambda_i \mathbf{I} - \mathbf{A})\mathbf{v}_i = \mathbf{0} .$$

Solving these, we get

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} , \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} .$$

4(b) (1 point) Is the system asymptotically stable? Explain your answer.

Because one eigenvalue is positive, the system is *not* asymptotically stable.

4(c) (3 points) The transfer function $H(s)$ of the system is:

$$H(s) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -6 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s+1}{(s+2)(s-3)} .$$

What feature of this $H(s)$ tells you the system is reachable and observable?

The fact that there are two poles in $H(s)$, neither one canceled by the zero, shows that this 2nd-order system has no hidden modes, i.e., is reachable and observable.

4(d) (4 points) When the initial state of the system is at zero, i.e., when $\mathbf{q}(0) = \mathbf{0}$, and when the input for $t \geq 0$ is $x(t) = 5e^{-t}$, the output $y(t)$ of the system for $t \geq 0$ is

$$y(t) = e^{3t} - e^{-2t} .$$

(You can use this fact as a check on some of your preceding answers!) What initial condition $\mathbf{q}(0)$ would we need in order to have $y(t) \equiv 0$ for $t \geq 0$, with this same input $x(t) = 5e^{-t}$ for $t \geq 0$. Be sure to explain your reasoning!

We need $\mathbf{c}^T \mathbf{q}(t)$ for the ZIR to be the negative of the given output ZSR. So with ZIR

$$\mathbf{q}(t) = \alpha_1 \mathbf{v}_1 e^{\lambda_1 t} + \alpha_2 \mathbf{v}_2 e^{\lambda_2 t}$$

we need

$$\alpha_1 (\mathbf{c}^T \mathbf{v}_1) = 1 , \quad \text{and} \quad \alpha_2 (\mathbf{c}^T \mathbf{v}_2) = -1 ,$$

from which $\alpha_1 = -1$ and $\alpha_2 = -1/4$, so

$$\mathbf{q}(0) = -\mathbf{v}_1 - (1/4)\mathbf{v}_2 = \frac{5}{4} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

4(e) (3 points) Suppose we implement a state feedback control of the form

$$x(t) = \mathbf{g}^T \mathbf{q}(t) + p(t)$$

on the given system, where the feedback gain vector is $\mathbf{g}^T = [g_1 \ g_2]$, and $p(t)$ is some new external input. Write down the resulting state-space model, with input $p(t)$, state vector $\mathbf{q}(t)$, and output $y(t)$. You need to show the model in detail, making explicit the entries of all matrices and vectors involved in the state evolution equation and the output equation.

$$\begin{aligned} \dot{\mathbf{q}}(t) &= \begin{bmatrix} 0 & 1 \\ 6 + g_1 & 1 + g_2 \end{bmatrix} \mathbf{q}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} p(t) \\ y(t) &= [1 \ 1] \mathbf{q}(t). \end{aligned}$$

4(f) (4 points) For your closed-loop system in (e), determine what choice of g_1 and g_2 will result in a closed-loop characteristic polynomial of

$$\nu(s) = s^2 + 3s + 2.$$

The closed-loop characteristic polynomial is

$$\nu(s) = s(s - (1 + g_2)) - (6 + g_1) = s^2 - (1 + g_2)s - (6 + g_1).$$

Setting this equal to the desired polynomial specified above, we find $g_1 = -8$ and $g_2 = -4$.

Is the resulting system observable? Be sure to show your reasoning.

We can write $\nu(s) = (s+1)(s+2)$, so the denominator of the closed-loop transfer function has a factor due to the pole at -1 that cancels with the numerator factor due to the (unchanged) zero of the open-loop transfer function at -1 . Since state feedback cannot modify reachability, the cause of this hidden mode must be a loss of observability.

Alternatively, we could compute the closed-loop eigenvectors and then test observability in the usual way.

4(g) (2 points) With the state feedback gains picked as in (f), suppose $p(t)$ in the closed-loop system is kept constant at the value $\bar{p} = 6$ for all time. What is the corresponding equilibrium value $\bar{\mathbf{q}}$ of the state vector $\mathbf{q}(t)$?

Setting $\dot{\mathbf{q}}(t) = 0$ in the closed-loop system of 4(e), with the gains picked as in 4(f) and with $p(t) = \bar{p} = 6$, we find

$$\bar{\mathbf{q}} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

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