

Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.007 Electromagnetic Energy: From Motors to Lasers
Supplemental Class Notes
Magnetic Circuit Analog to Electric Circuits *

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1 Introduction

In this chapter we describe an equivalence between *electric* and *magnetic* circuits and in turn a method of describing and analyzing magnetic field systems which can be described in magnetic circuit fashion. As it turns out, the equivalence is a fair approximation to reality and may be used with some confidence.

Magnetic circuits are those parts of devices that employ magnetic flux to either induce voltage or produce force. Such devices include transformers, motors, generators and other actuators (including things such as solenoid actuators and loudspeakers). In such devices it is necessary to produce and guide magnetic flux. This is usually done with pieces of ferromagnetic material (which has permeability very much larger than free space). In this sense, magnetic circuits are like electric circuits in which conductive material such as aluminum or copper has high electric conductivity and are used to guide electric current.

The analogies between electric and magnetic circuits are two: the electric circuit quantity of current is analogous to magnetic circuit quantity flux. (Both of these quantities are 'solenoidal' in the sense that they have no divergence). The electric circuit quantity of voltage, or electromotive force (EMF) is analogous to the magnetic circuit quantity of magnetomotive force (MMF). EMF is the integral of electric field \vec{E} , MMF is the integral of magnetic field \vec{H} .

2 Electric Circuits and Kirchoff's Laws

2.1 Conservation of Charge and KCL

To begin with, consider the law of Conservation of Charge:

$$\oint \vec{J} \cdot d\vec{a} = \frac{d}{dt} \int_{\text{vol}} q dv = 0$$

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This assumes, of course, that there is no accumulation of charge anywhere in the system. This is not a wonderful assumption for any systems with capacitor plates, but if one considers capacitors to be circuit elements so that both plates of a capacitor are part of any given element the right hand side of this expression really *is* zero.

Thsn, if we note current to be the integral of current density: Over some area, a fraction of the whole area around a node:

$$i_k = \iint_{A_k} \vec{J} \cdot d\vec{a}$$

then we have:

$$\sum_k i_k = 0$$

2.2 Faraday's Law

:

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{a}$$

The left-hand integral may be taken to be a number of sub-integrals, each denoted by a discrete fractional integral:

$$v_k = \int_{a_k}^{b_k} \vec{E} \cdot d\vec{\ell}$$

If we assume that there are no substantive flux linkages among the circuit elements:

$$\iint \vec{B} = 0$$

then we have KVL:

$$\sum_k v_k = 0$$

2.3 Ohm's Law

At this point it is probably appropriate to note that Ohm's Law can be used to derive the constitutive relationship for a resistor. Suppose we have a conductive element similar to the rectangular solid shown in Figure 1. Assume current is confined in this element and flowing perpendicular to the flat end shown in the figure. Current density is

$$J_x = \frac{I}{hw}$$

where I is to total current and h and w are height and width of the conductor, respectively.

Electric field along the length of the element is:

$$E_x = \frac{J_x}{\sigma}$$

where σ is the electrical conductivity of the material. Voltage developed is:

$$v_k = \int E_x dl = \frac{J_x l}{\sigma}$$

Which leads us to an expression for element resistance:

$$R = \frac{v}{I} = \frac{l}{hw\sigma}$$

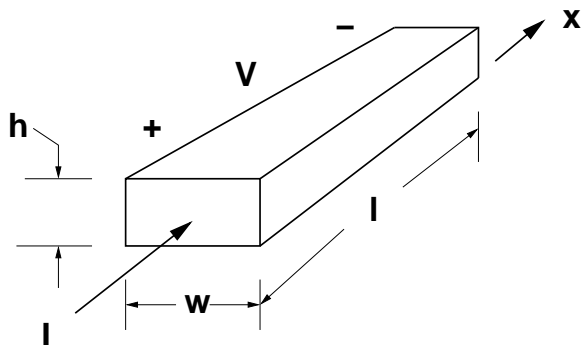


Figure 1: Simple circuit element

3 Magnetic Circuits

As it turns out, magnetic circuits are very similar and are governed by laws that are not at all different from those of electric circuits, with only one minor difference.

3.1 Conservation of Flux: Gauss' Law

To start, Gauss' law is:

$$\oint \vec{B} \cdot d\vec{a} = 0$$

This reflects that notion that there are no *sources* of flux: this is a truly sinusoidal quantity. It neither begins nor ends but just goes in circles.

If we note a fraction of the surface around a node and call it surface k , the flux through that surface is:

$$\Phi_k = \iint_{A_k} \vec{B} \cdot d\vec{a}$$

If we take the sum of all partial fluxes through a surface surrounding a node we come to the analog of KCL:

$$\sum_k \Phi_k = 0$$

3.2 MMF: Ampere's Law

Ampere's Law is simply stated as:

$$\oint \vec{H} \cdot d\vec{\ell} = \iint \vec{J} \cdot d\vec{a}$$

The integral of current density \vec{J} is current, quantified in the SI system as *Amperes*. As it is generally carried in wires which might number, say, N , it is often quantified as:

$$\iint \vec{J} \cdot d\vec{a} = NI$$

For this we will often use the term *MagnetoMotive Force* or MMF, which gets the symbol F . If we use that symbol to denote the integral of magnetic field over a magnetic circuit element:

$$F_k = \int_{a_k}^{b_k} \vec{H} \cdot d\vec{\ell}$$

Then, if we take enough of these subintegrals to cover the loop around a group of elements, we have

$$\sum_k F_k = NI$$

Note that this is not exactly the same as KVL, as it has a source term on the right.

3.3 Magnetic Circuit Element: Analogy to Ohm's Law

Magnetic circuits have an equivalent to resistance. It is the ratio of MMF to flux and has the symbol \mathcal{R} . In direct comparison with the derivation of resistance, a magnetic circuit element is shown in Figure 2.

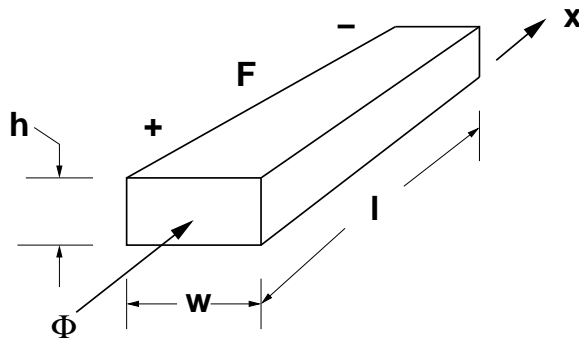


Figure 2: Magnetic circuit element

Assume that the material of this element has permeability $\mu > \mu_0$, so that it has a constitutive relationship:

$$B_x = \mu H_x$$

If flux density in the material is uniform, total flux through the element is:

$$\Phi = hwB_x = hw\mu H_x$$

And the MMF is simply the integral of magnetic field H from one end to the other: again we assume uniformity so that:

$$F = \ell H_x$$

The reluctance of this element is then:

$$\mathcal{R} = \frac{F}{\Phi} = \frac{\ell}{hw\mu}$$

3.4 Magnetic Gaps

In reality, magnetic circuits tend to be made up of very highly permeable elements (pieces of iron) and relatively small air-gaps. A sketch of such a gap is shown in Figure 3.

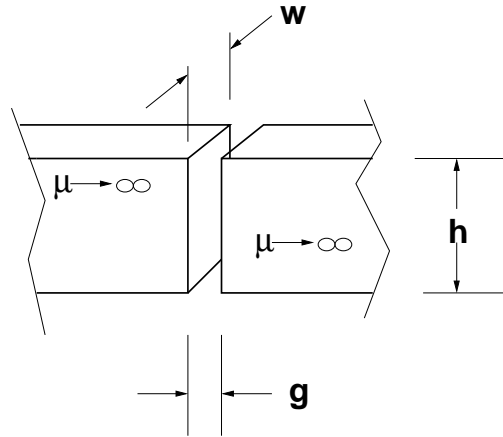
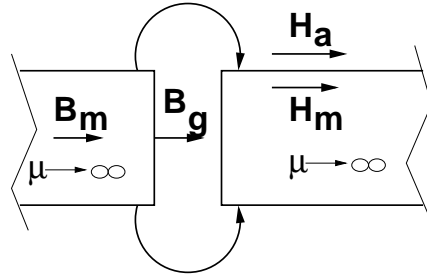


Figure 3: Gap between magnetic elements

It is usually permissible to assume that iron elements have very high permeability ($\mu \rightarrow \infty$), so that there is negligible MMF drop. In this sense the iron elements serve in the same role as copper or aluminum wire in electric circuits. The gap, on the other hand, has reluctance:

$$\mathcal{R}_g = \frac{g}{hw\mu_0}$$



Fringing Field

Figure 4: Gap details

3.5 Boundary Conditions

Shown in Figure 4 is the cross-section of a gap. It is assumed that the elements shown have some depth into the paper which is greater than the gap width g . If the permeability of the elements to the right and the left is very high, we say that magnetic flux is largely confined to those elements. Note that the boundary condition associated with Ampere's Law dictates that the magnetic field intensity H_a in the air adjacent to the permeable material and parallel with the surface must be equal to the magnetic field intensity H_m just inside the magnetic material and parallel with the surface. If the material is very highly permeable ($\mu \rightarrow \infty$), that magnetic field must be nearly zero: $H_a = H_m \rightarrow 0$. This means that magnetic field must be perpendicular to the surface of very highly permeable material. This is the case in the gap itself, where:

$$B_g = B_m$$

We should note, however, that there will be 'fringing' fields in the region near the gap, so that our expression for the reluctance of the gap will not be quite correct. The accuracy of the expression which ignores fringing is best for really small gaps and generally over-estimates the reluctance.

4 Faraday's Law and Inductance

Changing magnetic fields give rise to electric fields and consequently produce voltage. This is how inductance works. Consider the situation shown rather abstractly in Figure 5.

Faraday's Law is, in integral form:

$$\oint \vec{E} \cdot d\ell = -\frac{d}{dt} \iint_{\text{Area}} \vec{B} \cdot \vec{n} da$$

If the contour shown is highly conducting (say, if it is a wire), there is zero electric field over that part of the contour. Voltage across the terminals is:

$$V_{ab} = \int_a^b \vec{E} \cdot d\ell$$

and that is the whole of the integral above. Thus we may conclude that:

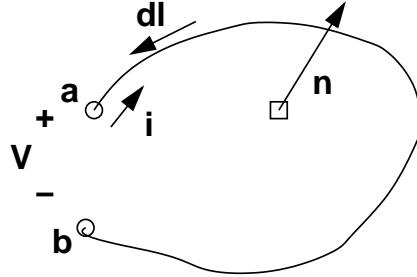


Figure 5: Loop for Faraday's Law

$$V_{ab} = -\frac{d}{dt} \iint_{\text{Area}} \vec{B} \cdot \vec{n} da$$

Now, if we define flux linked by this contour to be:

$$\lambda = - \iint_{\text{Area}} \vec{B} \cdot \vec{n} da$$

then voltage is, as we expect:

$$V_{ab} = \frac{d\lambda}{dt}$$

As it turns out, current flowing in the wire with sense shown by i in Figure 5 tends to produce flux with sense opposite to the normal vector shown in that figure, and so produces positive flux. Generally, in calculating inductance, one uses the 'right hand rule' in determining the direction of flux linkage: if the fingers of your right hand follow the direction of the winding, from the positive terminal, positive flux is in the direction of your thumb.

4.1 Example: Solenoid Actuator

Shown in Figure 6 is a representation of a common solenoid actuator. When current is put through the coil a magnetic flux appears in the gaps and pulls the plunger to the left. We will examine the force and how to calculate it in later chapters. For now, however, we are concerned with magnetic fields in the device and with the calculation of inductance. Assume that the stator and plunger are both made of highly permeable materials ($\mu \rightarrow \infty$). If the coil carries current I in N turns in the sense shown, magnetic flux will cross the narrow air-gap to the right in the direction *from* the stator *to* the plunger and then return in the sense shown in the variable length gap of width x . It is clear that this is also positive sense flux for the coil.

The magnetic circuit equivalent is shown in Figure 7. All of the flux produced crosses the variable width gap which has reluctance:

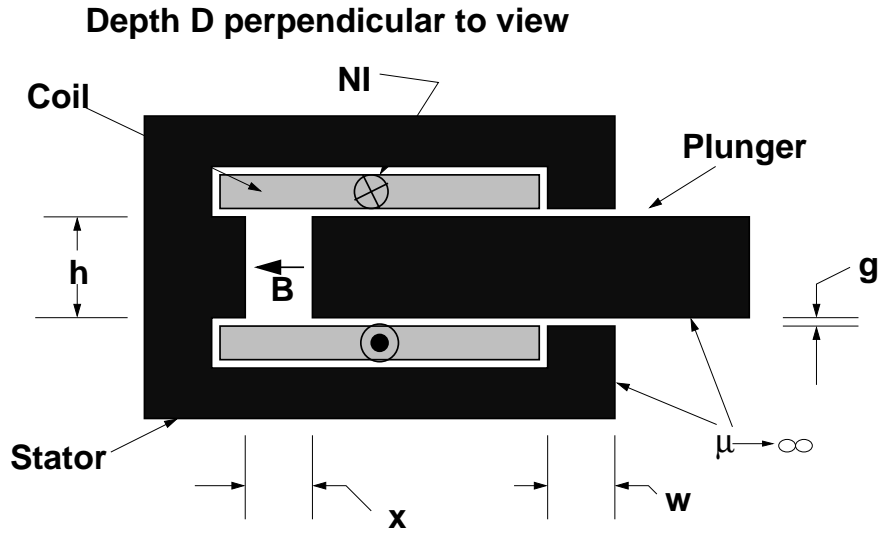


Figure 6: Cross-Section of Solenoid Actuator

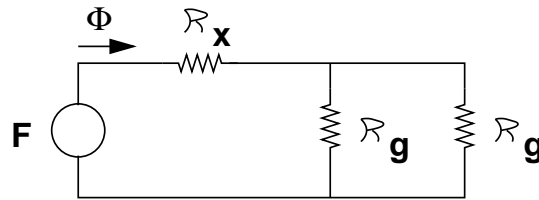


Figure 7: Magnetic Equivalent Circuit of Solenoid Actuator

$$\mathcal{R}_x = \frac{x}{\mu_0 h D}$$

Half of the flux crosses each of the other two gaps, which are in parallel and have reluctance:

$$\mathcal{R}_g = \frac{g}{\mu_0 w D}$$

Total flux in the magnetic circuit is

$$\Phi = \frac{F}{\mathcal{R}_x + \frac{1}{2}\mathcal{R}_g}$$

And since $\lambda = N\Phi$ and $F = NI$, the inductance of this structure is:

$$L = \frac{N^2}{\mathcal{R}} = \frac{N^2}{\mathcal{R}_x + \frac{1}{2}\mathcal{R}_g}$$

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