The Lorentz Oscillator and its Applications

described by I.F. Almog, M.S. Bradley, and V. Bulović

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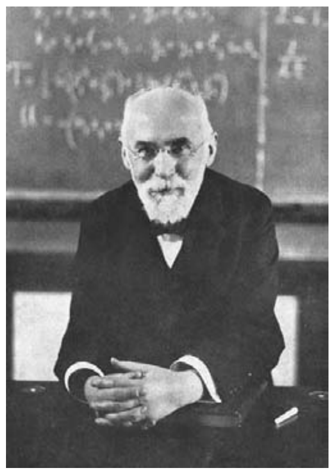
1 Introduction

In 1900, Max Planck presented his "purely formal assumption" that the energy of electromagnetic waves must be a multiple of some elementary unit and therefore could be described as consisting of small packets of energy. The term "quantum" comes from the Latin "quantus", meaning "how much", and was used by Planck in this context to represent "counting" of these elementary units. This idea was exploited by Albert Einstein, who in 1905 showed that EM waves could be equivalently treated as corpuscles - later named 'photons' - with discrete, "quantized" energy, which was dependent on the frequency of the wave.

Prior to the advent of quantum mechanics in the 1900s, the most well-known attempt by a classical physicist to describe the interaction of light with matter in terms of Maxwell's equations was carried out by a Hendrik Lorentz. Despite being a purely classical description, the Lorentz oscillator model was adapted to quantum mechanics in the 1900s and is still of considerable use today.



H. A. Lorentz

(1853-1928)

Hendrik Antoon Lorentz was a Dutch physicist in the late 19^{th} century, responsible for the derivation of the electromagnetic Lorentz force and the Lorentz transformations, later used by Einstein in the development of Special Relativity. Lorentz shared the 1902 Nobel Prize in Physics with Pieter Zeeman for the discovery and theoretical explanation of the Zeeman effect (splitting of spectral lines when a static magnetic field is applied).

In his attempt to describe the interaction between atoms and electric fields in classical terms, Lorentz proposed that the electron (a particle with some small mass) is bound to the nucleus of the atom (with a much larger mass) by a force that behaves according to Hooke's Law - that is, a spring-like force. An applied electric field would then interact with the charge of the electron, causing "stretching" or "compression" of the spring, which would set the electron into oscillating motion. This is the so-called **Lorentz oscillator model**.

2 Physical harmonic oscillators

In 8.01, 8.02 and 18.03, you analyzed multiple cases of harmonic oscillators. A second-order linear differential equation accurately describes the evolution (with respect to time) of the displacement of a mass attached to a spring, with/without a driving force and with/without a linear damping term (due to energy-dissipating forces). An analogous equation applies to resonant (LC or RLC) circuits. Let us first analyze the simplest case in order to obtain an expression for the resonant frequency.

2.1 Simple case and the resonant frequency

A simple spring oscillator is undriven and undamped. The forces governing the motion of the mass are only Newton's 2^{nd} Law and Hooke's Law (for springs that obey a linear relation). Putting both together, we obtain the equation for the mass' displacement with respect to time from the equilibrium position (assuming the system is initially perturbed out of equilibrium to initiate motion).

1. Hooke's Law: $F(y) = -ky$, where k is the spring constant and y is the displacement from the equilibrium position
2. Newton's 2^{nd} Law: $F(y) = m\frac{d^2y}{dt^2}$

Equating both:

$$m\frac{d^2y}{dt^2} = -ky \tag{2.1}$$

Or equivalently:

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y \tag{2.2}$$

The solution to this differential equation is a cosine (or a sine), of frequency $\sqrt{\frac{k}{m}}$.

That is called the **resonant frequency** - also called **natural frequency** or **fundamental frequency** - of an undamped spring-like oscillator. Let us call such frequency ω_0. Hence, $\omega_0^2 = \frac{k}{m}$, and we can solve for k, yielding $k = m\omega_0^2$.

The displacement is given by $y(t) = A\cos(\omega_0 t) + C$, where A and C are constants corresponding to the initial conditions of the system.

2.2 General case

Let us now include a driving force and a damping force. The simplest type of damping we can have is linearly proportional to the velocity of the mass, the force being of the form $F_{damping} = -m\gamma\frac{dy}{dt}$. The driving force can be of any kind, dependent or independent of time and displacement. Collecting all the terms, on the left-hand side we still have Newton's 2^{nd} Law, and on the right-hand side we have the summed contribution of all forces - using for the spring force the expression for k obtained above.

$$m\frac{d^2y}{dt^2} = F_{\text{driving}} + F_{\text{damping}} + F_{\text{spring}} = F_{\text{driving}} - m\gamma\frac{dy}{dt} - m\omega_0^2 y \tag{2.3}$$

Rearranging the equation:

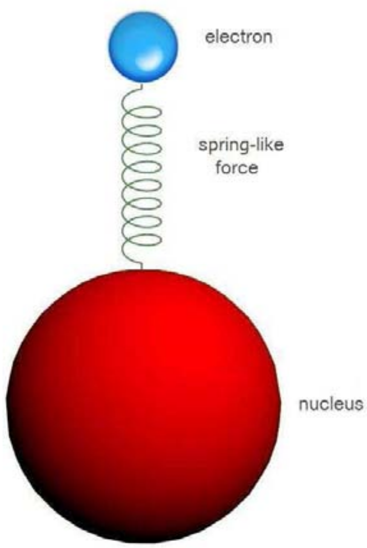
$$m\frac{d^2y}{dt^2} + m\gamma\frac{dy}{dt} + m\omega_0^2 y = F_{\text{driving}} \tag{2.4}$$

3 The Lorentz Oscillator Model

3.1 Lorentz oscillator equation

If we assume the nucleus of the atom is much more massive than the electron, we can treat the problem as if the electron-spring system is connected to an infinite mass, which does not move, allowing us to use the mass of the electron, $m = 9.11 \times 10^{-31} kg$. Depending on the case, this value may be substituted by the *reduced* or *effective electron mass* to account for deviations from this assumption.

Moreover, the assumption that the binding force behaves like a spring is a justified approximation for *any* kind of binding, given that the displacement is small enough (meaning that only the constant and linear terms in the Taylor expansion are relevant).



The damping term comes from internal collisions in the solid and radiation emitted by the electron (any accelerating charge emits radiation, as you will see later in this class). The relevance of the damping term will become clearer in section 4.5. It is also briefly mentioned in section 5.1.

All there is left for us to complete the Lorentz oscillator equation is to determine the driving force. In the case of a solid placed in an electric field varying in time with angular frequency ω but *independent* of the displacement in the y direction, we get:

$$\overrightarrow{E_y}(\omega, t) = \overrightarrow{E_{0y}} \cos \omega t = \Re e\{\overrightarrow{\widetilde{E_y}}(\omega, t)\} = \Re e\{\overrightarrow{E_{0y}} e^{j\omega t}\} \tag{3.1}$$

(with $\overrightarrow{E_{0y}}$ real and independent of time)

$$\overrightarrow{F_{\text{driving}}} = -q\overrightarrow{E_y} \tag{3.2}$$

This practice of writing the cosine as the real part of an exponential – by taking advantage of $e^{jx} = \cos(x) + j\sin(x)$, called **Euler's formula** – is quite common, and its practical advantage will become clear shortly. We've chosen here to make the driving field have zero phase arbitrarily–as you will see, if a phase is also present (meaning the field is not purely a cosine function but a mix of sine and cosine), then the time-independent constant (in the above equation, $\overrightarrow{E_{0y}}$) will be complex. A shorthand notation is to drop the $e^{j\omega t}$ term and the $\Re e\{\}$ throughout the derivations, and only "putting them back" for the final calculation. This shorthand approach is called the **phasor notation**.

We have learned two different, but equivalent, formulas for the polarization:

$$\text{I. } \overrightarrow{P} = N\delta\overrightarrow{x} = Nq(-\overrightarrow{y}) = -Nq\overrightarrow{y} \tag{3.3}$$

→The polarization vector is nothing but the density (per volume) of dipole moments, which in turn are defined simply as the product of the charge and the displacement vector *from the negative to the positive charge*, that is, *from the electron to the nucleus*. We are using q as the notation for the **elementary charge**, the absolute value of the charge of the electron ($q = 1.602 \times 10^{-19}C$).

Note the negative sign in the definition: our spring-mass system displacement vector was taken to point *to the end of the spring* (where the electron sits.) In order to keep the sign convention, our dipole displacement is the **opposite** of the spring-mass displacement.

$$\text{II. } \overrightarrow{P} = \epsilon_0\chi_e\overrightarrow{E} \tag{3.4}$$

→The polarization of a material is related to the applied electric field by this quantity we called the material's **electric susceptibility**: applying an electric field to a wide range of materials will cause the electrons in the material to be displaced, creating multiple positive/negative charge dipoles. The more electrically susceptible the material, the larger the displacement and/or the greater the number of dipoles created, as given by the first definition of the polarization.

Let us write the general harmonic oscillator equation driven by the an electric field in terms of the electric polarization vector $\overrightarrow{P}$.

For clarity, the arrows that indicate vector quantities were eliminated and we are working exclusively with magnitudes now - given the previously assumed orientations. However, this calculation could be performed in vector form.

From the first definition of the polarization above: $y(t) = \frac{-P(t)}{Nq}$

$$m\frac{d^2y}{dt^2} + m\gamma\frac{dy}{dt} + m\omega_0^2 y = F_{\text{driving}}$$

$$-\frac{m}{Nq}\frac{d^2P}{dt^2} - \frac{m\gamma}{Nq}\frac{dP}{dt} - \frac{m\omega_0^2}{Nq}P = F_{\text{driving}}$$

$$\frac{d^2P}{dt^2} + \gamma\frac{dP}{dt} + \omega_0^2 P = -\frac{Nq}{m}F_{\text{driving}}$$

Substituting our expression for $F_{driving}$:

$$\frac{d^2P}{dt^2} + \gamma\frac{dP}{dt} + \omega_0^2 P = \frac{Nq^2}{m}E_y \tag{3.5}$$

Let us digress for a bit to introduce a new term.

3.2 The plasma frequency

Simply stated, a plasma is an ionized, electrically conducting gas of charged particles, usually occurring under conditions of very high temperature and/or very low particle density.

Plasmas exhibit many cool effects, as you probably have seen, in *aurora* (polar light) or in a plasma ball.

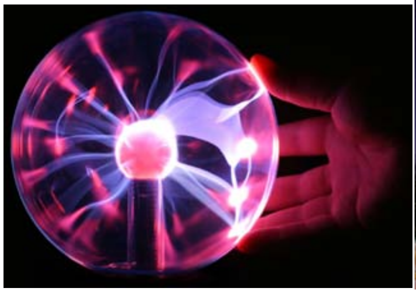


Image by Jean-Jacques Milan, from Wikipedia.

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Many of these effects take place *collectively.* One of the most fundamental collective effects of a plasma is the *plasma oscillation.*

In equilibrium, the electric fields of the electrons and the ionized nuclei cancel each other out, but this equilibrium is hardly maintained.

Instead of dealing with the individual (chaotic!) motion of electrons and nuclei, consider the center of mass of the nuclei and the center of mass of the electrons. In equilibrium, they coincide. However, when they shift with respect to each other, a Coulomb force arises trying to restore their position, initiating an oscillatory behavior (think of a blob of fluid floating at zero gravity). The frequency at which these oscillations resonate is called the **plasma frequency**.

The magnitude of this frequency has highly significant implications with respect to the propagation of electromagnetic waves through the plasma.

Plasma exists naturally in what we call the **ionosphere** (80 km $\sim$ 120 km above the surface of the Earth). There, UV light from the Sun ionizes air molecules.

There are several ways to determine, or estimate, the plasma frequency, and those are beyond the scope of this short introduction. In 6.007, we will use the simplest and most convenient of them, namely:

$$\omega_p^2 = \frac{Nq^2}{m\epsilon_0} \tag{3.6}$$

or equivalently

$$\omega_p = \sqrt{\frac{Nq^2}{m\epsilon_0}} \tag{3.7}$$

Note that the plasma frequency is proportional to the electron density, and can be calculated for any material for which such density is known. The plasma frequency for "non-plasma" materials stands for the natural *collective* oscillation frequency of the "sea" of electrons in the material, not of individual dipoles.

As before, q stands for the absolute value of the charge of the electron, and m stands for the standard mass of the electron, or sometimes the effective mass.

In the ionosphere, $N = 10^{12}\,\frac{\text{electrons}}{m^3}$, so $\omega_p = 5.64 \times 10^7\,\frac{\text{rad}}{\text{s}}$ and $f_p = \frac{\omega_p}{2\pi} = 9$Mhz.

As Richard Feynman stated in his "Lectures of Physics, Vol. II":

"This natural resonance of a plasma has some interesting effects. For example, if one tries to propagate a radio wave through the ionosphere, one finds that it can penetrate only if its frequency is higher than the plasma frequency. Otherwise the signal is reflected back. We must use high frequencies if we wish to communicate with a satellite in space. On the other hand, if we wish to communicate with a radio station beyond the horizon, we must use frequencies lower than the plasma frequency, so that the signal will be reflected back to the earth."

3.3 Obtaining permittivity

We are now able to complete the model. Recall our differential equation for the polarization:

$$\frac{d^2P(\omega,t)}{dt^2} + \gamma\frac{dP(\omega,t)}{dt} + \omega_0^2 P(\omega,t) = \frac{Nq^2}{m}E_y(\omega,t) \tag{3.8}$$

Where ω is the frequency with witch the electric field varies. We can rewrite the equation substituting the plasma frequency in the following way:

$$\frac{d^2P(\omega,t)}{dt^2} + \gamma\frac{dP(\omega,t)}{dt} + \omega_0^2 P(\omega,t) = \epsilon_0\omega_p^2 E_y(\omega,t) \tag{3.9}$$

Since we chose to work with a sinusoidally-varying electric field, let us make the educated assumption that the polarization which solves that equation also varies sinusoidally and is of the form:

$$P(\omega,t) = P(\omega)\cos(\omega t) = \Re e\{\widetilde{P}(\omega)e^{j\omega t}\} \tag{3.10}$$

With $\widetilde{P}(\omega)$ being independent of time and allowed to be a complex quantity (that is what the $\sim$ stands for) in order to account for any *phase-lag* between the driving electric field and the polarization.

Another way of writing this would be $\widetilde{P}(\omega) = P(\omega)e^{j\phi_P(\omega)}$, where $\phi_P(\omega)$ is the phase difference between the electric field and the polarization vector. This phase difference could also be dependent on the angular frequency of the electric field.

Including the electric field expression from section 2.3.1 and the above expression for the polarization in our Lorentz oscillator equation ($\Re e\{\}$'s were eliminated as the variable is now complex):

$$\tfrac{d^2[\widetilde{P}(\omega)e^{j\omega t}]}{dt^2} + \gamma\tfrac{d[\widetilde{P}(\omega)e^{j\omega t}]}{dt} + \omega_0^2\widetilde{P}(\omega)e^{j\omega t} = \epsilon_0\omega_p^2 E_{0y}e^{j\omega t}$$

Taking derivatives of exponential terms (since $\widetilde{P}(\omega)$ is independent of time):

$$(j\omega)^2\widetilde{P}(\omega)e^{j\omega t} + j\gamma\omega\widetilde{P}(\omega)e^{j\omega t} + \omega_0^2\widetilde{P}(\omega)e^{j\omega t} = \epsilon_0\omega_p^2 E_{0y}e^{j\omega t}$$

The exponentials cancel, and by solving for the complex polarization we obtain:

$$\widetilde{P}(\omega) = \frac{\epsilon_0\omega_p^2}{(\omega_0^2-\omega^2)+j\gamma\omega}E_{0y} \tag{3.11}$$

Bear in mind that through all these derivations, the polarization and electric field are vector quantities: arrows are just being omitted to prevent the derivation to become clumsier than necessary.

Recall now the second definition of polarization: $\overrightarrow{P} = \epsilon_0 \chi_e \overrightarrow{E}$

You have been introduced to the concept of **relative permittivity**, also called the **dielectric constant** $\frac{\epsilon}{\epsilon_0} = 1 + \chi_e$. We can rewrite the above definition of polarization as:

$$\overrightarrow{P} = \epsilon_0 (\frac{\epsilon}{\epsilon_0} - 1) \overrightarrow{E} \tag{3.12}$$

We will make a slight modification to accommodate our complex result. Instead of a real ϵ, let there be a complex $\widetilde{\epsilon}$ such that:

$$\overrightarrow{\widetilde{P}}(\omega) = \epsilon_0 (\frac{\widetilde{\epsilon}}{\epsilon_0} - 1) \overrightarrow{E_{0y}} \tag{3.13}$$

(*Note*: this requires that χ_e and $P(\omega)$ be also complex! Being such, we can call them $\widetilde{\chi}_e$ and $\widetilde{P}(\omega)$)

Combining this with our previous result:

$$\frac{\widetilde{\epsilon}}{\epsilon_0} = 1 + \frac{\omega_p^2}{(\omega_0^2 - \omega^2) + j\omega\gamma} \tag{3.14}$$

The above equation shows that *permittivity depends on the frequency of the electric field*, besides the plasma frequency and damping (which are properties of the medium). A medium displaying such behavior (that is, whose permittivity depends on the frequency of the wave) is called **dispersive**, named after "dispersion," which is the phenomenon exhibited in a prism or raindrop that causes white light to be spread out into a rainbow of colors (white light is a mixture of beams of many different colors–all traveling at the same speed, but having different frequencies and wavelengths).

Image in the public domain, by NASA.

3.4 Quick review on notation

Just to get things straight:

- ϵ is the electric permittivity of the medium
- ϵ_0 is the electric permittivity of vacuum
- $\frac{\epsilon}{\epsilon_0}$ is the *relative* electric permittivity of the medium
- μ is the magnetic permeability of the medium
- μ_0 is the magnetic permeability of vacuum
- μ_r is the *relative* magnetic permeability of the medium, and it is equal to $\frac{\mu}{\mu_0}$

- χ_m is the magnetic susceptibility of the medium, and $\mu_r = 1 + \chi_m$
- χ_e is the electric susceptibility of the medium, and $\frac{\epsilon}{\epsilon_0} = 1 + \chi_e$
- $\overrightarrow{E_y}(\omega, t)$ is the time-varying electric field driving the oscillator
- $\overrightarrow{E_{0y}}$ is the time-independent part of the electric field
- ω is the angular frequency of the time-varying electric field driving the oscillator
- ω_o is the *resonance (angular) frequency* of the oscillator
- ω_p is the *plasma frequency* of the material

3.5 Frequency analysis

Since our electric permittivity is a complex quantity, we can break it down into a real and an imaginary part:

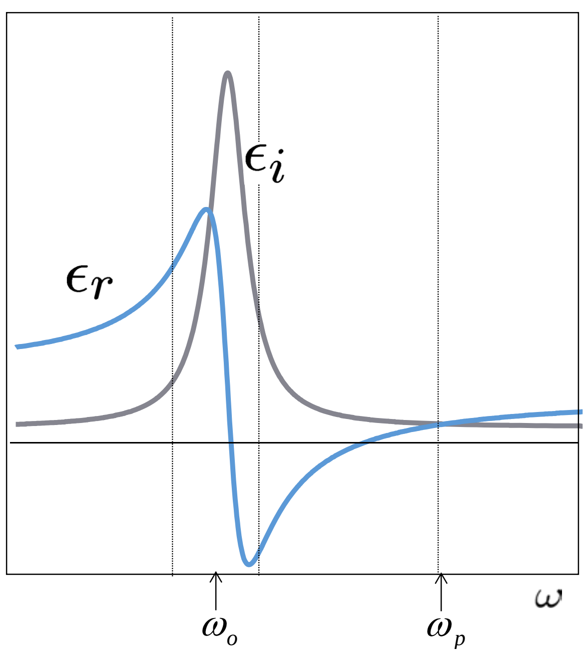
$$\frac{\widetilde{\epsilon}}{\epsilon_0} = \epsilon_r - j\epsilon_i \tag{3.15}$$

From this definition, and remembering these quantities are a function of the frequency of the driving electric field, we can obtain the magnitude and phase of the polarization (and hence the amplitude and phase of the displacement of the oscillator) with respect to the electric field. We just need some algebraic manipulation to remove the imaginary number from the denominator in equation 3.14:

$$\epsilon_r(\omega) - 1 = \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \tag{3.16}$$

$$\epsilon_i(\omega) = \frac{\omega_p^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} \tag{3.17}$$

The form of equation for ϵ_i is often referred to as a "Lorentzian."



Note that since the *polarization vector* and the *electric field* are related by the *electric susceptibility* we should analyze the magnitude and phase of $\widetilde{\chi}_e = (\frac{\widetilde{\epsilon}}{\epsilon_0} - 1) = (\epsilon_r - 1) - j\epsilon_i$, instead of $\frac{\widetilde{\epsilon}}{\epsilon_0}$ (the difference being just the -1 term).

$$\overrightarrow{\widetilde{P}}(\omega) = \epsilon_0 \widetilde{\chi}_e \overrightarrow{E_{0y}}$$

$$\widetilde{\chi}_e(\omega) = |\chi_e(\omega)| e^{j\phi_{\chi_e}(\omega)}$$

The magnitude is given by:

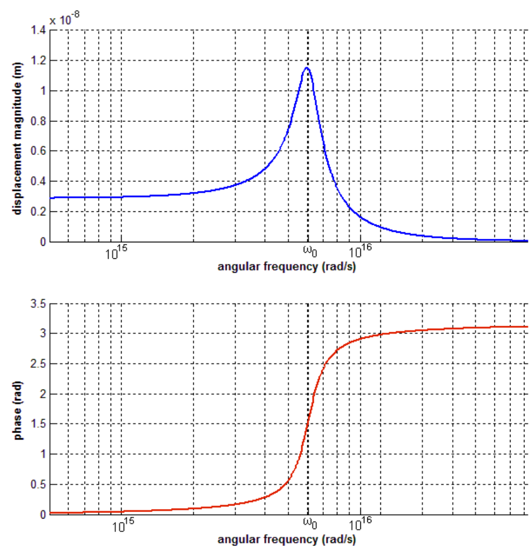
$$|\chi_e(\omega)| = \sqrt{(\epsilon_r - 1)^2 + \epsilon_i^2} \tag{3.18}$$

With the assumed notation of $\overrightarrow{E_{0y}}$ being a real entity (as opposed to complex), the phase difference between the polarization vector and the electric field will be equal to the phase of the the complex susceptibility, which is given by:

$$\phi_P = \phi_{\chi_e}(\omega) = \arctan(\frac{\epsilon_i}{\epsilon_r - 1}) = \arctan(\frac{\omega\gamma}{\omega_0^2 - \omega^2}) \tag{3.19}$$

Remember that $\overrightarrow{P} = -Nq\overrightarrow{y}$. Therefore, the magnitude of the displacement will be given by $|\overrightarrow{y}| = \frac{|\overrightarrow{P}|}{qN}$, and the negative sign contributes with an inversion of phase ($\phi_P = -\phi_y$).

To illustrate, the magnitude of the displacement and its phase (relative to the phase of the electric field) were plotted for a made-up material with $N = 10^{28} cm^{-3}$, $\omega_p = 1.3 \times 10^{16} \frac{\text{rad}}{s}$, $\omega_0 = 6.077 \times 10^{15} \frac{\text{rad}}{s}$ and $\gamma = 1.519 \times 10^{15} \frac{\text{rad}}{s}$, as follows.



3.5.1 Low frequencies

For $\omega \approx 0$:

$\epsilon_i(0) \approx 0$

$|\chi_e(0)| \approx |\epsilon_r - 1| \approx \frac{\omega_p^2}{\omega_0^2}$

$\phi_{\chi_e}(0) \approx \arctan(0) = 0$

Therefore, the amplitude of the displacement is "medium" (compared to the peak and to zero) and the displacement is in phase with the varying electric field.

3.5.2 Near the resonant frequency

For $\omega \approx \omega_0$:

$\epsilon_r(\omega_0) - 1 \approx 0$

$|\chi_e(\omega_0)| \approx |\epsilon_i| \approx \frac{\omega_p^2}{\gamma\omega_0}$

$\phi_{\chi_e}(\omega_0) \approx \arctan(\frac{\gamma\omega_o}{0}) \approx \arctan(\infty) = \frac{\pi}{2}$

Here, the amplitude is much larger than the low frequency scenario (since $\gamma \ll \omega_0$: if this was not the case, damping would dissipate the energy too fast and prevent motion from occurring). The displacement is 90^o out of phase with the electric field.

3.5.3 High frequencies

For $\omega \to \infty$:

Since ω shows up to the fourth power in the denominator of both $\epsilon_r$and ϵ_i and not even his cube appears on the numerators, they vanish.

$|\chi_e(\infty)| = 0$

$\phi_{\chi_e}(\infty) \approx \arctan(0) = 0$ or π

To figure out the phase, we go back to the result derived at the end of section 3.3.

The term that dominates the denominator is $(-\omega^2)$, so the whole thing becomes a tiny, but *negative*, number. A negative real number has phase equal to 180^o.

Hence, the displacement is minimal and it is 180^o out of phase with the electric field.

As an example of each frequency regime described, an analogy can be made with pushing someone on a swing at low frequencies, near the resonance frequency, and at high frequencies, as shown in the cartoon.

4 Electromagnetic Waves

4.1 The 1-D Wave Equation and plane waves

A physical wave can be described by a partial differential equation called the **wave equation**. Let us consider a general uni-dimensional wave, whose shape is given by a function $F(z,t)$, propagating along the z-direction (either towards its positive or negative orientation) with **phase velocity** v_p (which is the **propagation velocity** for waves of a single frequency. The phase velocity is distinct from the **group velocity** of a collection of waves that make up a pulse or other type of **wavepacket**.

The general wave equation is:

$$\frac{\partial^2 F}{\partial z^2} = \left(\frac{1}{v_p^2}\right)\frac{\partial^2 \mathrm{F}}{\partial t^2} \tag{4.1}$$

Note that this same 1-D equation for a uni-dimensional wave also describes a **3-D uniform plane wave**, only requiring us to request that the value of F at some z_0 and t_0 - call it $F(z_0, t_0)$ - holds for *any* x or y, that is, it has the same value F on all of the xy planar slice passing through z_0, perpendicular to the direction of propagation, which contains z_0.

As shown by mathematician Jean d'Alembert, the general solution to this equation is of the form:

$$F(z,t) = f_+(z - v_p t) + f_-(z + v_p t) \tag{4.2}$$

(where $f_+$and $f_-$are uniform waves moving in the +z and -z directions, respectively)

*(**Hint**: try applying the partial derivative operators $\frac{\partial^2}{\partial t^2}$ and $\frac{\partial^2}{\partial z^2}$ to **any** $F(z,t)$, $f_+(z - v_p t)$ or $f_-(z + v_p t)$ in order to verify that they all satisfy the 1-D wave equation)*

It should be noted that the argument of f_+ or f_- can be multiplied (or divided) by a non-zero constant, and those functions would still be solutions to the wave equation. That is, if instead of $f(z - v_p t)$ we have $f(az - av_p t)$, we still get a solution! A particularly useful case happens when $a = \frac{1}{v_p}$, yielding $f_+(\frac{z}{v_p} - t)$ and $f_-(\frac{z}{v_p} + t)$. This case will be further explored in section 4.4.

In the particular case of electromagnetic waves, the phase velocity is dependent on the *magnetic permeability* as well as on the *electric permittivity* of the medium of propagation, according to the relation:

$$v_p = \frac{1}{\sqrt{\mu\epsilon}} \tag{4.3}$$

This comes from the 1-D wave equation for electromagnetic waves, which was derived in lecture through the recursive application of Maxwell's equations, representing the coupled behavior of varying electric fields that generate magnetic fields and vice versa. The resulting wave equation for the electric field is:

$$\frac{\partial^2 E}{\partial z^2} = \mu\epsilon \frac{\partial^2 E}{\partial t^2} \tag{4.4}$$

And its magnetic counterpart is:

$$\frac{\partial^2 H}{\partial z^2} = \mu\epsilon \frac{\partial^2 H}{\partial t^2} \tag{4.5}$$

4.2 Index of refraction

We will return to the wave equation shortly, but a digression will serve us well at this point. You are probably familiar with the **refractive index n** of a medium. **Refraction** refers to the bending of rays of light when they pass from one medium to another, and the refractive index determines how sharp the bending will be. It is defined as the ratio between the propagation speed of light in vacuum and the propagation speed of light in the medium in question:

$$n \equiv \frac{c}{v_p} \tag{4.6}$$

A medium in which light travels very slowly will have a very large index of refraction.

We can substitute the equation in the section above to obtain:

$$n = \frac{c}{\frac{1}{\sqrt{\mu\epsilon}}} = c\sqrt{\mu\epsilon} \tag{4.7}$$

The speed of light in vacuum, c, is also related to the properties of the medium, and the properties of this medium (vacuum) are well known:

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{4.8}$$

Putting this back into the equation for the index of refraction:

$$n = \frac{\sqrt{\mu\epsilon}}{\sqrt{\mu_0\epsilon_0}} = \sqrt{(\frac{\mu}{\mu_0})(\frac{\epsilon}{\epsilon_0})} \tag{4.9}$$

That is: the index of refraction of a medium is the square root of the product of that medium's relative magnetic permeability and relative electric permittivity.

Assuming that for the purpose of optics and wave propagation we will stay away from magnetic materials (ones with high magnetic susceptibility, and hence high relative magnetic permeability), in most cases $\mu \approx \mu_0$, and applying this approximation to our equation, we get simply:

$$n = \sqrt{\frac{\epsilon}{\epsilon_0}} \tag{4.10}$$

In a non-magnetic medium, the index of refraction is just the square root of the relative electric permittivity!

As mentioned previously at the end of section 3.3, as the permittivity depends on the frequency of the driving electric field, so will the refractive index.

4.3 Complex refractive index

To describe materials that absorb light, we must introduce a complex index of refraction, $\tilde{n}$. The real index of refraction will be the real part of this complex quantity:

$$\widetilde{n} = n_r - jn_i \tag{4.11}$$

$$n_r = \Re e\{\widetilde{n}\} = n \tag{4.12}$$

For reasons that will become clear (see section 4.5), let us make a small modification, renaming the imaginary part as:

$$n_i \equiv \kappa \tag{4.13}$$

κ is called the **extinction coefficient** of the medium for a particular wavelength of light.

Taking the square of the complex refractive index (to eliminate the square root on the right-hand side):

$$(\widetilde{n})^2 = (n - j\kappa)^2 = n^2 + -2j\kappa n + (j\kappa)^2 = (n^2 - \kappa^2) - j2\kappa n \tag{4.14}$$

This whole thing must be equal to our relative complex permittivity.

Bringing back our definition from section 3.5:

$$\frac{\widetilde{\epsilon}}{\epsilon_0} = \epsilon_r - j\epsilon_i \tag{4.15}$$

Then, equating the real and imaginary parts, we get:

$$\epsilon_r = n^2 - \kappa^2 \tag{4.16}$$

$$\epsilon_i = 2\kappa n = 2n_r n_i \tag{4.17}$$

From which we can also derive (assuming non-magnetic materials):

$$n = \frac{1}{\sqrt{2}}\sqrt{\epsilon_r + \sqrt{\epsilon_r^2 + \epsilon_i^2}} \tag{4.18}$$

$$\kappa = \frac{1}{\sqrt{2}}\sqrt{-\epsilon_r + \sqrt{\epsilon_r^2 + \epsilon_i^2}} \tag{4.19}$$

4.4 Solving the Wave Equation

Now we will combine the concepts of complex permittivity, the Lorentz oscillator model, and the wave equation to describe how electromagnetic fields propagate inside materials.

As shown in section 4.1, there are *infinitely* many functions that solve the wave equation, as long as the arguments obey a certain form. Since we cannot solve all of them in a simple and straightforward manner, we will focus on sinusoidal solutions.

As you know, the electromagnetic wave *always* has both an electric and a magnetic field component. However, let us for a second consider only its electric part.

Assume, without loss of generality, that this electric field is traveling along the $+z$ direction, varying along the y axis.

That is, we have $E_y(z, t)$, which must also be a solution to the wave equation (think linear **superposition**).

The simplest guess for the form of such a wave would be:

$$E_y(z, t) = A\cos(t - \tfrac{z}{v_p})$$

(Notice that we did not assume the phase velocity to be c, since in general the wave could be propagating in a material other than air, which we generally regard as vacuum.)

However, there is a problem with the suggested form: physical units.

Dimensional analysis of the argument prevents this from being valid, since the argument to a cosine must have units of radians or degrees, but this argument has units of time!

How do we fix it? We use the fact that the argument of a solution to the wave equation can be multiplied by a constant so as to have the units combine properly and still be a solution.

What entity would have the unit of $\frac{\text{rad}}{\text{s}}$? Angular frequency!

Multiplying the argument by the angular frequency of the wave, a valid form for our solution would now be:

$$E_y(z, t) = A\cos(\omega t - \frac{\omega}{v_p} z) \tag{4.20}$$

The next equation, called the **dispersion relation**, introduces another term: the **angular wave-number**. The angular wave-number is also known as the **wave-vector** when the wave-numbers from multiple directions are combined in vector form. The wave vector is a vector whose direction is the same as the propagation direction of the wave, with magnitude equal to the total angular wave-number.

$$k = \frac{\omega}{v_p} \tag{4.21}$$

$$k = \omega\sqrt{\mu\epsilon} \tag{4.22}$$

Its units are $\frac{radian}{m}$(if the velocity is given in $\frac{m}{s}$). In vacuum, when the velocity is c, it gets special treatment, and it is referred to as β instead of k.

$$\beta = \frac{\omega}{c} \tag{4.23}$$

Since the speed of light in a medium is related to the speed in vacuum by the index of the refraction, we can derive the following relation:

$$n = \tfrac{c}{v_p}$$

$$v_p = \frac{c}{n} \tag{4.24}$$

$$k = \frac{\omega}{v_p} = \frac{\omega}{\frac{c}{n}} = \frac{\omega n}{c} = n\beta \tag{4.25}$$

Moreover, we can derive another expression using the traveling wave relationship:

$$v_p = \lambda f \tag{4.26}$$

$$k = \frac{\omega}{v_p} = \frac{2\pi f}{\lambda f} = \frac{2\pi}{\lambda} \tag{4.27}$$

Writing the general solution for our electric wave, using the wave-number:

$$E_y(z,t) = A_1 \cos(\omega t - kz) + A_2 cos(\omega t + kz) \tag{4.28}$$

Or in vacuum (where $k = n\beta$ still holds, but $n = 1$):

$$E_y(z,t) = A_1 \cos(\omega t - \beta z) + A_2 cos(\omega t + \beta z) \tag{4.29}$$

We can also use our now familiar phasor notation:

$$E_y(z,t) = \Re e\{A_1 e^{j(\omega t - kz)}\} + \Re e\{A_2 e^{j(\omega t + kz)}\} \tag{4.30}$$

Note that, from the beginning, we have been assuming that the electric field has only a single angular frequency. These EM waves are therefore called **monochromatic**. In the case of a **polychromatic** wave (i.e., a pulse or wavepacket), we could split the wavepacket into multiple monochromatic components and then carry out the analysis for each of the individual frequencies (thanks to linear superposition and **Fourier analysis**!

4.5 Beer's law and the absorption coefficient

So far, these wave solutions deal only with loss-less materials (that is, (ideal) materials through which an electromagnetic wave passes without dissipation of its energy). This is often not the case, and thus our model should account for lossy media as well. As you recall, our Lorentz oscillator model already accounted for losses via the damping factor γ, which was responsible for the electric permittivity becoming a complex quantity in section 3.3.

Before we expand on the concepts derived from the Lorentz oscillator model, let us conduct an intuitive analysis of a dissipating medium.

Absorption of light in a medium is usually defined as the fraction of the power dissipated per unit length of the medium. If a beam is propagating in the $+z$ direction and the intensity (optical power per unit area) at position z is $I(z)$, then the decrease of the intensity across an incremental slice of thickness dz is given by:

$$dI = (-\alpha)dz \times I(z) \tag{4.31}$$

By integrating this, we obtain **Beer's law**, also known as the **Beer-Lambert law**, namely:

$$I(z) = I(0)e^{-\alpha z} \tag{4.32}$$

The coefficient α is called the **absorption coefficient** or **absorption constant**, which we will now analyze in more detail. It is dependent on the wavelength of light, causing some wavelengths to be absorbed in a medium, while the same medium is transparent to other wavelengths. The absorption coefficient has units of m^{-1} or cm^{-1}.

Let us revisit our results from section 4.3. We introduced a complex refractive index of the form:

$$\widetilde{n} = n - j\kappa \tag{4.33}$$

We can generalize the wave-vector in the previous section, making it complex, and substituting that in our plane wave solutions:

$$\widetilde{k} = \widetilde{n}\beta = \widetilde{n}\frac{\omega}{c} \tag{4.34}$$

$$\widetilde{k} = (n - j\kappa)\frac{\omega}{c} = \frac{n\omega}{c} - j\frac{\kappa\omega}{c} = k - j\kappa\beta \tag{4.35}$$

$$E_y(z,t) = \Re e\{A_1 e^{j(\omega t - \widetilde{k}z)}\} + \Re e\{A_2 e^{j(\omega t + \widetilde{k}z)}\} \tag{4.36}$$

$$E_y(z,t) = \Re e\{A_1 e^{j[\omega t - (k - j\kappa\beta)z]}\} + \Re e\{A_2 e^{j[\omega t + (k - j\kappa\beta)z]}\}$$

$$E_y(z,t) = \Re e\{A_1 e^{j(\omega t - kz + j\kappa\beta z)}\} + \Re e\{A_2 e^{j(\omega t + kz - j\kappa\beta z)}\}$$

$$\text{(noting that } j^2 = -1\text{)}$$

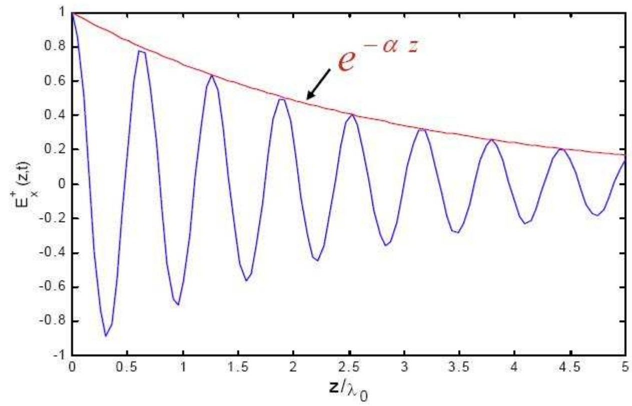
$$E_y(z,t) = \Re e\{A_1 e^{[j(\omega t - kz) - \kappa\beta z]}\} + \Re e\{A_2 e^{[j(\omega t + k) + \kappa\beta z]}\}$$

$$E_y(z,t) = \Re e\{A_1 e^{-\kappa\beta z} e^{j(\omega t - kz)}\} + \Re e\{A_2 e^{\kappa\beta z} e^{j(\omega t + kz)}\} \tag{4.37}$$

Observing that the exponential factor containing κ is purely real, let us evaluate the real part of the expression above:

$$E_y(z,t) = A_1 e^{-\kappa\beta z}\cos(\omega t - kz) + A_2 e^{\kappa\beta z}\cos(\omega t + kz) \tag{4.38}$$

If the wave is traveling in the $+z$ direction, corresponding to the first term, z is increasing, and the exponential factor's exponent is decreasing, "becoming more negative". If it is traveling in the $-z$ direction, z is decreasing, and the exponent of the second term is also becoming more negative. In either case, the magnitude of the exponential is decreasing, and therefore the electric field is being *attenuated*.



Intensity is proportional to the square of the magnitude of the electric field. Taking the first term only, for the sake of simplicity, we can say that if the electric field is decreasing with $e^{-\kappa\beta z}$, then the intensity of this wave is decreasing proportionally to $(e^{-\kappa\beta z})^2 = e^{-2\kappa\beta z}$. Comparing this to Beer's law, we obtain the relationship between the absorption coefficient α and the extinction coefficient κ:

$$\alpha = 2\kappa\beta = \frac{2\kappa\omega}{c} = \frac{4\pi\kappa}{\lambda} \tag{4.39}$$

A medium in which α is negative is called a **gain medium**, as the electric field increases by acquiring energy previously stored in the medium. Lasers are produced by stimulated emission in a gain medium. When $\alpha(\omega)$ is very small (close to 0), the medium is **transparent** for that frequency.

4.6 Another quick review of terms

- λ is the wavelength of the electromagnetic wave
- τ is the period of the wave, the inverse of its frequency f
- ω is the angular frequency of the wave, and it is equal to $2\pi f = \frac{2\pi}{\tau}$
- β is the wave-vector in vacuum, with magnitude $\frac{\omega}{c}$
- $\widetilde{k}$ is the wave-vector in materials, equal to $\widetilde{n}\beta$
- k is the real part of $\widetilde{k}$, and it is equal to $\frac{\omega}{v_p} = \frac{2\pi}{\lambda} = n\beta$
- α is the absorption coefficient of the medium for some wavelength
- κ is the extinction coefficient of the medium for some wavelength, and it is the imaginary part of $\widetilde{n}$

5 Optical properties of materials

5.1 Other EM-Wave Phenomena

Up to now, we have specifically investigated refraction and absorption. These two processes are a subset of phenomena that EM waves exhibit. In the most general division, there are three phenomena: **propagation**, **reflection** and **transmission**.

Propagation is simply the movement of a wave through a medium. When a wave hits a boundary, that is, an interface between two media of different refractive indices, the wave can either bounce back (reflection) or pass through the boundary (transmission), either completely or partially.

Refraction happens to a *transmitted* wave, its change in direction (bending) being due to a change in phase velocity.

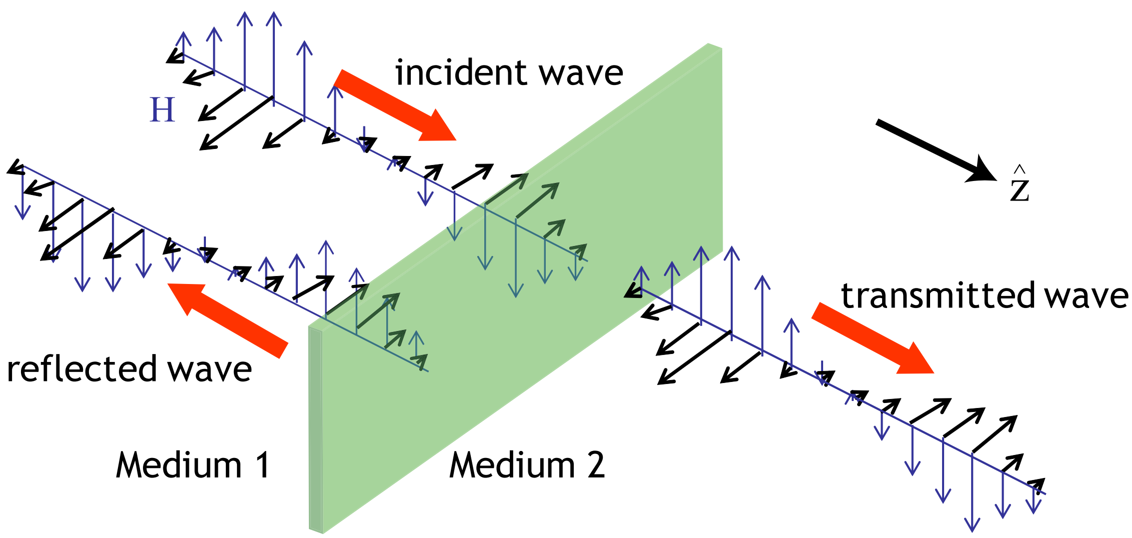
When the wave is propagating through a medium, **absorption** occurs if the frequency of the wave's photons (particles of electromagnetic radiation) is near the resonant frequency of some type of excitation in the material. Once a photon has been absorbed, the energy can be dissipated through heating the medium or reemitted in either **spontaneous emission** (e.g., what occurs in a light-emitting diode, LED) or **stimulated emission** (e.g., what occurs in a LASER).

Scattering is the name given to the phenomenon in which the medium causes part of the propagating radiation to change direction. In general, scattering is one of the factors responsible for the damping represented in the Lorentz oscillator model, but for simplicity we assume here that all attenuation is due to absorption.

A number of other phenomena can also occur to a propagated wave, but these generally belong to the field of **nonlinear optics**.

5.2 Optical coefficients

5.2.1 Reflection and transmission coefficients in loss-less media



In the picture above, assume that both media are dielectrics (i.e., poor conductors–the importance of this assumption will be seen when we discuss the Drude model).

We have n_1 and n_2 as the indices of refraction of mediums 1 and 2, respectively. The corresponding wave-vectors are $k_1 = n_1\beta$ and $k_2 = n_2\beta$. As you recall, $\beta = \frac{\omega}{c}$. Therefore, in order for us to accept that the waves have identical β in both mediums 1 and 2, their angular frequencies must be identical. The frequency of an electromagnetic wave *does not change* when passing from a medium to another. The wavelength changes, due to the different phase velocity (think $v_p = \lambda f$), but the frequency is kept unaltered unless in the case of scattering or other nonlinear phenomena that require quantum physics for an explanation. In short: the frequency defines the photons carrying that wave, their energy being given by the equation that Planck experimentally derived, $\varepsilon = hf$, where h is **Planck's constant**. Therefore, for the frequency to change, the photons would have to become different photons! Photons are self-contained and independent of other photons, as you will see later in the course, so it is not trivial for a change in frequency to occur.

We will treat only the case of **normal incidence** (perpendicular to the boundary plane, as opposed to **oblique incidence**, when waves hit the plane at an angle). The general scenario will be analyzed in class through the derivation of **Fresnel's equations**.

Assume that the electric field is confined to the xz-plane, meaning the electric field vector always has its direction only along the x-axis, as indicated by the subscript (which implies that this wave must be **linearly polarized** along the x direction). Let us call $E_x^i(z,t)$ the magnitude of the incident electric field, $E_x^r(z,t)$ for the reflected electric field and $E_x^t(z,t)$ for the transmitted electric field. We can write equations for these waves:

$\overrightarrow{E_x^i}(z,t) = \hat{x}E_0^i e^{j(\omega t - k_1 z)}$: traveling in the $+z$ direction, within medium 1
$\overrightarrow{E_x^r}(z,t) = \hat{x}E_0^r e^{j(\omega t + k_1 z)}$: traveling in the $-z$ direction, within medium 1
$\overrightarrow{E_x^t}(z,t) = \hat{x}E_0^t e^{j(\omega t - k_2 z)}$: traveling in the $+z$ direction, within medium 2

Dropping the $e^{j\omega t}$terms for our phasor notation (the t in the left-hand side of the equation was kept to indicate that we are using phasor notation):

$$\overrightarrow{E_x^i}(z,t) = \hat{x}E_0^i e^{-jk_1 z} \tag{5.1}$$

$$\overrightarrow{E_x^r}(z,t) = \hat{x}E_0^r e^{+jk_1 z} \tag{5.2}$$

$$\overrightarrow{E_x^t}(z,t) = \hat{x}E_0^t e^{-jk_2 z} \tag{5.3}$$

Note that these are vectors, all along the direction of the x-axis. We now impose a boundary condition of **continuity**, derived from Maxwell's equations applied to the interface considering the parallel (tangential) and perpendicular (orthogonal) portions of the electric and magnetic fields: at $z = 0$, where the three of them (incident, reflected and transmitted fields) meet, the *tangential electric field* is always continuous, and the *perpendicular electric field* is continuous if there is no surface charge at the boundary. In our case, with no surface charge, whatever is on the left side of the interface (at $z = 0^-$) must be equal to whatever is on the right side (at $z = 0^+$). Thus, for any time t:

$$\overrightarrow{E_x^i}(0,t) + \overrightarrow{E_x^r}(0,t) = \overrightarrow{E_x^t}(0,t) \tag{5.4}$$

The left-hand side of this equation shows the superposition of the incident and reflected waves. The exponentials all take the value of 1 for $z = 0$. Eliminating the vectors (again, for any time t):

$$E_0^i + E_0^r = E_0^t \tag{5.5}$$

Now, assume we know the value of the incident wave (for example, because we sent it from some transmitter), but we do not know what will go through the interface and what will bounce back. Divide the whole equation by E_0^i:

$$\frac{E_0^i}{E_0^i} + \frac{E_0^r}{E_0^i} = \frac{E_0^t}{E_0^i} \tag{5.6}$$

We define the **reflection coefficient** r as:

$$r \equiv \frac{E_0^r}{E_0^i} \tag{5.7}$$

And the **transmission coefficient** t as:

$$t \equiv \frac{E_0^t}{E_0^i} \tag{5.8}$$

Therefore, we can rewrite our equation as:

$$1 + r = t \tag{5.9}$$

This relation is *always* valid, even for oblique incidence or lossy media, as it is mentioned in section 5.2.4.

For loss-less media, r and t are *real numbers*. Note that the reflected electric field cannot be larger in magnitude than the incident electric field, but they can have equal magnitudes (in the case of total reflection), and can also have opposite signs. Hence:

$$-1 \leq r \leq 1 \tag{5.10}$$

Together with the equation above, this gives us:

$$0 \leq t \leq 2 \tag{5.11}$$

5.2.2 Intrinsic impedance

At some point in the derivation of the 1-D Wave Equation for electromagnetic waves, the corrected **Ampère's circuital law** was used, namely:

$$\oint_C \overrightarrow{H} \cdot \overrightarrow{dl} = \int_S \overrightarrow{J} \cdot \overrightarrow{dA} + \frac{d}{dt} \int_S \epsilon \overrightarrow{E} \cdot \overrightarrow{dA} \tag{5.12}$$

From this law, assuming a medium with no free current-sources, that is, with $\overrightarrow{J} = 0$, it was concluded that, at any time t, for some point z_0 (the square brackets indicate we are evaluating the result of the derivative at point $z = z0$):

$$\left[\frac{\partial B_y(z,t)}{\partial z} \right]_{z_0} = \mu\epsilon \left[\frac{\partial E_x(z,t)}{\partial t} \right]_{z_0} \tag{5.13}$$

Therefore, since the **magnetic flux density** $\overrightarrow{B}$ and the **magnetic field** $\overrightarrow{H}$ are related by:

$$\overrightarrow{B} = \mu \overrightarrow{H} \tag{5.14}$$

We can reduce that equation to:

$$\left[\frac{\partial H_y(z,t)}{\partial z} \right]_{z_0} = \epsilon \left[\frac{\partial E_x(z,t)}{\partial t} \right]_{z_0} \tag{5.15}$$

If we have an electromagnetic wave traveling in the $+z$ direction, whose electric component is given (in phasor notation) by:

$$\overrightarrow{E_x}(z,t) = \hat{x} E_0 e^{j(\omega t - kz)} \tag{5.16}$$

Then, following the right-hand rule, its magnetic portion must be given by:

$$\overrightarrow{H_y}(z,t) = \hat{y} H_0 e^{j(\omega t - kz)} \tag{5.17}$$

Now, applying the reduced equation derived from Ampère's circuital law:

$$\left[\frac{\partial H_y(z,t)}{\partial z}\right]_{z_0} = \epsilon\left[\frac{\partial E_x(z,t)}{\partial t}\right]_{z_0} \tag{5.18}$$

$$H_0 j(-k)e^{j(\omega t - kz_0)} = \epsilon E_0 j\omega e^{j(\omega t - kz_0)} \tag{5.19}$$

Canceling the exponentials and the j's, and taking only the *absolute values* (ignoring the direction for the moment):

$$kH_0 = \epsilon\omega E_0 \tag{5.20}$$

But $k = \frac{\omega}{v_p}$, so:

$$\frac{\omega}{v_p}H_0 = \epsilon\omega E_0 \tag{5.21}$$

Canceling ω's and writing $v_p = \frac{1}{\sqrt{\mu\epsilon}}$, we get:

$$H_0 = \frac{\epsilon}{\sqrt{\mu\epsilon}}E_0 \tag{5.22}$$

Which is the same as:

$$H_0 = \sqrt{\frac{\epsilon}{\mu}}E_0 \tag{5.23}$$

The quantity $\eta = \frac{|E|}{|H|}$ is called the **electromagnetic wave impedance**, **intrinsic impedance** or **characteristic impedance**, namely**:**

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \tag{5.24}$$

Like the velocity of propagation for EM waves and the index of refraction, the impedance depends exclusively on the properties of the medium. Since the electric field has units of $[\frac{\text{V}}{\text{m}}]$ and the magnetic field has units of $[\frac{\text{A}}{\text{m}}]$, the impedance has units of $\left[\frac{\frac{V}{m}}{\frac{A}{m}}\right] = [\frac{V}{A}] = [\Omega]$ (resistance!), and in vacuum its value is:

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 377\Omega \tag{5.25}$$

From the intrinsic impedance, we can always determine the electric field of an EM wave if we know the magnetic field, and vice versa. Note, however, that this relates the *amplitude* of the electric and magnetic fields at a point, but we still have to account for the *direction* of the fields: our equation derived from Ampère's law had some time and spatial derivatives in it, which will bring down j's and negative signs.

There is also a straightforward relationship between the impedance and the index of refraction of a medium:

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \frac{\mu}{\sqrt{\mu\epsilon}} = \mu v_p = \mu\frac{c}{n} \tag{5.26}$$

Which can also be written as:

$$\eta = \frac{(\frac{\mu}{\mu_0})\eta_0}{n} \tag{5.27}$$

Thus, for non-magnetic media ($\mu \cong \mu_0$), the intrinsic impedance of the medium is simply:

$$\eta = \frac{\eta_0}{n} \tag{5.28}$$

5.2.3 Reflection and transmission coefficients revisited

Let us now repeat what we did in section 5.2.1, now for the corresponding magnetic wave. We have the incident, reflected and transmitted magnetic waves ($\overrightarrow{H_y^i}$, $\overrightarrow{H_y^r}$and $\overrightarrow{H_y^t}$, respectively), in the yz-plane (linearly polarized along the y direction:

$\overrightarrow{H_y^i}(z,t) = \hat{y}H_0^i e^{j(\omega t - k_1 z)}$: traveling in the $+z$ direction, within medium 1
$\overrightarrow{H_y^r}(z,t) = \hat{y}H_0^r e^{j(\omega t + k_1 z)}$: traveling in the $-z$ direction, within medium 1
$\overrightarrow{H_y^t}(z,t) = \hat{y}H_0^t e^{j(\omega t - k_2 z)}$: traveling in the $+z$ direction, within medium 2

Applying to each of these (and their respective electric field counterparts) the equation derived from Ampère's law:

$$\left[\frac{\partial H_y(z,t)}{\partial z}\right]_{z_0} = \epsilon \left[\frac{\partial E_x(z,t)}{\partial t}\right]_{z_0}$$

$$H_0^i j(-k_1) e^{j(\omega t - k_1 z_0)} = \epsilon E_0^i j\omega e^{j(\omega t - k_1 z_0)} \tag{5.29}$$

$$H_0^r j k_1 e^{j(\omega t + k_1 z_0)} = \epsilon E_0^r j\omega e^{j(\omega t + k_1 z_0)} \tag{5.30}$$

$$H_0^t j(-k_2) e^{j(\omega t - k_2 z_0)} = \epsilon E_0^t j\omega e^{j(\omega t - k_2 z_0)} \tag{5.31}$$

Again, canceling all the j's and exponentials:

$$H_0^i(-k_1) = \epsilon\omega E_0^i \tag{5.32}$$

$$H_0^r k_1 = \epsilon\omega E_0^r \tag{5.33}$$

$$H_0^t(-k_2) = \epsilon\omega E_0^t \tag{5.34}$$

And using our definition for impedance:

$$\frac{k}{\epsilon\omega} = \frac{1}{\epsilon v_p} = \frac{\sqrt{\mu\epsilon}}{\epsilon} = \sqrt{\frac{\mu}{\epsilon}} = \eta \tag{5.35}$$

We conclude:

$$H_0^i = -\frac{E_0^i}{\eta_1} \tag{5.36}$$

$$H_0^r = \frac{E_0^r}{\eta_1} \tag{5.37}$$

$$H_0^t = -\frac{E_0^t}{\eta_2} \tag{5.38}$$

We observe the following pattern (where the $+$ and $-$ subscripts refer to propagation in the $+z$ or $-z$ direction, respectively):

$$\eta H_{y+} + \eta H_{y-} = -E_{x+} + E_{x-} \tag{5.39}$$

Recall our equation:

$$\overrightarrow{E_x^i}(0,t) + \overrightarrow{E_x^r}(0,t) = \overrightarrow{E_x^t}(0,t)$$

or

$$E_0^i + E_0^r = E_0^t$$

This equation was obtained by imposition of continuity at the boundary (at $z = 0$). Analogously, we can do the same for the magnetic field:

$$\overrightarrow{H_y^i}(0,t) + \overrightarrow{H_y^r}(0,t) = \overrightarrow{H_y^t}(0,t) \tag{5.40}$$

$$H_0^i + H_0^r = H_0^t \tag{5.41}$$

Substituting the relations we just derived between the magnetic and electric fields:

$$-\frac{E_0^i}{\eta_1} + \frac{E_0^r}{\eta_1} = -\frac{E_0^t}{\eta_2} \tag{5.42}$$

Rearranging:

$$\eta_2 E_0^i - \eta_2 E_0^r = \eta_1 E_0^t \tag{5.43}$$

And using our previous continuity equation:

$$E_0^i + E_0^r = E_0^t$$

We get:

$$\eta_2 E_0^i - \eta_2 E_0^r = \eta_1 (E_0^i + E_0^r)$$

Divide both sides by E_0^i and apply the definition for the reflection coefficient r (in this case, r_{12}, specifying that the wave is coming *from medium 1 into medium 2*):

$$\eta_2 - \eta_2 \tfrac{E_0^r}{E_0^i} = \eta_1 (1 + \tfrac{E_0^r}{E_0^i})$$

$$\eta_2 - \eta_2 r_{12} = \eta_1 + \eta_1 r_{12}$$

From which we get:

$$r_{12} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{5.44}$$

Using $1 + r = t$:

$$t_{12} = \frac{2\eta_2}{\eta_2 + \eta_1} \tag{5.45}$$

This being the transmission coefficient *from medium 1 into medium 2*.

For yet another form for the coefficients, apply the substitution $\eta = \frac{\eta_0}{n}$ (which can only be done assuming $\mu_1 = \mu_2 \cong \mu_0$) :

$$r_{12} = \frac{\frac{\eta_0}{n_2} - \frac{\eta_0}{n_1}}{\frac{\eta_0}{n_2} + \frac{\eta_0}{n_1}} = \frac{\eta_0}{\eta_0}\left(\frac{\frac{n_1-n_2}{n_1 n_2}}{\frac{n_1+n_2}{n_1 n_2}}\right) = \frac{n_1 - n_2}{n_1 + n_2} \tag{5.46}$$

$$t_{12} = \frac{\frac{2\eta_0}{n_2}}{\frac{\eta_0}{n_2} + \frac{\eta_0}{n_1}} = \frac{\eta_0}{\eta_0}\left(\frac{\frac{2}{n_2}}{\frac{n_1+n_2}{n_1 n_2}}\right) = \frac{2n_1}{n_1 + n_2} \tag{5.47}$$

Since $\frac{\eta_1}{\eta_2} = \frac{n_2}{n_1}$, to switch between the impedance form and the refractive index form, simply exchange $\eta_1 \Leftrightarrow n_2$ and $\eta_2 \Leftrightarrow n_1$.

From the formula for the reflection coefficient, one thing can be noted: if $n_1 > n_2$, if the wave is traveling from a more refractive medium to a less refractive one (called a **soft boundary**), $r > 0$, and a positive r means that the E_0^i and E_0^r have the same direction; therefore, no inversion of the field occurs. Conversely, when $n_1 < n_2$ (a **hard boundary**), $r < 0$ and the directions of the incident and reflected electric fields are opposite; put another way, the directions are 180^o out of phase.

That is, the electric field *flips* when the wave hits a hard boundary, and only in that case. The transmitted electric field, however, always has the same direction as the incident electric field; they are always in phase.

5.2.4 Complex reflection and transmission coefficients

Following the trend, we generalize so as to include lossy media as well by making these coefficients complex.

The nice thing is that it is quite trivial: all n's become $\widetilde{n}$'s, all η's become $\widetilde{\eta}$'s and we got out $\widetilde{r}$ and $\widetilde{t}$.

In order to show it, we would just repeat the derivation using $\widetilde{k}$'s as our wave-vectors.

The full expression for $\widetilde{r}$ is:

$$\widetilde{r}_{12} = \frac{\widetilde{n}_1 - \widetilde{n}_2}{\widetilde{n}_1 + \widetilde{n}_2} = \frac{(\widetilde{n}_{1_r} - \widetilde{n}_{2_r}) - j(\widetilde{n}_{1_i} - \widetilde{n}_{2_i})}{(\widetilde{n}_{1_r} + \widetilde{n}_{2_r}) - j(\widetilde{n}_{1_i} + \widetilde{n}_{2_i})} = \frac{(n_1 - n_2) - j(\kappa_1 - \kappa_2)}{(n_1 + n_2) - j(\kappa_1 + \kappa_2)} \tag{5.48}$$

Similarly, for $\widetilde{t}$:

$$\widetilde{t}_{12} = \frac{2\widetilde{n}_1}{\widetilde{n}_1 + \widetilde{n}_2} = \frac{2\widetilde{n}_{1_r} - j2\widetilde{n}_{1_i}}{(\widetilde{n}_{1_r} + \widetilde{n}_{2_r}) - j(\widetilde{n}_{1_i} + \widetilde{n}_{2_i})} = \frac{2n_1 - j2\kappa_1}{(n_1 + n_2) - j(\kappa_1 + \kappa_2)} \tag{5.49}$$

Again, the magnitude of the reflected electric field cannot be larger than the incident one. However, since we are now dealing with complex numbers:

$$|\widetilde{r}| \leq 1 \tag{5.50}$$

$$|\widetilde{t}| \leq 2 \tag{5.51}$$

5.3 Reflectivity and Transmittance

The concepts of reflectivity and transmittance are not very distant from coefficients analyzed in the previous section. The only difference is they are determined in terms of the *power* (or, equivalently, in terms of the *intensity)* of the wave that gets reflected or transmitted, respectively.

The intensity ($\frac{\text{power}}{\text{area}}$, or power density) of an electromagnetic wave is given by the magnitude of the **Poynting vector** $\overrightarrow{S}$ (the notation $\overrightarrow{S}$ was chosen by John Henry Poynting himself when the concept was introduced in his original 1884 paper proving the now called Poynting's theorem):

$$\vec{S} = \frac{\vec{P}}{A} = \vec{E} \times \vec{H} \tag{5.52}$$

The Poynting vector *points* in the direction of power transfer (which happens to be the direction of propagation of the wave).

This being a cross-product, we can simplify the equation for orthogonal electric and magnetic fields and can therefore say its magnitude is:

$$\text{Intensity(t)}=|\vec{S}(t)| = |\vec{E}(t)||\vec{H}(t)|\sin(\frac{\pi}{2}) = |\vec{E}(t)||\vec{H}(t)| \tag{5.53}$$

We recently showed that the magnitude of the magnetic field at any point is related to the magnitude of the electric field by the impedance:

$$|\vec{E}||\vec{H}| = \frac{|\vec{E}|^2}{\eta} = \eta|\vec{H}|^2 \tag{5.54}$$

Generally, the intensity is taken as the time-average of the Poynting vector since both $\vec{E}$ and $\vec{H}$ vary with time.

$$I = <|\vec{S}(t)|> = \frac{1}{2}\frac{|\vec{E}(t_0)|^2}{\eta} \tag{5.55}$$

The factor of $\frac{1}{2}$ comes from averaging the *squared* sinusoidal part over a period (the angle $<$ $>$ brackets denote time-average; a sinusoidal, instead of its square, has 0 average over a period). For this choice of definition for intensity, the electric field can be taken at any time-point for calculations.

Therefore, if we would like to calculate the **reflectance**, that is, the portion of the incident power that was reflected:

$$R = \frac{\text{reflected power density}}{\text{incident power density}} = \frac{\frac{1}{2}\frac{|\vec{E^r}|^2}{\eta}}{\frac{1}{2}\frac{|\vec{E^i}|^2}{\eta}} = \left|\frac{\vec{E^r}}{\vec{E^i}}\right|^2 = |\tilde{r}|^2 \tag{5.56}$$

Using our complex reflection coefficient from section 5.2.4, along with the fact that the magnitude of the ratio of two complex numbers is the ratio of their magnitudes:

$$R = \frac{(n_1 - n_2)^2 + (\kappa_1 - \kappa_2)^2}{(n_1 + n_2)^2 + (\kappa_1 + \kappa_2)^2} \tag{5.57}$$

A typical reflectivity curve for insulating materials is plotted in section 5.4.2.

In a *loss-less medium*, all power from the incident wave must be either reflected or transmitted. Therefore, applying conservation of energy:

$$\text{reflected power} + \text{transmitted power} = \text{incident power}$$

$$\tfrac{\text{reflected power}}{\text{incident power}} + \tfrac{\text{transmitted power}}{\text{incident power}} = 1$$

$$R + T = 1 \tag{5.58}$$

Where T stands for the **transmittance**, the portion of the power that passes through the boundary.

From this, using our previous formula for reflectivity, we obtain:

$$T = \frac{4n_1n_2 + 4\kappa_1\kappa_2}{(n_1 + n_2)^2 + (\kappa_1 + \kappa_2)^2} \tag{5.59}$$

Note that $T \neq |\tilde{t}|^2$!!! (Can you figure out why? See question 5)

5.4 T-A-R-T

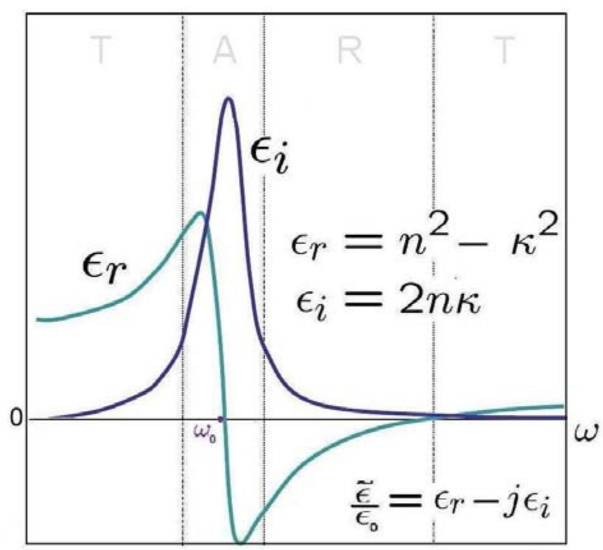
Our final section deals with the most practical aspect of all the theory discussed. Materials will be split in three groups for analysis: insulators (dielectrics), free-electron metals and plasmas.

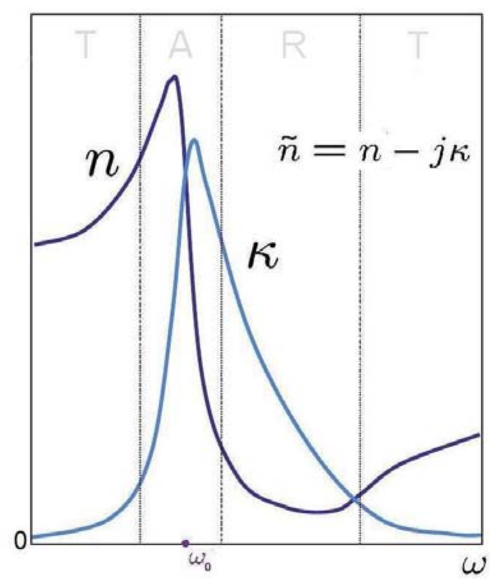
5.4.1 A more intuitive frequency analysis

In section 3.5, we explained how the magnitude and phase of the displacement (of the electron, in our Lorentz oscillator model) varied with the frequency of the driving electric field. We did not have the tools yet to explain the response of the material to EM waves of different frequencies, apart from the effect on the displacement of the electron. That analysis will be carried out in this section.

As mentioned in section 5.1, *absorption* takes place when the frequency (or frequencies) of an EM wave matches one (or more) of the resonant frequencies of the material, what we called ω_0 in the derivation of the Lorentz oscillator model. In fact, *resonance* is exactly this phenomenon of frequency-matching when driving any kind of oscillator. Therefore, when the frequency of the electromagnetic wave is in the vicinity of ω_0, the predominant behavior in the material is absorption. What about before or after that frequency?

5.4.2 Insulators





The optical response is quite typical for a wide range of dielectric materials. The plots above represent such typical behavior.

From the plot on the left, we can see there is a peak in ϵ_i around $\omega = \omega_0$ (the resonance frequency for the material), and this peak levels off quickly for both lower or higher frequencies. The peak corresponds to the strong absorption of the electromagnetic radiation by the material near that frequency.

We also notice that, as we increase frequency, both ϵ_r and n increase before ω_0, reach a minimum then start rising again. **Normal dispersion** is the name given to the phenomenon of rising permittivity and refractive index with frequency, while **anomalous dispersion** is the decrease in permittivity and index of refraction that takes place around the resonance frequency.

The region of anomalous dispersion can be determined by setting to 0 the derivative of $\frac{\tilde{\epsilon}}{\epsilon_0}$ with respect to frequency (i.e., by finding the frequencies for which $\frac{d\tilde{\epsilon}}{d\omega} = 0$, which correspond to the points of maximum and minimum). These points are found to be $\omega = \omega_0 + \frac{\gamma}{2}$ and $\omega = \omega_0 - \frac{\gamma}{2}$, comprising a region of width equal to the damping constant, γ.

The frequency-dependent optical behavior of a material is divided in regions: **T** for transmission (or "transparency"), **A** for absorption and **R** for reflection. The absorption behavior has been explained, but we are left to understand the reflection region. The following reflectivity curve corresponds to the same insulator whose optical behavior is shown in the previous plots:

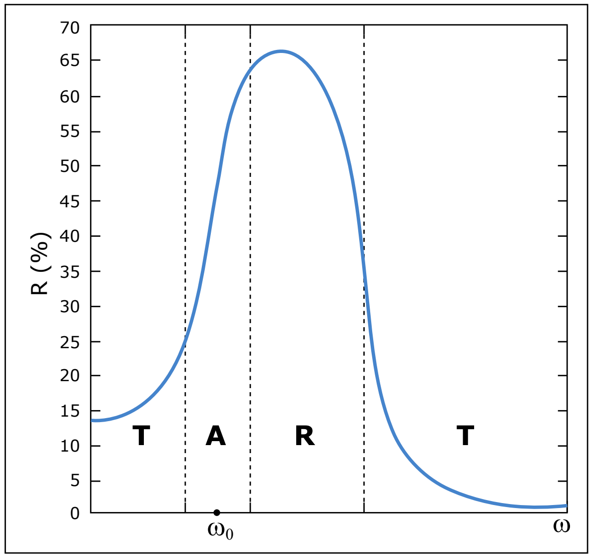


Image by MIT OpenCourseWare.

Remember that absorption only takes place around the resonant frequency. Thus, energy is conserved in the other regions: so they are either predominantly transmissive, or predominantly reflective. As it was stated above, the region of anomalous dispersion has a width of value γ. Some authors choose to define the anomalous dispersion region as the absorption region, but in order to cover a larger portion of the absorption curve, we define it in this plot as centered at ω_0, but with width equal to double the anomalous dispersion's, namely 2γ.

From the plot above, we can see that reflectivity starts increasing in the absorption region. Although there may indeed be appreciable reflectivity (of the portion of light that is not absorbed) in this region, the reflectivity only becomes proportionally significant for frequencies in the region labeled R. Referring to the previous plots, we notice that in that region $n < \kappa$, which, as shown by our equation, yields $\epsilon_r < 0$ - which is always true within the reflective region, but the converse is not true: not all frequencies for which $\epsilon_r < 0$ will be predominantly reflected, as they could also belong to the absorption region. Alluding to Feynman's statement in section 3.2, the reflection region has its outer boundary defined at $\omega = \omega_p$. In the plot, the boundary was taken at the point where ϵ_r crosses the x-axis ($\epsilon_r(\omega) = 0$ for $\omega > \omega_0$). One can verify that these two statements are equivalent if we consider that $\omega \gg \omega_0 \gg \gamma$ in the formula for ϵ_r.

The last region is another transmission region: the insulator is completely transparent at very high frequencies.

To summarize the regions for a dielectric material:

- **T**ransmissive for $0 < \omega < \omega_0 - \gamma$
- **A**bsorptive for $\omega_0 - \gamma < \omega < \omega_0 + \gamma$
- **R**eflective for $\omega_0 + \gamma < \omega < \omega_p$
- **T**ransmissive for $\omega > \omega_p$

5.4.3 Metals: the Drude-Lorentz model

In 1900, Paul Drude proposed the **Drude model for electrical conduction**, the result of applying kinetic theory to electrons in a solid. In metals, the electrons are not bound to the nuclei - the potential energy distribution of a metallic lattice makes it as *energetically favorable* for electrons to jump around from side to side - from one nucleus to another - as staying bound to a single nucleus. These delocalized electrons constitute which is known as the “sea of electrons” that flows freely around the lattice of nuclei, and are the one thing that allows metals to conduct electricity.

Consider a particular change in our Lorentz oscillator model: in this metallic bond, if the electrons are not bound, then there is no analogous of a restoring “spring” force. That is, $F_{\text{spring}} = 0$, hence the equivalent spring constant associated with it is equal to 0.

Going back to one of our first equations, $k = m\omega_0^2$, we see that this condition yields ω_0=0.

Nevertheless, there is still a damping term, due mostly to the collisions within the electron cloud and with the nuclei.

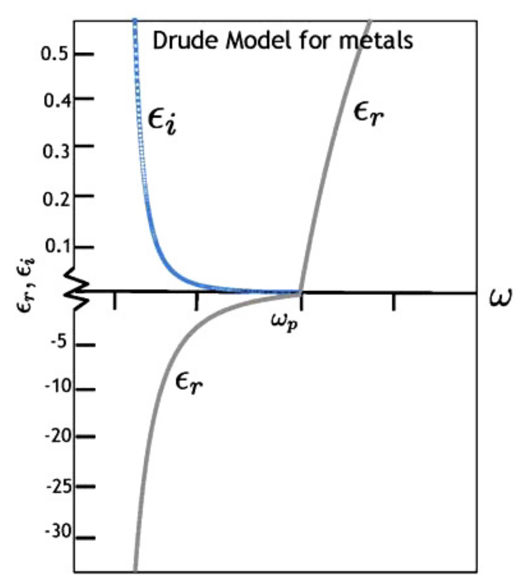
Our model then becomes:

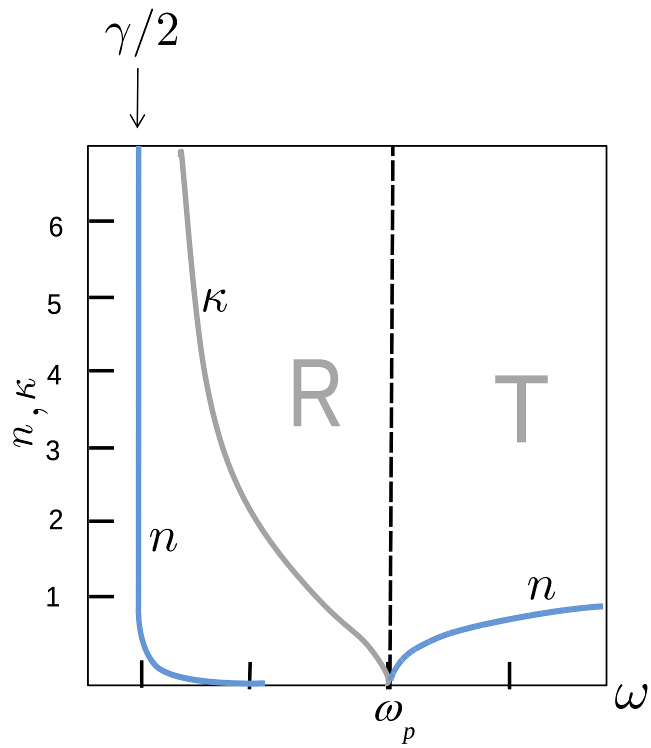
$$\frac{\widetilde{\epsilon}}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2 - j\omega\gamma} \tag{5.60}$$

(Note that the negative signs agree with the ones in the original equation.)

This is the so-called **Drude-Lorentz model** for metals.

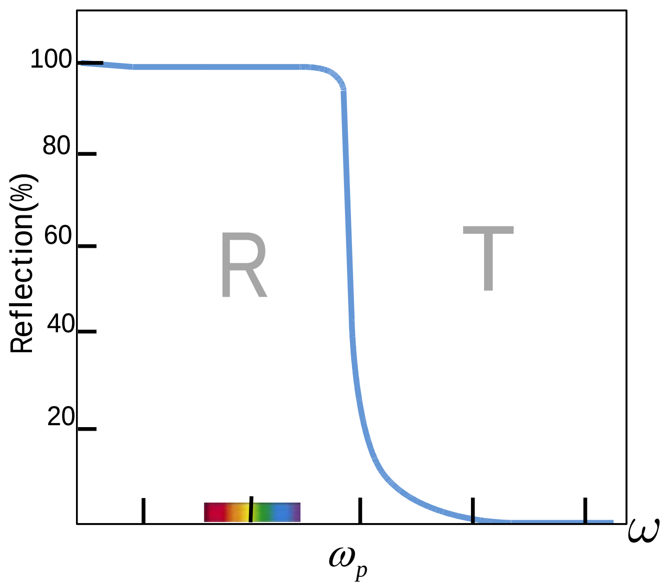
Although updated models exist for metals, which take advantage of quantum theory and Boltzmann statistics, the Drude model is very useful to develop accurate intuitions regarding the optical behavior of metals.





Metals do not exhibit the first T region; they start at the absorption region for low frequencies. That absorption region, however, is quite limited, ending at $\omega = \gamma$, which is usually a relatively small number compared to the other parameters. After that, metals exhibit regions like the insulator, a reflective region and a transparent region (in fact, the electrons of the insulator are the ones that behave like a metal's free electrons for frequencies in those two regions).

Note the different y-axis scales on the left plot: the *magnitude* of ϵ_r is much greater than ϵ_i. At the plasma frequency, we observe that $n = 0$. But what does that mean? Given the definition of the real index of refraction, it means that the velocity in the medium approaches ∞, as well as infinite wavelength for the wave. An infinite wavelength means that all the electron dipoles in the material are oscillating in phase, but this does not suit as a good intuition. In order to understand this, one would have to compare the Drude-Lorentz model to the behavior of real metals, but that shall not be covered here. Let us continue with the practical insights that we can indeed obtain from these results. Namely, the reflectivity curve of a metal is:



The rainbow-colored strip is included to indicate where the visible light range belongs in such a curve, calculated for a typical metal.

Summarizing these results, we have the following optical regions in a metal:

- **A**bsorptive, for $0 < \omega < \gamma$
- **R**eflective, for $\gamma < \omega < \omega_p$
- **T**ransmissive, for $\omega > \omega_p$

5.4.4 Plasmas

In the case of a plasma, our model gets simplified even further. As in the case of metals, there is no restoring force, so $\omega_0 = 0$ here as well. In a plasma, however, the electrons in the "cloud" are far enough from each other and from the ionized nuclei that they do not collide, statistically speaking (i.e., the plasma has a very small value of mean free collision time). Then, the only loss of energy would be due to re-radiation, but then that re-radiation would be absorbed by some other dipole and re-re-radiated, "*ad infinitum*". For this reason, the damping term γ is null as well.

The simplified permittivity equation for plasmas is then just:

$$\frac{\widetilde{\epsilon}}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \tag{5.61}$$

This result does not depart much from the results obtained for metals. If you recall, when the T-A-R-T regions for a metal (in fact, only A-R-T), it was stated that although metals do exhibit an absorptive region, it is very narrow, extending only up to the value of the damping factor γ, which was also assumed to be very small (and, in the real world, it is indeed). Therefore, as we make $\gamma = 0$ here, we get exactly the same response, except that for a plasma there is not even an absorption region–we are left with only R-T.

- **R**eflective, for $0 < \omega < \omega_p$
- **T**ransmissive, for $\omega > \omega_p$

When the plasma frequency was introduced and Richard Feynman was quoted regarding its significance for the behavior of signal transmission, it was exactly this behavior that was being described: plasmas (and metals to some extent) are entirely reflective for electromagnetic waves of frequency less than the medium's plasma frequency, and entirely transmissive for frequencies above it.

This completes the introduction to the Lorentz oscillator and the applications derived from it. The next section includes some problems for practice.

6 Questions

1. A damped electron (mass m, charge q) oscillator of natural frequency ω_0, and damping constant γ is being driven by a electric field of magnitude E_0 varying with frequency ω.
   a) What is the motion equation governing the displacement x of the electron?
   b) In terms of the given parameters, what is the phase of x relative to the phase of the electric field?

2. For the monochromatic electromagnetic wave described by the electric field
   $\overrightarrow{E}(x,t) = 5\cos(2\pi 10^{13}t + 14\pi 10^6 x)\hat{y} + 3\sin(2\pi 10^{13}t + 14\pi 10^6 x - \frac{5\pi}{3})\hat{z}$
   Determine, in MKS units:
   (a) its propagation direction
   (b) its phasor representation
   (d) the phase of its $\hat{z}$ component, i) at $t = 2\mu s$ and $|x| = 5m$ and ii) relative to the $\hat{y}$ component
   (c) the magnitude of the electric field at $t = 2\mu s$ and $|x| = 5m$
   (e) its angular wavenumber
   (f) its angular frequency and linear frequency
   (g) its phase velocity
   (h) its wavelength. How does it relate to the linear wavenumber? (Obs: the angular wavenumber is the linear wavenumber multiplied by 2π)
   (i) the index of refraction in the medium
   (j) the impedance of the medium (assume it is non-magnetic)
   (k) the magnetic field associated with this wave (in both real vector form, as above, and phasor notation)
   (l) the time-averaged power hitting a surface of area $3m^2$

3. An electromagnetic wave of a certain frequency f is traveling in a medium whose relative electric permittivity is equal to 16, and encounters a medium X, perpendicularly. You observe that absolutely no light of that frequency was absorbed or reflected at the boundary. Both media are non-magnetic. What is the phase velocity of the wave in medium X?

4. A uniaxial birefringent crystal made from quartz has $n_o = 1.5443$ and $n_e = 1.5534$ (meaning that the crystal has a single special axis, called the *optical axis*, for which the index of refraction is different from the rest of the material) . A wave plate is made by cutting the crystal so that the optic axis is parallel to the surfaces of the plate. The crystal will function as a quarter-wave plate if the phase difference between the o- and e- rays is 90^o, turning light polarized at 45^o to the optic axis into circularly polarized light. Calculate the thickness of the crystal if it behaves as a quarter-wave plate for 500nm light.

5. In section 5.3, we derived the equation for the transmittance T in a loss-less medium from conservation of energy, subtracting the reflectivity from unity. Actually, this calculation is done by using Stokes' method of time-reversal of light rays: considering the scenario in which the transmitted ray is going in the reverse direction (hence, being the new "incident ray"), then adding the respective components of its refracted and reflected rays, and making everything equal to what was was there in the original "before time-reversal" scenario. Re-derive the formula for transmittance in a loss-less medium,
(a) using the Poynting vector, by writing the electric and magnetic components at each point (Hint: the transmitted wave lies in a different medium, with different properties from the incident medium).
(b) by using Stokes' time-reversal principal. The "time-reversed coefficients" are called r' and t'. (i) Show that the Stokes definition of reflectivity, $R = -rr'$, is identical to $|\tilde{r}^2|$. (ii) Calculate the transmittance $T = tt'$. (iii) Show that $tt' - rr' = 1$.

6. Take an insulator whose resonance frequency is $\omega_0 = 5.4 \times 10^{14} \frac{\text{rad}}{\text{s}}$, damping factor $\gamma = 0.18\omega_0$, plasma frequency $\omega_p = 6.8 \times 10^{16} \frac{\text{rad}}{\text{s}}$.
(a) What is the electron density of this material?
For the following, use MATLAB or your favorite plotting program. Make sure to adjust the plots to include all the relevant features. Label the axes carefully, include appropriate units and indicate the position of ω_0. Calculate all local maxima and minima and include in your answer the frequency for which they occur as well as the value of the function at those points. Identify the T-A-R-T regions in all plots. Each item should have one plot (i.e., plot both quantities in the same graph). If necessary, assume that this material is surrounded by vacuum.
(b) Plot ϵ_r and ϵ_i, the real and imaginary parts of the complex permittivity.
(c) Plot n and κ, the real refractive index and the extinction coefficient. What is the phase velocity of light traveling in this material?
(d) Plot R, the reflectivity. Also plot the transmittance T, but not in the absorption region (A).
(e) What happens - predominantly - to the individual frequency components of a polychromatic electromagnetic wave composed of soft X-rays ($\lambda_1 = 3.75nm$), red light ($\lambda_2 = 646.5nm$) and FM radio ($\lambda_3 = 1.2m$), when the wave hits this insulator?

7. Germanium - named after his homeland by Clemens Winkler, the first person to isolate the element, in 1886 - is an important semiconductor material used in transistors and various other electronic devices. For example, germanium is the substrate of the wafers for high-efficiency multijunction photovoltaic cells for space applications. Germanium-on-insulator substrates are seen as a potential replacement for silicon on miniaturized chips, and its compound $CdGeAs_2$ (cadmium-germanium-arsenide) is considered a very promising material in the field of optoelectronics. The complex refractive index of germanium at

400nm is given by $\widetilde{n} = 4.141 - j2.215$. Calculate for germanium at 400nm, in MKS units:
a) The phase velocity of light propagating in it
b) Its absorption coefficient
c) Its complex electric permittivity
d) Its impedance
e) Its reflection and transmission coefficients for incident light propagating in air
f) Its reflectivity (percentage) for incident light as above

8. (a) Sea water has a refractive index of 1.33 and absorbs 99.8% of red light of wavelength 700nm in a depth of 10m. What is the complex dielectric constant at this wavelength?
(b) Explain why ice is birefringent, but water is not.

9. A beam of light in incident perpendicularly to a plate of thickness l. The reflectivity of the front and back surfaces is R and the absorption coefficient is α.
(a) Show that the *intensity* of the beam exiting the sample after having been reflected from the back surface *once* is smaller than that of the beam that has suffered no reflections by a factor of $R^2 e^{-2\alpha l}$
(b) Crown glass has a refractive index of approximately 1.51 in the visible spectral region. i) Calculate the reflectivity of the air-glass interface and ii) the transmission of a typical glass window. iii) Also, calculate the factor in part (a) if the plate is made of glass.
(c) Calculate the ratio in part (a) for the *electric fields* of the beams, rather than intensities, for a glass plate.

10. Terahertz lasers (T-rays) are a hot area of research these days. Terahertz radiation, whose frequencies lie in the 300GHz to 3THz range (between infrared and microwave), passes through clothing, plastic paper, wood, plastic, among other materials, but it does not pass through water or metals. For that reason, *airport security T-ray scanners* are being developed, so that a metal (gun, knife etc) would be visible to the detector even if hidden under clothing or inside objects.
The National Security Agency, skeptical, needs more information regarding the technology. Learning that you are a 6.007 graduate, they hire you to determine if the concept is indeed fail-proof.
a) Assuming a Lorentz oscillator model with standard values for the electron mass and charge, and using your knowledge of reflectivity, determine what would be the maximum electron density of a material so that T-rays can pass through it.
b) The plasma frequency of a typical metal is within the range of 1 to 4 PHz (PetaHertz, 10^{15}Hz, in the ultraviolet range). Would you be able to make a knife out of these metals so that it would not show up in the detector? Explain your position to the NSA administrator, using T-A-R-T and reflectivity arguments.

11. (a) Zinc is a divalent metal with 6.6×10^{28} *atoms* per unit volume. In what region of the electromagnetic spectrum lies the plasma frequency of zinc?
(b)Assume that most metals also have their plasma frequencies in that region. How does this relate to the fact that metals are shiny?
(c) In your UROP with Professor Bulovic, you are given top-secret techniques of doping a standard non-magnetic metal such that you have some control over its Drude-Lorentz parameters directly. (i) What would you do to either or both of its electron density N and damping constant γ in order to make the metal less shiny? (ii) Assume the metal's original plasma frequency to be 2PHz, and that due to physical limitations you can only change the electron density N by $\pm 70\%$ and the damping constant by $\pm 30\%$. For what colors of (visible) light, if any, are you able to make the metal significantly less shiny? Show your calculations.

12. The (pulsed) radiation from a pulsar (a type of star), at 300MHz, is delayed 0.1s with respect to radiation at 900MHz. If the delay is caused by electrons in space, estimate the number of electrons in a $1m^2$ column along the line of sight (assume that the plasma frequency at all points along the line of sight is much less than 300MHz).