**DENNIS FREEMAN:** So today's topic is to talk about Z-transforms. But before I start, I want to mention that we've already covered a great deal of material. Here, I've made a map just of things we've done in DT, and there's a fair number of entries. More importantly, there's a fair number of connections between those entries.

And if you're thinking about systems, you should be able to think about, what do each one of those arrows stand for? The reason that's important is that we like to have multiple representations for the system so that we're always able to choose the one that lets us work the most simply. But sometimes that involves moving between the squares. So the way the problem was posed to you may not be the simplest way to solve it, and that involves, then, going across one of those arrows.

So for example, we know that there's a simple relationship between block diagrams and system functionals. All you need to do is think about delays as being the right shift operator. That's a way of thinking about the system functional is just a formula picture of the block diagram. Similarly, we can think about moving between the block diagram and the difference equation by thinking about delay being a shift of the index. So you should be able to think of ways of thinking about all of the transformations between the boxes.

Over here, we thought about system functionals. How do you think about a system as a sequence of operations that you do to a signal? We also thought about system functions, and there's a very simple relationship between those. You think about the operator expression that involves Rs and replace each R with 1/z.

OK, that was all a big set up, because now I want to ask a question. Using your vast knowledge of how all these things interrelate, what's the relationship between the system functional, a function of R, and the unit sample response? As usual, I'd like you to talk to your neighbor to figure this out, and as usual, you won't do that unless I tell you to do something trivial first. So look toward your neighbor, say, "Hi." Wonderful, good. Now, figure out this problem.

OK, how many of you-- I want a show of hands. How many of you know a way of getting between those boxes? Raise your hand. How many of you are sitting beside somebody who knows the way to get between those boxes? No. OK, that didn't work either.

OK, can somebody tell me, if I told you a system functional, say I told you that one, how would I figure out the unit sample response from the system function-- functional? Yes.

**AUDIENCE:** Don't you take the inverse Laplace of it?

**DENNIS FREEMAN:** Inverse Laplace. OK, that's kind of right, but not quite. So usually, the answer to the question is either the slide before this one or the slide after this one. So that doesn't quite fit this. The slide before this one said something about Z-transforms.

Can somebody think of a way, if I told you the system functional is this thing, how would you derive the unit sample response? Yes.

**AUDIENCE:** You could convert it into a difference equation and then just work it out manually.

**DENNIS FREEMAN:** Convert it into a difference equation. That would work. So what you could do is go from this box to that box. Then, if you had this box how would you get the unit sample response? Yes.

**AUDIENCE:** [INAUDIBLE] delta.

**DENNIS FREEMAN:** Put a delta into the difference. Think about the difference equation being a system. Put a delta in and see what comes out. Exactly. Can somebody think of a more direct way that doesn't skirt through the difference equation? Yes.

**AUDIENCE:** Multiply the system functional by a delta function.

**DENNIS FREEMAN:** Multiply the functional by a delta function. I'm not quite sure what that means, but I'm willing to try anything. 1 over 1 minus R minus R squared. Multiply that times a delta function. What do I do next?

OK, as I said, the answer to the question is usually something that we just did or something we're just about to do. Where did we spend the last three weeks doing? No, it's not three weeks. It's only a week, but, you know, I exaggerate. What have we been doing? Yes.

**AUDIENCE:** We could use long division [INAUDIBLE]

**DENNIS** One way would be use long division. How would that work? If we did 1/1-- maybe I'd better use

**FREEMAN:** a bigger blackboard space-- if I tried to do 1 over 1 minus R minus R squared, if I tried to think about that by long division, I would do 1 minus R minus R squared into 1. How would that work?

Well, I'd go 1-- I'd go 1 goes into 1 once. I get 1 minus R minus R squared. This minus this gives me R plus R squared. This goes into this plus R. That gives me whatever.

Having done it in the solitude of my own breakfast this morning, here's an answer that I got there. So if I perform long division on the functional, I get this kind of an answer. The question was, how do I relate the functional to the unit sample response? How do I-- how do I relate this to the unit sample response?

**AUDIENCE:** The coefficient [INAUDIBLE]

**DENNIS FREEMAN:** Excuse me?

**AUDIENCE:** The coefficients [INAUDIBLE]

**DENNIS FREEMAN:** The coefficients of this expression are the unit sample response. OK? So there's a very straightforward way. If I tell you the system functional, if I tell you this representation, there's a very simple way that we've thought about by which you can automatically calculate what is the unit sample response? OK?

In fact, it's so simple that we can write a formula for it. Think about the unit sample response being h of n. Associate the h 0 term with a constant, the h 1 term with an R, with the h 2 term with an R squared. Add them all together, that's the answer.

So the way you get between these representations is very simple. Here's the unit sample response, here's the system functional, and here's an equation that relates the two representations. That all completely clear? Simple. Yes, hi. Everything clear? Questions, comments?

OK, a follow-up question. So here's a relationship between these two representations. How about a relationship between these two representations? Yes.

**AUDIENCE:** Same relationship. Just use one of the place markers.

**DENNIS** Precisely. So it's really-- so we got this one in the previous step. R is just 1/z, so we get this.

**FREEMAN:** So we get a relationship that looks like that.

OK, that relationship is the basis of today's lecture. We call that relationship-- so that relationship represents a mathematical relationship between a function of z and a function of n, and we call that relationship the Z-transform. So that's the topic for today. We're going to think about systems, not as an operator, but as a mathematical function, h of z.

So the first thing to realize is that this transform then, this thing that we're going to be worried about today is a map between a discrete function and a continuous function of z. And even though it's been motivated, I motivated it by thinking about how it would relate to a system, that relationship between a function of z and a function of n is something that you could do on any discrete time signal. So in fact, if you have any signal x of n, we can think about the Z-transform of it by simply thinking about what that equation is telling us. How does it map from a function of n to a function of z?

I've written it in a little-- in a way that you might not have anticipated. You might have thought I would have started at 0. That's just because we're going to do what's called the bilateral transform. There are many kinds of Z-transforms. We're going to do this particular one.

Other classes that you might take may use something called a unilateral. If you're not taking such a class, there's no good reason for you to know that. If you are taking such a class, we're doing bilateral. That just means two-sided, and we'll see by the end of the hour how that makes some things simpler. The big picture is identical whether you do unilateral or bilateral, but there are differences.

OK. Simple Z-transforms. What's the Z-transform of the simplest signal that we can imagine? Simplest signal that we can imagine is the unit sample signal. It's 0 everywhere except n equals 0. At n equals 0, it's the simplest possible non-zero answer, which is 1.

What's the Z-transform? Well, that's trivial. You stick it in the formula. The only non-zero answer is at n equals 0. Stick in n equals 0, and we get that the answer is the Z-transform is 1. What could be easier? Good. Simple signal in time, simple signal in the Z-transform.

What's the Z-transform for a delayed unit sample signal? What happens if the only place that it's not 0 is shifted to n equals 1? Not a big deal. Again, there's a single non-zero answer, it's just that it's now at n equals 1. So this answer is z to the minus 1.

We took a signal that was a function of n, and we represent it by a Z-transform, which is a function of z. Delta of n is 1. Delta of n minus 1 is z to the minus 1.

OK, with that vast knowledge of Z-transforms, figure out the Z-transform for a slightly more complicated sequence of the type-- of the type that we saw when we were looking at systems. What if the signal that we're interested in is 7/8 to the n, u of n, where I'm using the u of n just to cut off the negative parts of 7/8 to the n. So find the Z-transform, and find if it looks like one of those four answers or none, which is number five.

So what answer best represents the Z-transform of the signal x? Raise your hand, number one through five. It's a participatory sport. Good 97% correct, I think.

So OK. Everybody says two. How do you get two? What do you do? Plug in the formula, right? It's a very simple-minded thing.

If you were to think about this signal and think about the definition of the Z-transform, just substitute this particular signal, 7/8 to the n, u of n, in where there would have been the x of n and try to close the sum, right? It's a geometric sequence. We've had lots of experience with geometric sequences. It was in the homework, and so we all know that the answer to summing a geometric-- a one-sided geometric sequence is something of the form 1 over 1 minus a. OK, easy, right?

So the idea, then, is that we understand a general rule by which we can map a function of time into a function of z. So the function of time has to make sense everywhere. That's-- we want that to be true. We want to know what was the system's unit sample response, for example, or what was the input to the system, or whatever. We want to know that at all possible values of n.

How about the other case? Does it make sense? Is x of z makes sense for all possible values of z?

And since I ask the question, the answer is no. Yes, of course. I wouldn't have asked the question if the answer were yes, right? So by the theory of lecturers, you can tell the answer has to have been no. So the question is, what values of z don't make sense? Shout.

**AUDIENCE:** The summation of the [INAUDIBLE]

**DENNIS FREEMAN:** The summation has to converge, so it's only going to be defined if this sum, which happens to be an infinite sum and therefore may not converge, it's going to have to be the case that that infinite sum does converge. Otherwise, it won't make sense to talk about it.

So then the question is, for what values of z will that converge? And again, we know from our experience with geometric sequences that the base of the geometric, a, when we did homework one, the base of the geometric has to have what? What's the property of the base of the-- the geometric that'll make it work?

So if I want to sum some series of the form n equals 0 to infinity a to the n, what's the values-- what's the limitation on a that makes it work?

**AUDIENCE:** [INAUDIBLE]

**DENNIS FREEMAN:** OK, you're not saying it loud enough or clearly enough, or I'm too dense or something.

**AUDIENCE:** Abs less than 1?

**DENNIS FREEMAN:** So we need something-- we need abs less than 1, right. So we do the same thing here. We're going to have to have that the geometric base, 7/8 1 over z has to be less-- has to have a magnitude that is less than 1. And if we torture our minds with inequalities and magnitude signs, that says that the magnitude of z has to be bigger than 7/8. OK?

The important idea is that when we characterize the Z-transform, we should be expecting it to work for all values of n, but not necessarily all values of z. So we have to be cognizant of that. We have to know which values of z are you talking about. We call that idea the region of convergence. We say the Z-transform converged for all z inside the region of convergence.

So when you specify a Z-transform, generally, you have to tell me not just some functional form, but you also have to tell me what was the region for which that functional form converged. So in general, you have to tell me the region of convergence. In this particular case, you have to tell me that you should restrict your attention to values of z whose magnitude is bigger than 7/8. OK?

OK, that's completely the whole definition of Z-transforms. We're done from a point of view of mathematics, right? So all we need to know is that the Z-transform of a signal is some sum of the signal z to the minus n, and we're done. We need to know about the region of convergence, but mathematically, we've completely specified the problem at this point.

To make it useful, however, what we need to do is investigate properties of that transformation. If the transformation were not easy to manipulate, we wouldn't bother with it. So we want to know what's easy to do with the Z-transform and what's not easy to do with the Z-transform.

When something's easy to do with the Z-transform, we'll use it. When things are easier to do some other way, we use the other way. That's the game plan.

So there's a number of properties of the Z-transform that make it easy or hard to do things. So we'll talk about the easy ones. The most fundamental one is linearity. Basically, this entire subject is about linear things. If it's not linear, largely, we don't talk about it.

The reason is we have such powerful tools for thinking about things that are linear. This is one of them. If-- so we say that the Z-transform is a linear operator, and all that means is if you apply the Z-transform to the sum of two things, you get the sum of the things out.

It's a little more complicated than that. If I were being a little more careful-- in fact, I didn't quite say a complete definition here. This is certainly true, but there are additional things that are also true. We say that the Z-transform is linear because if we knew the z-transform for X 1, that includes a functional form and a region of convergence, and if we knew the Z-transform for X 2, again, a functional form and a region of convergence, then by the linearity of the operator, we can figure out just from the two Z-transforms, what is the Z-transform of the sum.

And that's trivial to see. It's easy to see why it ought to be that way. Just look at-- let's think about y, which is the signal, which is the sum. By the definition of Z-transform, the Z-transform of the sum is that formula. Y, by definition, is that thing. Z commutes-- distributes over addition, and the sum separates. So we can always reduce this to this, at least when the case that z is in both regions of convergence.

Though later in the course, we'll see that that's a little overly restrictive. Sometimes it works for more zs than in the intersection. But it's guaranteed to work in the intersection, because I know that I can do this sum if z is in ROC 1. I know I can do this sum if z is in ROC 2.

So I know I can do both of them if it's in the intersection. We'll see a little later that sometimes you can do it even when it's not in the intersection. But for the time being, we know you can do it if it's in the intersection.

So because of linearity, we know that the Z-transform of a sum is the sum of the Z-transforms. I didn't realize when I made this slide, linearity implies more than that. I could have done a weighted sum. I could have said if X 1 goes to X1 and X 2 goes to X 2, then alpha 1, X 1 plus alpha 2, X 2 goes to alpha 1, X 1 plus alpha 2, X 2. I should have put that in the slide, and it just didn't occur to me.

So the idea of linearity is slightly more powerful than the example that I gave here. But again, linearity is the most fundamental property of Z-transforms. If the Z-transform weren't linear, we wouldn't bother with it. But it is, so we do.

Another important property is the delay property. Think about what we do with discrete signals. We put them through discrete systems. Discrete systems of the type we've looked at so far have adders, delays, gains. Delays. If the Z-transform couldn't handle delays, again, we wouldn't do it, right?

Delays are so fundamental to the way we think about discrete systems that it must be the case that Z-transforms deal with delays well, and they do. We've already seen two examples. The first example I did was the unit sample signal has a transform of 1, and the delayed unit sample's signal has transform of z to the minus 1. So there's a simple relationship-- if I knew the first, how I could find the second.

More generally, if I had a signal X and I knew its transform X, then I could readily compute the transform of the shifted version just from the transform of the unshifted version, and that's-- you can see how that would work mathematically. Let's call the shifted version Y, then the transform of the shifted version is given by this expression. By the definition of shift, y of n is the same as x of n minus 1.

And now all we need to do is massage the math so it ends up looking like the definition of a Z-transform. So the way to do that would be to substitute m for n minus 1. We get something that looks more like a Z-transform. And if we bring this minus 1 out front, it looks exactly like a Z-transform pre-multiplied by z to the minus 1. So the delay theorem just says if I already know the Z-transform of a signal, then to find the Z-transform of the delayed version of that signal, simply multiply the original Z-transform by z to the minus 1.

OK? So so far, I've talked about two properties of Z-transforms, linearity and the delay property. And in combination, they let me do a lot of stuff with discrete systems. So think, for example, of a system, a discrete system that can be described by a linear difference equation with constant coefficients. If you can describe a system that way, then you can write the relationship between the input signal X and the output signal Y that looks like a difference equation.

Then, if you take into account that the Z-transform is both linear and has a simple representation for delays, I can take the Z-transform of that difference equation and get a new expression. So the difference equation represents an equality between two sums of time domain signals. Taking the Z-transform of that equality tells me some equivalent relationship of the Z-transforms.

So if I-- so I think about the left-hand side by linearity, I can find the Z-transform of this sum by finding the sum of the Z-transforms. And so this one's easy, right? The Z-transform of y of n is, by assumption, y of z. By linearity, the part I didn't show you, if I pre-multiply by b nought, it's pre-multiplied by b nought.

Because it's linear, I can just add the next term to it. The next term looks a lot like the previous term except that it's shifted, but shift is easy. You just put z to the minus 1 in front. So this term just becomes b one, z to the minus 1. This just becomes b 2, z to the minus 2, and the whole thing factors.

Same sort of thing happens on the X side, and you end up with a very simple statement. The Z-transform of a system that can be represented by a difference equation with constant coefficients is the ratio of two polynomials in z to the minus 1-- this is a polynomial in z to the minus 1. Or if I multiply top and bottom by z to the k, I can make it look like a ratio of polynomials in z. Yeah.

**AUDIENCE:** Why is it y to the x on the b 0 [INAUDIBLE]

**DENNIS FREEMAN:** Why is it y the x? Because of my poor typography. Thank you very much. That's completely wrong. OK, so I'll try to remember that when I get to my office and change it before I post it on the web. Thank you.

So this is y of z, this is y of z, this is y of z. That is-- and you can see I used [? e-max, ?] so I copied the formula and I got the single error to turn into two. It's always very good when you can do that, right? Something like that.

OK, so the point then is that simply by knowing that the Z-transform is linear and has a delay property, it results in a very simple statement about the Z-transform for a system that can be represented by such a difference equation. OK. Now we use a little bit of knowledge about polynomials. We use the idea that the fundamental theorem in algebra-- anybody have any idea what that is? Nah. Fundamental theorem of algebra, what?

**AUDIENCE:** That was a long time ago.

**DENNIS FREEMAN:** Long time ago. If it was long for you, think about when it was for me.

[LAUGHTER]

No, you can't think that way. Yes?

**AUDIENCE:** Are we [INAUDIBLE]

**DENNIS FREEMAN:** Wonderful. Everybody hear that? Because I can't repeat it. But everybody hear that?

An nth order polynomial has n roots. Roughly speaking. I'm not a mathematician, right? I'm an engineer. Right? An nth order polynomial has n roots.

OK, and then the factor theorem. Surprisingly enough, that says you can factor things. So the idea then is that if I can represent the Z-transform as a ratio of polynomials and z, there's a factored form. And that's the basis of a decomposition that we will make extensive use of.

You've already seen, extensively, the roots of the denominator. They're the poles. We will similarly define, because of this manipulation, the roots of the numerator. They're the zeros. OK?

So-- and it's pretty easy to think through, then, how there is a relationship-- it's all about relationships, right? When I introduce something, I want to think about how the thing I just said relates to everything else I've ever said. There's a simple relationship between poles and regions of convergence. Turns out the regions of convergence are always going to be circles. That has-- circles in the z plane. That has to do with things like convergence of geometric sequences.

If I build my system out of adders and delays and gains, then I have that complex set of reasoning that gives rise to the idea that I have polynomials. That's going to be-- that's going to give rise to, if you think about poles, partial fractions, each of those is going to have some kind of a characteristic response. It's going to be a geometric sequence. Each of those is going to have a convergence property that has something to do with a circle in the z plane. Each of those circles is going to be bounded by a pole.

So the upshot of all that stuff is there's a relationship between poles and regions of convergence. Regions of convergence are always going to be circles in the z plane, and they're always going to be bounded by a pole. And we've seen an example of that already.

If we had a geometric sequence that was defined only to the right of n equals 0-- if it is non-zero, only at n equals 0 or bigger-- then we get convergence inside some region defined by when the base has an absolute value that's less than 1. That's a circle in the z plane. And the pole, which is alpha, turns into the edge of that circle of convergence. So the regions of convergence for these kinds of systems will always be circles in the z plane, always bounded by a pole.

OK, enough of my talking. What DC-- what DT signal has the following Z-transform? I want to know a DT signal that has transform of the form z over z minus 7/8 with a region of convergence inside absolute value of z equals 7/8.

So what's the DT signal that has that Z-transform?

**AUDIENCE:** It would be Y to the n is equal to x to the n plus 7/8y to the n minus 1?

**DENNIS FREEMAN:** Sounded like a difference equation. Say it again.

**AUDIENCE:** Oh yeah. Isn't that what you're looking for?

**DENNIS FREEMAN:** No. I wanted the signal.

**AUDIENCE:** Oh, OK.

**DENNIS FREEMAN:** So I want to think about-- this is a little confusing. I apologize if I didn't say this clearly. We motivated the idea of Z-transform by looking at systems, but the result, the Z-transform's just a relationship. It's a map between a function of n and a function of z. So we can do that for every signal.

So if I tell you the function of z, you can figure out the function of n. The question here is intended-- what I intended was, what's the function of n that corresponds to that function of z? Yes?

**AUDIENCE:** [INAUDIBLE]

**DENNIS FREEMAN:** 7/8 to the n, u of n. That sounds like something we did before. 7/8 to the n, u of n. That sounds like something we did before, actually. Yes, no?

Something we did before is a good thing, right? That's one of the general rules, right? When I ask a question, look at what we did before. That's a good rule. Is that the same Z-transform we did before? Yes?

**AUDIENCE:** [INAUDIBLE]

**DENNIS FREEMAN:** Yes, yes. The region of convergence that we did before was outside 7/8, and the region of convergence I asked for in this problem is inside 7/8 So what's the effect of switching the region of convergence? What happens if I switch the region of convergence?

**AUDIENCE:** Does z change? Does the value of z change?

**DENNIS FREEMAN:** Z. It's hard to talk about the value of z, right? Z.

**AUDIENCE:** [INAUDIBLE]

**DENNIS FREEMAN:** Excuse me. So z is defined-- the Z-transform is defined this way. So I want to say that h of z is always z over z minus 7/8. What happens if I say the region switched?

Well, it says something about convergence. What's convergence have to do with? Convergence has to do with, well, I'm thinking about h of z is some sum over n of h of n, z to the minus n. So convergent has to do with which ones of those ns can be in the sum, because some of these terms, z to the minus n, when I switch the region, I consider a different family of z.

In the first one, that sum had to converge outside the circle. And in the question of interest now, it has to converge inside. So it's a different set of zs for which the sum has to converge.

There's a way you can think about that. Think about that sum-- think about exploding that sum. In general, we're going to go from minus infinity to infinity, so explode that. So we get a whole bunch of stuff.

Then we get up to h of minus 2, z squared. Then we have h of minus 1, z. Then we have h of 0, z to the 0, which is 1. Then we have h of 1, z to the minus 1. Then we have h of 2, z to the minus 2, et cetera. Right? That's what the Z-transform always looks like.

Which of those terms are the most convergent when I have a z with a large magnitude? So I'm thinking about the n equals minus 2, n equals minus 1, n equals 0. I'm thinking about all the terms and Ys. Which of those terms is the most convergent if z has a large magnitude?

**AUDIENCE:** I think it would be z of negative n. Like, any negative.

**DENNIS FREEMAN:** So they get increasingly convergent as I go to the right. If I have z with a big magnitude, each term is getting increasingly convergent as I go to the right. That means that these numbers h 0, h 1, h 2, h 3, they can keep being some finite number. They don't need to be 0. And it will increasingly come closer to 0 as I keep going to the right.

The implication of that is something that's very important. So the implication of that is that a right-sided signal-- signal, not single-- has a-- maps to an outside region. If I want the sum to get increasingly convergent for large values of z, outside regions-- outside regions have to take all the values, no matter how big they get. OK, I want things to be on the right. I don't want things to be on the left. Things on the left become decreasingly convergent.

And a corollary of that, left-sided signals map to inside regions. So in fact, it's not a fluke that this right-handed signal mapped to this outside region. That's where the convergence for things to the right are the best.

OK. Now I've just told you everything you need to know. What's it going to-- what do I need to have happen if I want it to converge to the inside region? Flip the signal. More or less, I want it to go to the left. I don't want it to go to the right.

So the way I can think about that is by thinking about the functional form is the same for the inside and the outside region. That means they have the same difference equation, because you can find the difference equation from the functional form. But the way I want to think about it is propagating the signal that came in to the right or to the left. So rather then iterating forward in time, which is what we did before, one way I can think about it is, let's iterate backwards in time.

Rather than solving for y of n plus 1 in terms of y of n, solve instead for y of n in terms of y of n plus 1. Run it backwards. Difference equation doesn't care. If the difference equation doesn't care, the functional form will be the same. So run the difference equation backwards.

So if you think about doing that, rest starts to say, the system starts-- starts means future times, OK, because we're flipped. So the signal starts at 0. That means it starts-- that means at large values of n, the signal y is 0. So I fill in this table with a bunch of zeros for big values of n. So n's decreasing this way. I'm working from toward-- 0 toward the left.

So I assume that I start at rest. That means the output is 0 for times on the right. I want to find the unit sample response, so x is 1 only at n equals 0. And now I just compute each entry by sticking it into the difference equation.

And so I've done that. So you stick these values into the difference equation, and that lets you compute y of minus 1. So substitute y of minus 1. Y of minus 1, it depends on y of 0 and x of 0. Similarly for all the entries, and then I can make a plot of that, and I get a function that looks like this.

It's a geometric. I'm not surprised by that. The geometric is kind of facing the wrong way. It was 7/8 to the end. It looked convergent. This one looks divergent that way, but I'm multiplying these h numbers, which are blowing up that way, by numbers that look like z squared. So if I make z squared small enough, it'll still converge.

That's the idea. OK? By flipping the region of convergence, I've changed the magnitudes of these numbers, and I've changed which side converges best. I've made something that-- it's an inside region. It was converging for z values inside some region, so it's become left-sided.

So the idea then is that I have two different kinds of time signals that are both associated with the same functional form of z. They differ by the region of convergence. That's why when you tell me a Z-transform, you have to tell me the region of convergence. OK?

OK. So I've given some exercises. I'm running out of time. The best thing to do is to think about the exercises offline.

I could lead you through the math. That's never very inspirational. So what I wanted to show you is that there's many ways that you can go about expressing a complicated form by partial fractions to get a simpler form that can then be inverted this way. So that's the point of this exercise.

So I went through in the notes, and the notes are online. So in the notes, I went through three different ways you can think of the answer, and all of those answers give you the same functional form, right? So it was an exercise in thinking through how you do partial fractions.

But there's one more thing that I want to talk about, and that is that a lot of these problems-- one of the biggest uses of Z-transforms is to solve difference equations. Z-transforms are great for that. I've already talked about that.

With-- by using a Z-transform, you can take a difference equation, think about the difference equation, think about the input, take the Laplace transform of everything you get-- Laplace transform-- a Z-transform, sorry. Slipped. Next time it will be Laplace transform. Take a Z-transform. You end up with a Z-transform, and then the trick is to recognize the inverse Z-transform.

There are ways of thinking about that as a mathematician. Those ways are not easy, so by and large, we will never do this. If you would like to do that, I highly recommend course 18. OK? It is certainly something that people in course 18 do all the time, but there are always simpler ways that we will do it, and those simpler ways derive from thinking about properties of the z-transform. And we'll think more about those as we go forward in the course.

So the point of today was to emphasize that there are lots of different ways of thinking about DT systems. You should be able to think of all the relations between them. And in particular, today we talked about this thing, a mathematical relationship for how you can go from a unit sample response to a system function. See you next week.