6.003: Signals and Systems

CT Frequency Response and Bode Plots

October 18, 2011

Mid-term Examination #2

Wednesday, October 26, 7:30-9:30pm,

No recitations on the day of the exam.

Coverage: Lectures 1–12
Recitations 1–12
Homeworks 1–7

Homework 7 will not be collected or graded. Solutions will be posted.

Closed book: 2 pages of notes ($8\frac{1}{2} \times 11$ inches; front and back).

No calculators, computers, cell phones, music players, or other aids.

Designed as 1-hour exam; two hours to complete.

Review sessions during open office hours.

Conflict? Contact before Friday, Oct. 21, 5pm.

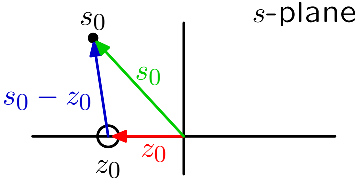
Review: Frequency Response

Complex exponentials are eigenfunctions of LTI systems.

$$e^{s_0 t} \longrightarrow \boxed{H(s)} \longrightarrow H(s_0)\, e^{s_0 t}$$

$H(s_0)$ can be determined graphically using vectorial analysis.

$$H(s_0) = K \frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$

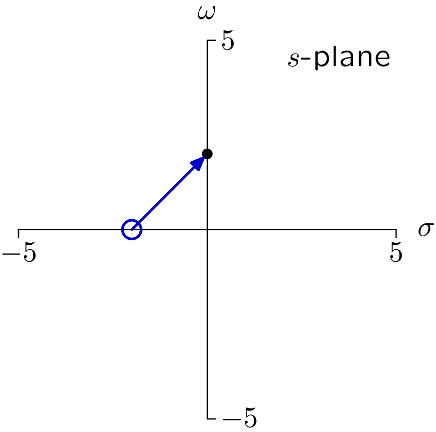


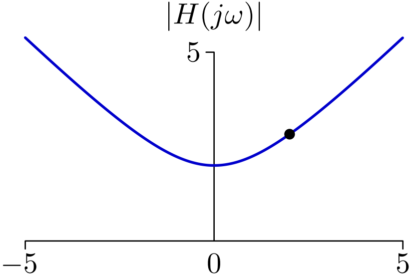
Response of an LTI system to an eternal cosine is an eternal cosine: same frequency, but scaled and shifted.

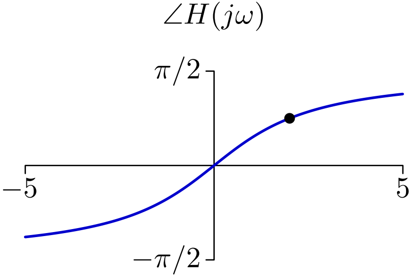
$$\cos(\omega_0 t) \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega_0)| \cos\big(\omega_0 t + \angle H(j\omega_0)\big)$$

Frequency Response: $H(s)|_{s\leftarrow j\omega}$

$$H(s) = s - z_1$$

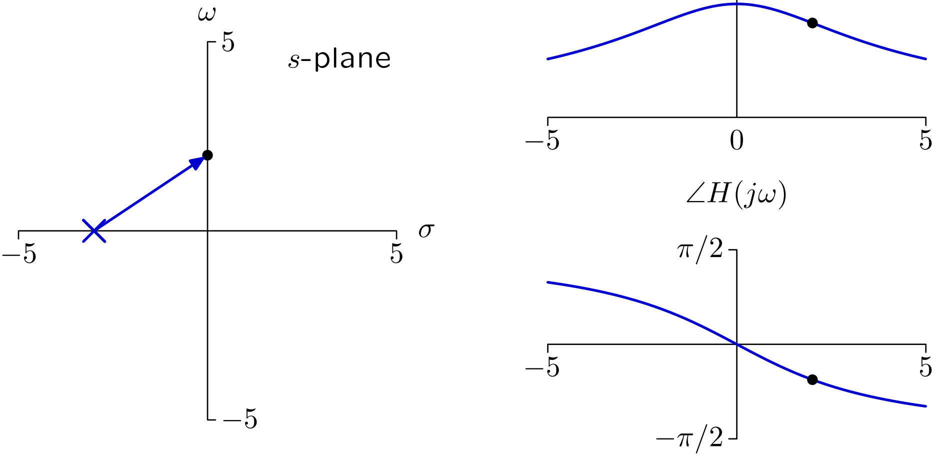






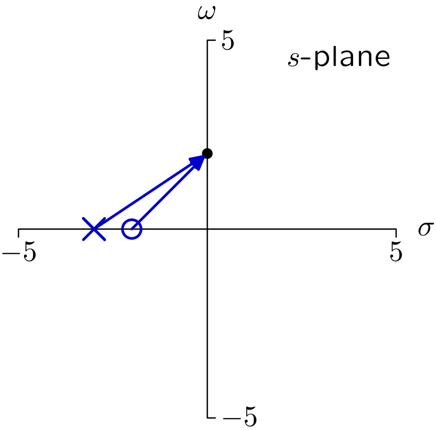
Frequency Response: $H(s)|_{s\leftarrow j\omega}$

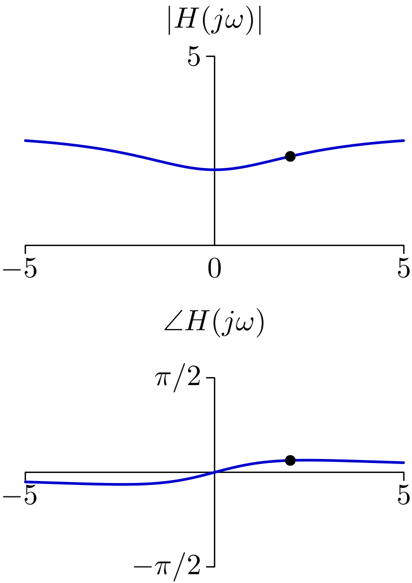
$$H(s) = \frac{9}{s - p_1}$$



Frequency Response: $H(s)|_{s \leftarrow j\omega}$

$$H(s) = 3\,\frac{s - z_1}{s - p_1}$$





Poles and Zeros

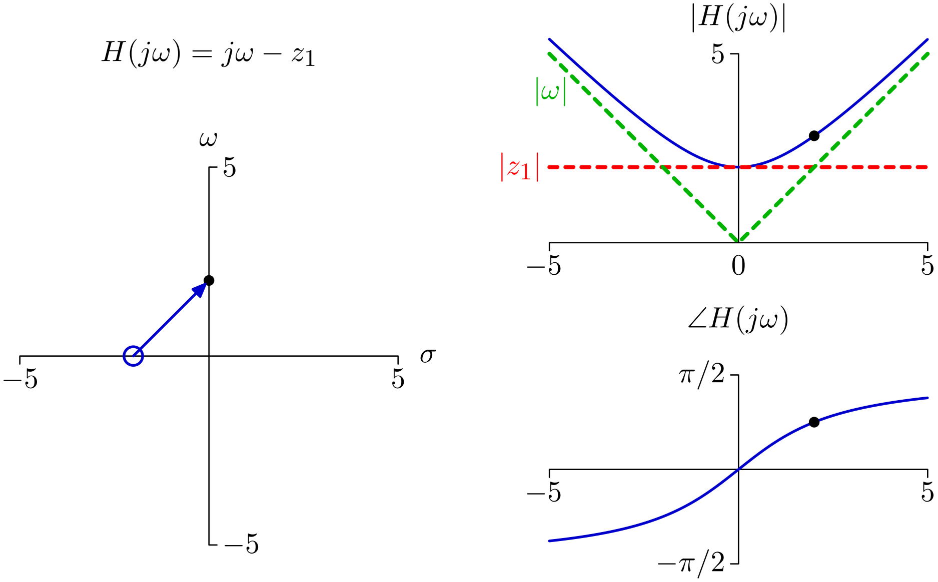
Thinking about systems as collections of poles and zeros is an important design concept.

- simple: just a few numbers characterize entire system
- powerful: complete information about frequency response

Today: poles, zeros, frequency responses, and Bode plots.

Asymptotic Behavior: Isolated Zero

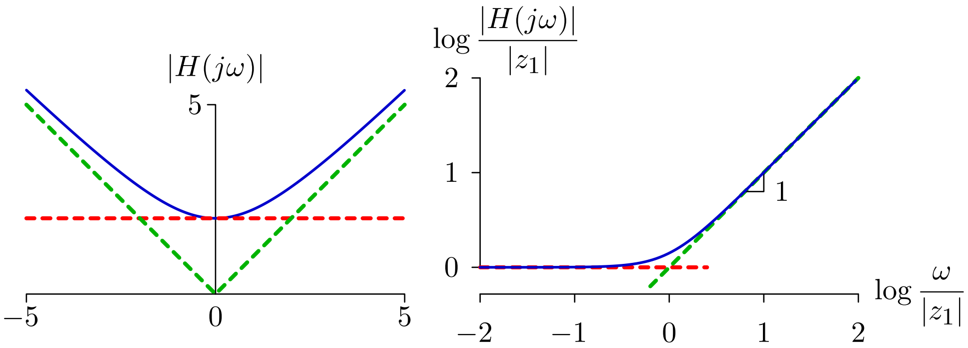
The magnitude response is simple at low and high frequencies.



Asymptotic Behavior: Isolated Zero

Two asymptotes provide a good approxmation on log-log axes.

$$H(s) = s - z_1$$



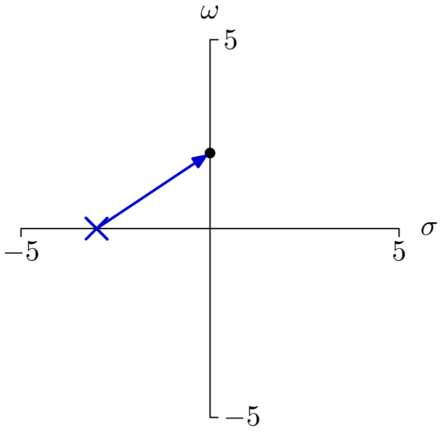
$$\lim_{\omega \to 0} |H(j\omega)| = |z_1|$$

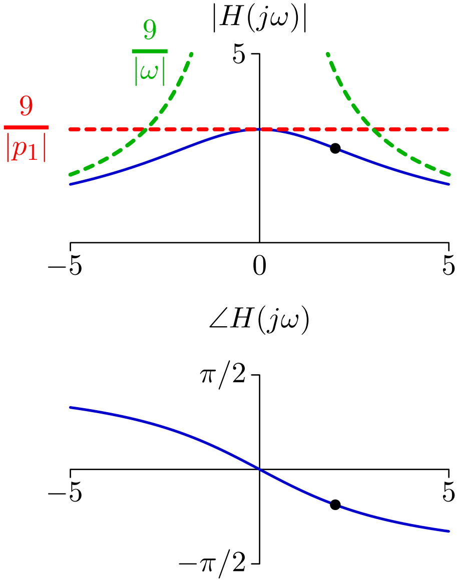
$$\lim_{\omega \to \infty} |H(j\omega)| = \omega$$

Asymptotic Behavior: Isolated Pole

The magnitude response is simple at low and high frequencies.

$$H(s) = \frac{9}{s - p_1}$$





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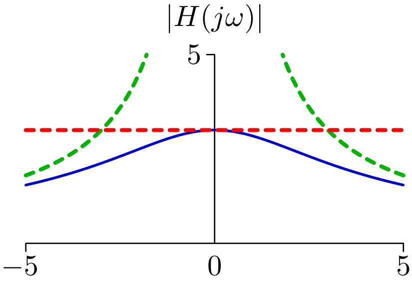

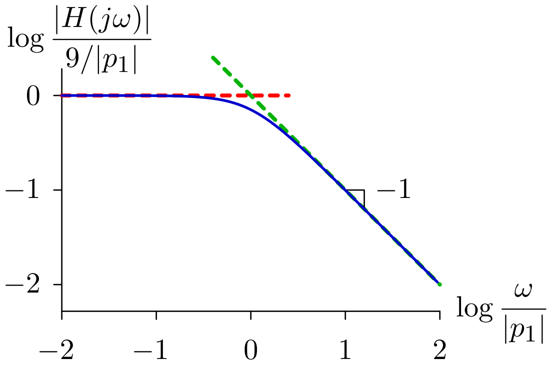

$$\lim_{\omega \to 0} |H(j\omega)| = \frac{9}{|p_1|}$$

$$\lim_{\omega \to \infty} |H(j\omega)| = \frac{9}{\omega}$$

Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

$$H_1(s) = \frac{1}{s+1} \qquad \text{and} \qquad H_2(s) = \frac{1}{s+10}$$

The former can be transformed into the latter by

1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

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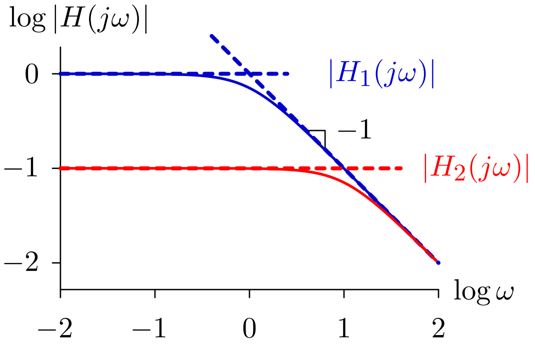

Check Yourself

Compare log-log plots of the frequency-response magnitudes of the following system functions:

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The former can be transformed into the latter by 3

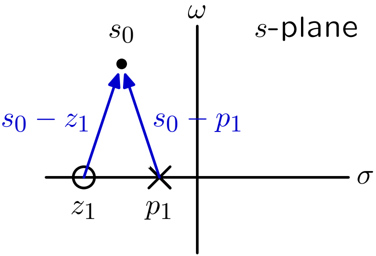
1. shifting horizontally
2. shifting and scaling horizontally
3. shifting both horizontally and vertically
4. shifting and scaling both horizontally and vertically
5. none of the above

no scaling in either vertical or horizontal directions !

Asymptotic Behavior of More Complicated Systems

Constructing $H(s_0)$.

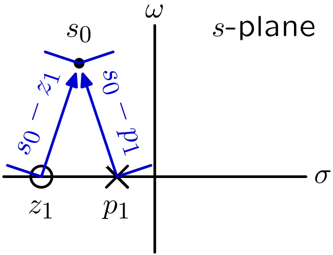
$$H(s_0) = K \frac{\displaystyle\prod_{q=1}^{Q}(s_0 - z_q)}{\displaystyle\prod_{p=1}^{P}(s_0 - p_p)} \quad \begin{array}{l} \leftarrow \text{ product of vectors for zeros} \\ \\ \leftarrow \text{ product of vectors for poles} \end{array}$$



Asymptotic Behavior of More Complicated Systems

The magnitude of a product is the product of the magnitudes.

$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q}(s_0 - z_q)}{\prod_{p=1}^{P}(s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q}|s_0 - z_q|}{\prod_{p=1}^{P}|s_0 - p_p|}$$



Bode Plot

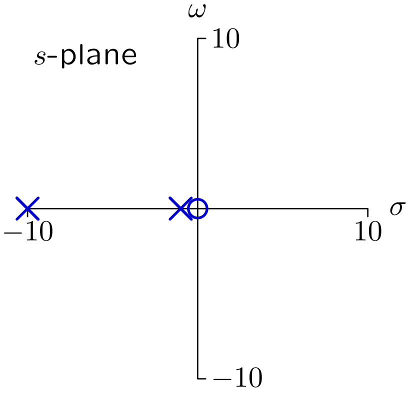
The log of the magnitude is a sum of logs.

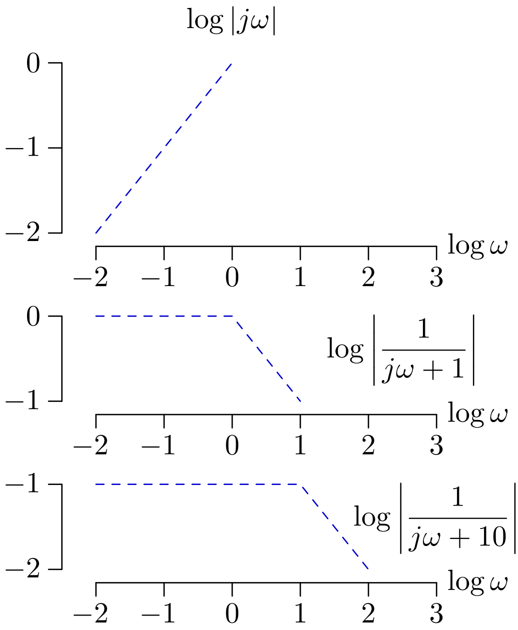
$$|H(s_0)| = \left| K \frac{\prod_{q=1}^{Q} (s_0 - z_q)}{\prod_{p=1}^{P} (s_0 - p_p)} \right| = |K| \frac{\prod_{q=1}^{Q} \left| s_0 - z_q \right|}{\prod_{p=1}^{P} \left| s_0 - p_p \right|}$$

$$\log |H(j\omega)| = \log |K| + \sum_{q=1}^{Q} \log \left| j\omega - z_q \right| - \sum_{p=1}^{P} \log \left| j\omega - p_p \right|$$

Bode Plot: Adding Instead of Multiplying

$$H(s) = \frac{s}{(s+1)(s+10)}$$





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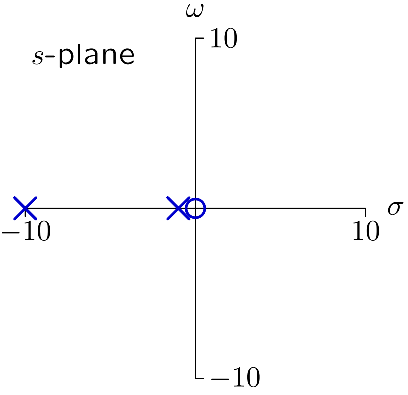

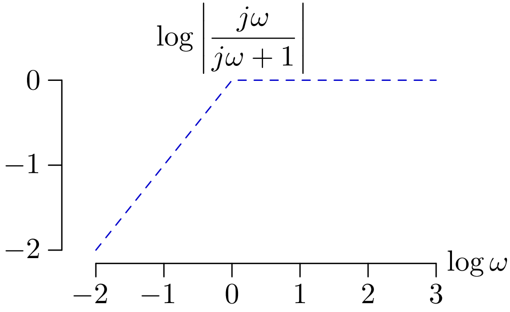

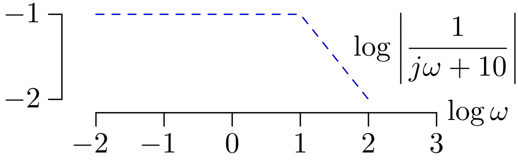

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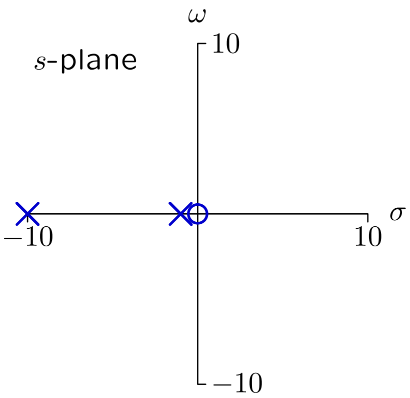

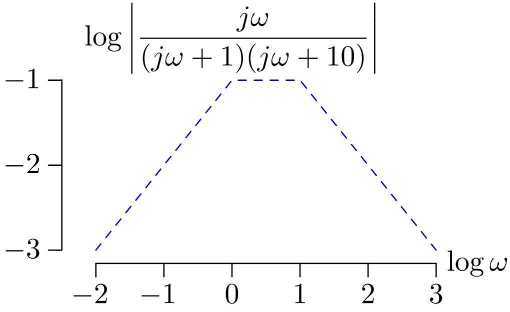

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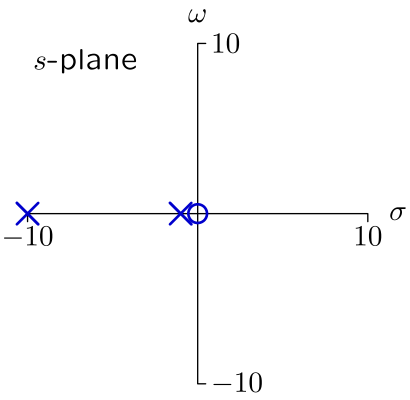

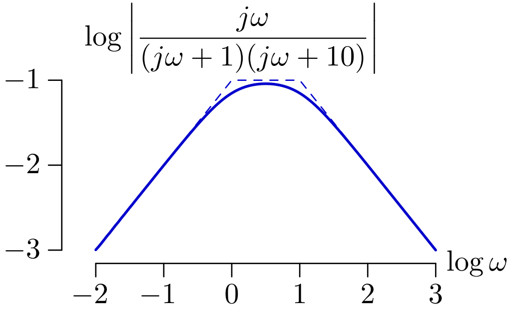

Asymptotic Behavior: Isolated Zero

The angle response is simple at low and high frequencies.

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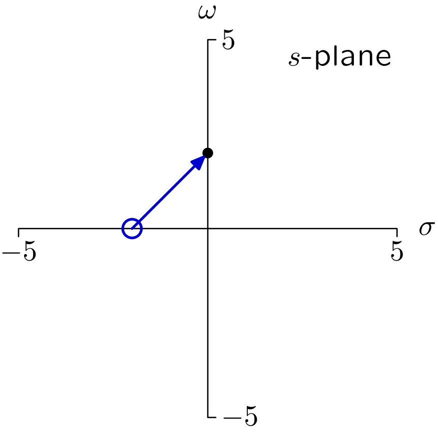
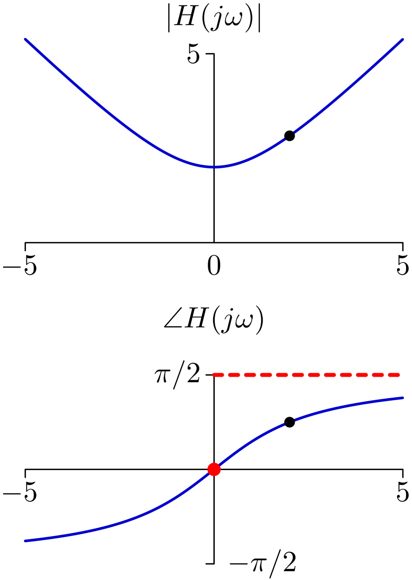
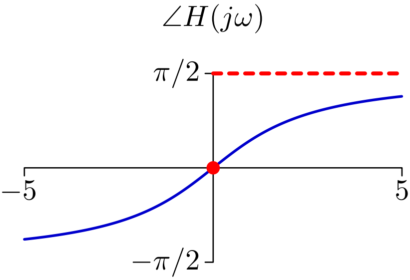


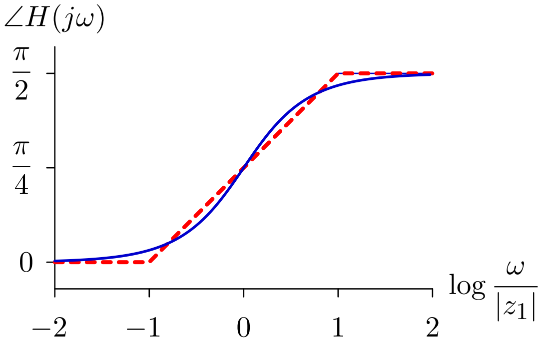


Asymptotic Behavior: Isolated Zero

Three straight lines provide a good approxmation versus log ω.

$$H(s) = s - z_1$$





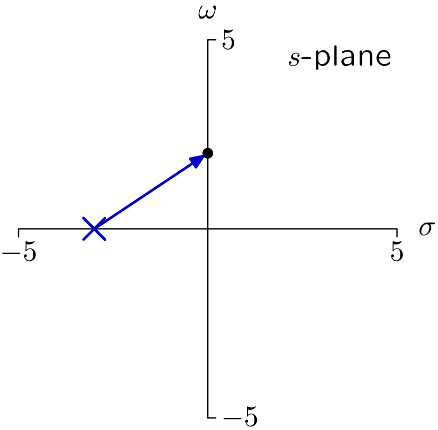
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

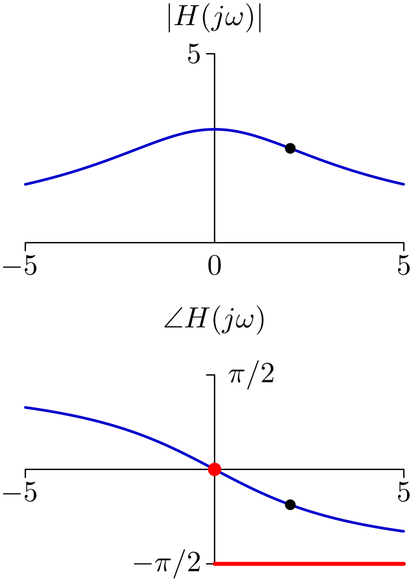
$$\lim_{\omega \to \infty} \angle H(j\omega) = \pi/2$$

Asymptotic Behavior: Isolated Pole

The angle response is simple at low and high frequencies.

$$H(s) = \frac{9}{s - p_1}$$





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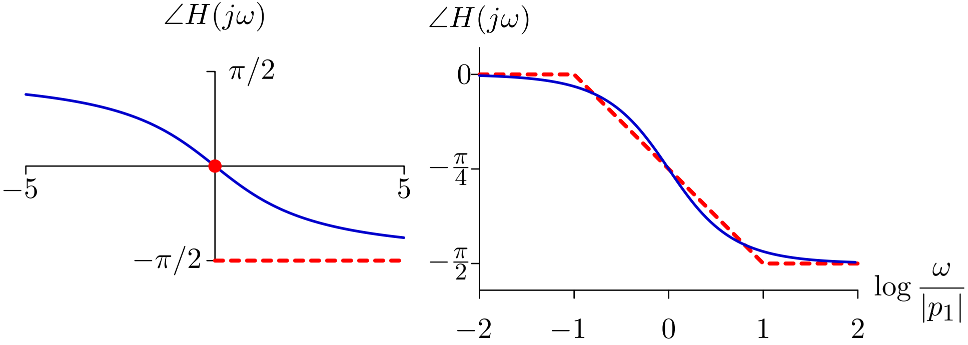


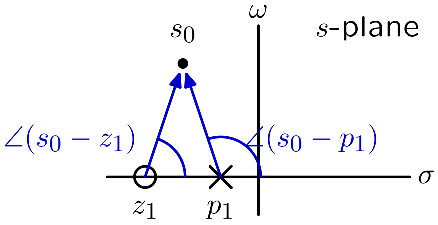
$$\lim_{\omega \to 0} \angle H(j\omega) = 0$$

$$\lim_{\omega \to \infty} \angle H(j\omega) = -\pi/2$$

Bode Plot

The angle of a product is the sum of the angles.

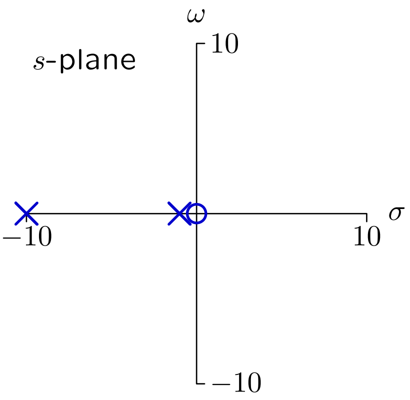
$$\angle H(s_0) = \angle \left(K \frac{\prod_{q=1}^{Q} (s_0 - z_q)}{\prod_{p=1}^{P} (s_0 - p_p)} \right) = \angle K + \sum_{q=1}^{Q} \angle \left(s_0 - z_q\right) - \sum_{p=1}^{P} \angle \left(s_0 - p_p\right)$$

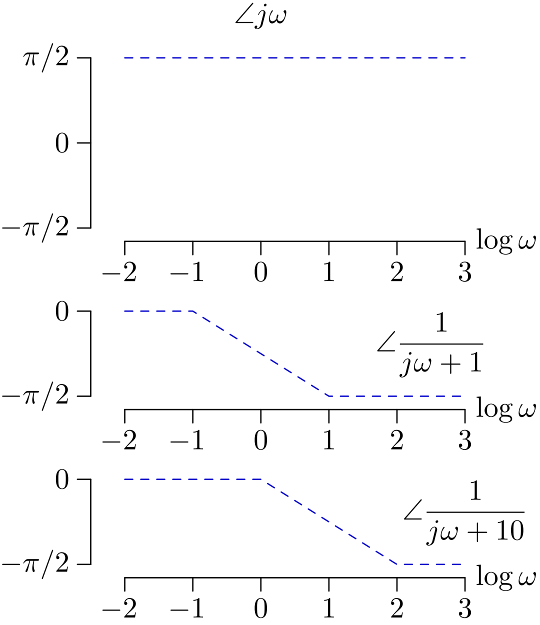


The angle of K can be 0 or π for systems described by linear differential equations with constant, real-valued coefficients.

Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$





Bode Plot

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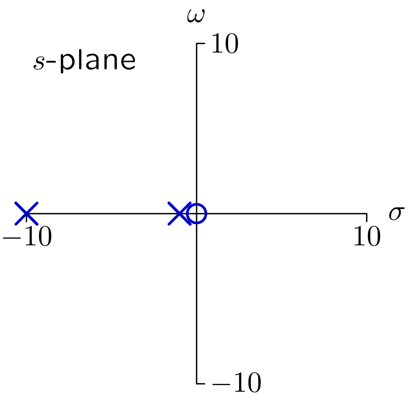
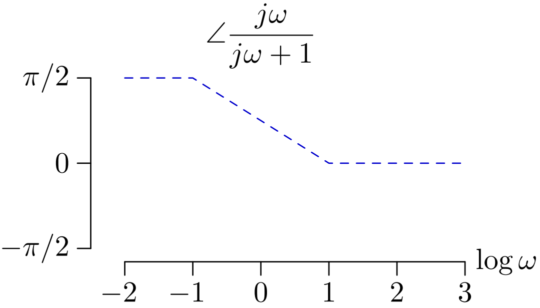
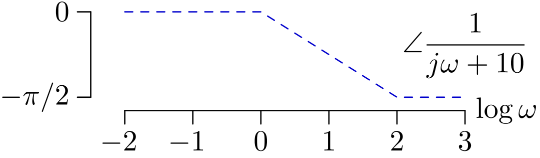
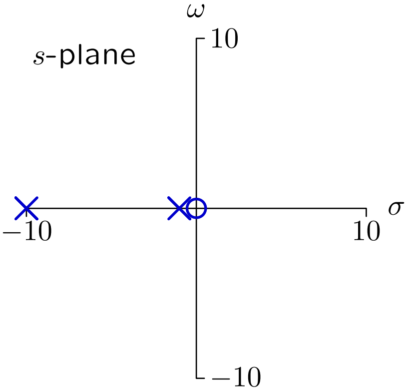




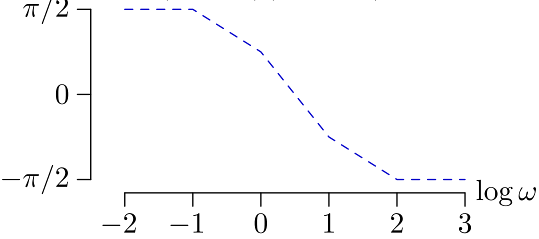


Bode Plot

$$H(s) = \frac{s}{(s+1)(s+10)}$$



$$\angle \frac{j\omega}{(j\omega+1)(j\omega+10)}$$



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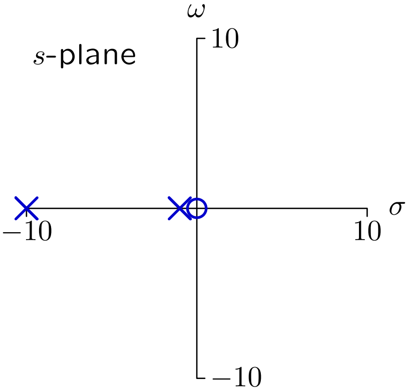
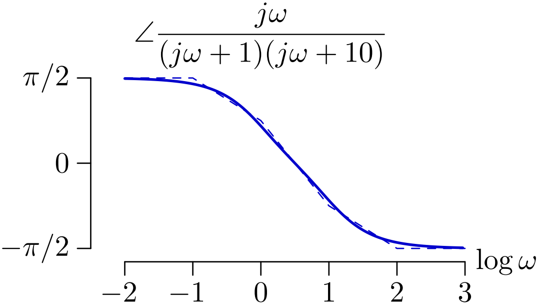




From Frequency Response to Bode Plot

The magnitude of $H(j\omega)$ is a product of magnitudes.

$$|H(j\omega)| = |K| \frac{\displaystyle\prod_{q=1}^{Q} \left|j\omega - z_q\right|}{\displaystyle\prod_{p=1}^{P} \left|j\omega - p_p\right|}$$

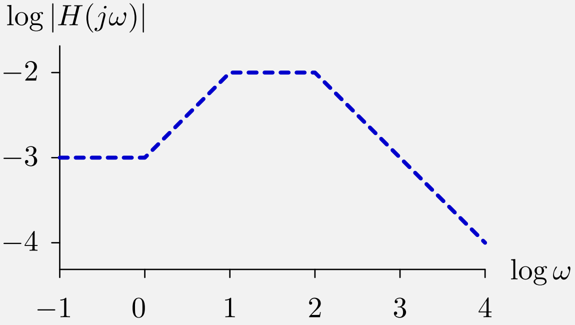
The log of the magnitude is a sum of logs.

$$\log|H(j\omega)| = \log|K| + \sum_{q=1}^{Q} \log\left|j\omega - z_q\right| - \sum_{p=1}^{P} \log\left|j\omega - p_p\right|$$

The angle of $H(j\omega)$ is a sum of angles.

$$\angle H(j\omega) = \angle K + \sum_{q=1}^{Q} \angle\left(j\omega - z_q\right) - \sum_{p=1}^{P} \angle\left(j\omega - p_p\right)$$

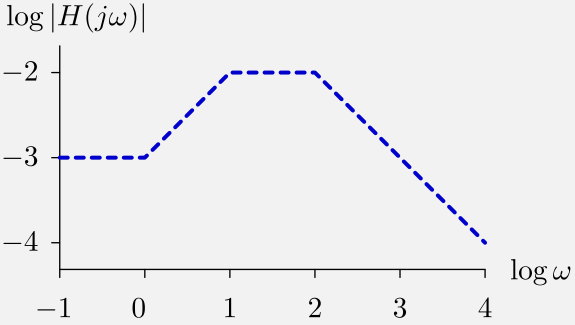
Check Yourself



Which corresponds to the Bode approximation above?

1. $\dfrac{1}{(s+1)(s+10)(s+100)}$
2. $\dfrac{s+1}{(s+10)(s+100)}$
3. $\dfrac{(s+10)(s+100)}{s+1}$
4. $\dfrac{s+100}{(s+1)(s+10)}$
5. none of the above

Check Yourself



Which corresponds to the Bode approximation above? 2

1. $\dfrac{1}{(s+1)(s+10)(s+100)}$

2. $\dfrac{s+1}{(s+10)(s+100)}$

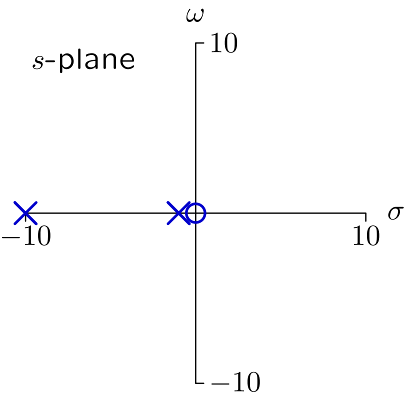
3. $\dfrac{(s+10)(s+100)}{s+1}$

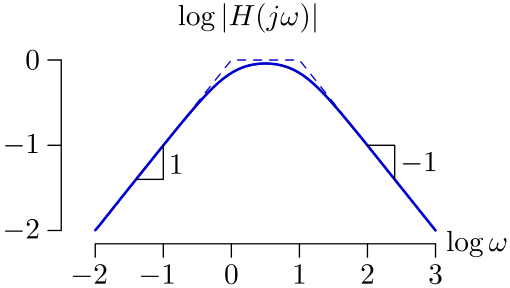
4. $\dfrac{s+100}{(s+1)(s+10)}$

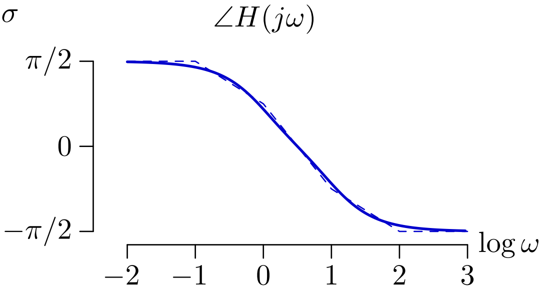
5. none of the above

Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$







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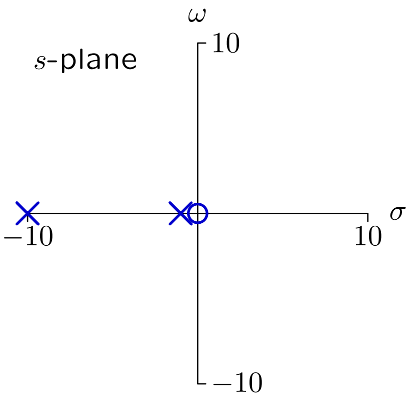

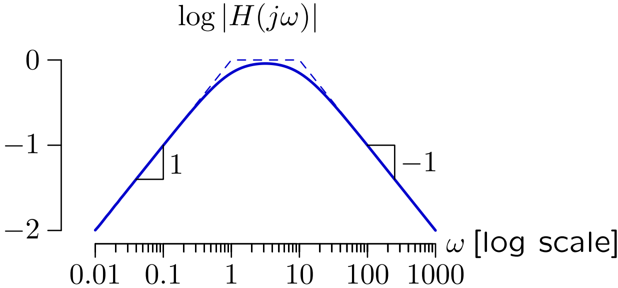

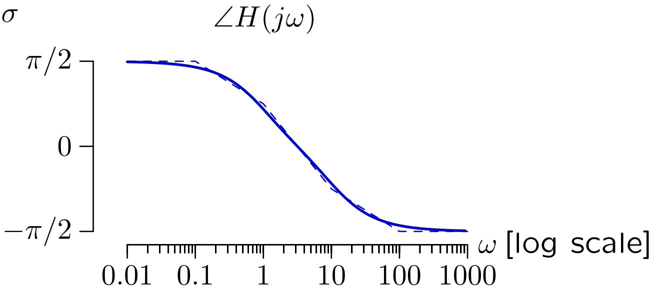

Bode Plot: dB

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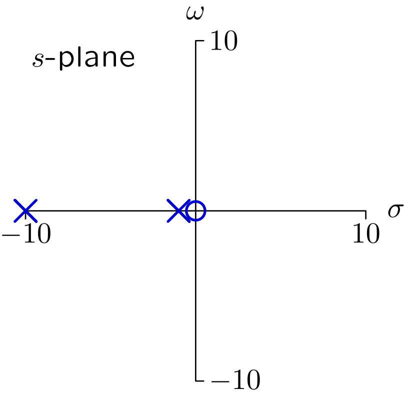
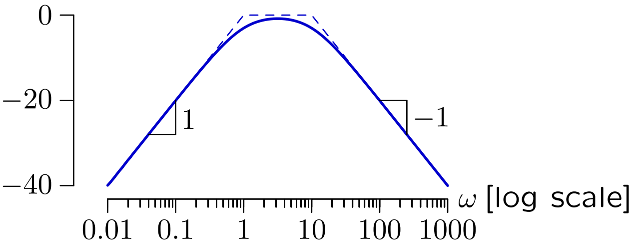
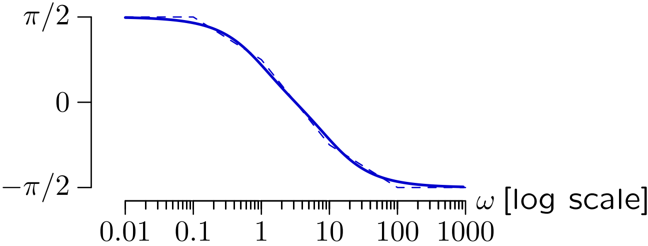


$|H(j\omega)|$[dB]$= 20\log_{10}|H(j\omega)|$

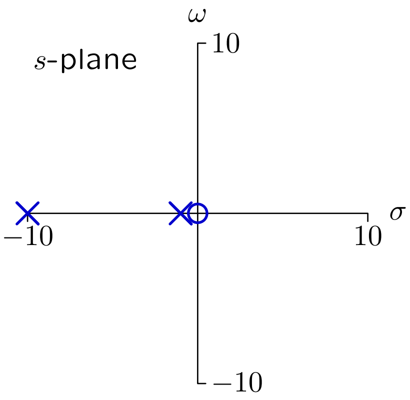


$\angle H(j\omega)$



Bode Plot: dB

$$H(s) = \frac{10s}{(s+1)(s+10)}$$



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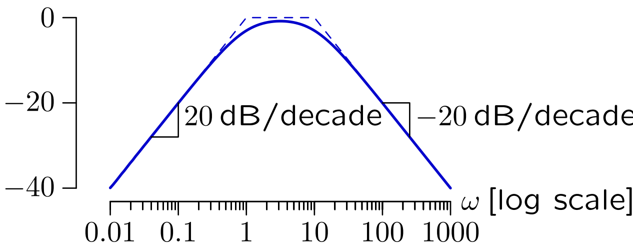

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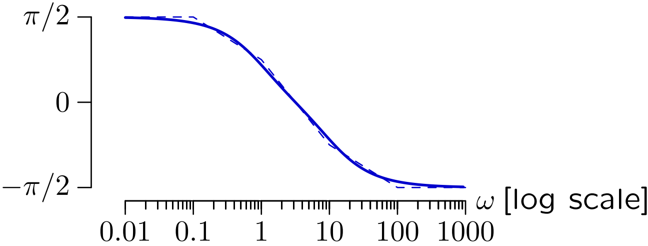
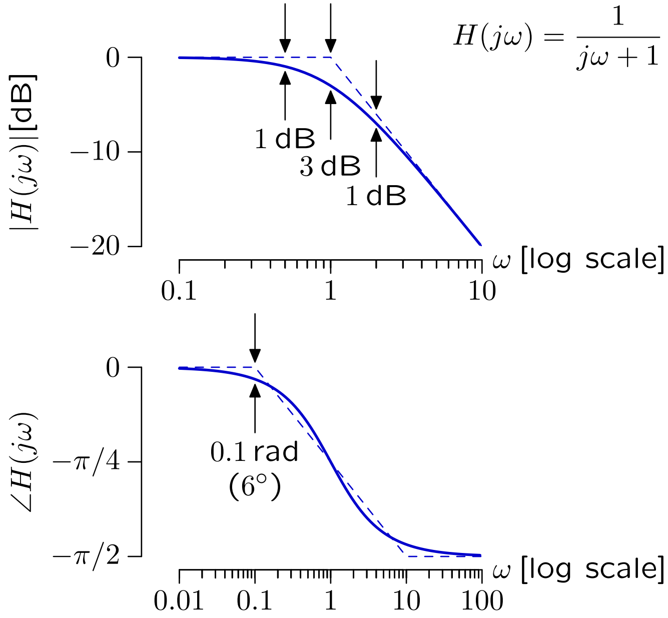


Bode Plot: Accuracy

The straight-line approximations are surprisingly accurate.

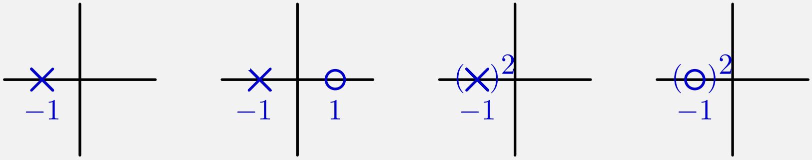
$$H(j\omega) = \frac{1}{j\omega + 1}$$



X	$20\log_{10} X$
1	0 dB
$\sqrt{2}$	≈ 3 dB
2	≈ 6 dB
10	20 dB
100	40 dB

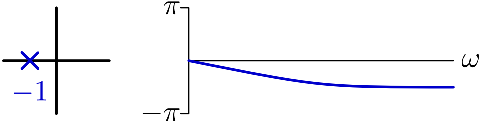
Check Yourself

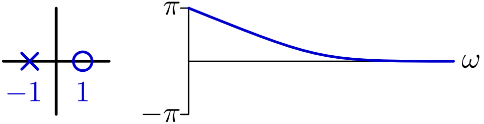
Could the phase plots of any of these systems be equal to each other? [caution: this is a trick question]

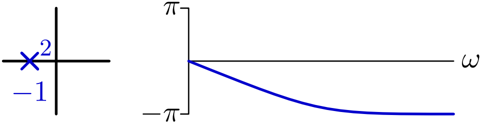


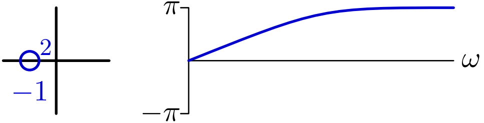
1 2 3 4

Check Yourself

1. 

2. 

3. 

4. 

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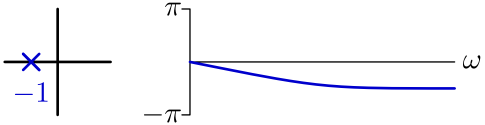
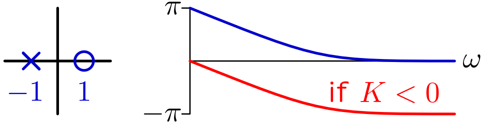
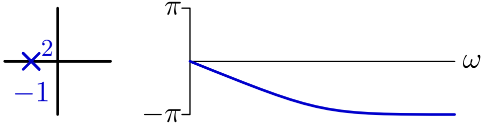
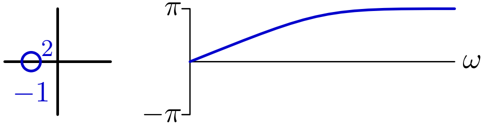
1. 

2. 

3. 

4. 

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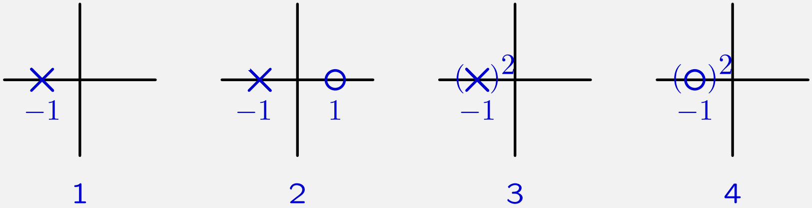
Could the phase plots of any of these systems be equal to each other? [caution: this is a trick question] yes

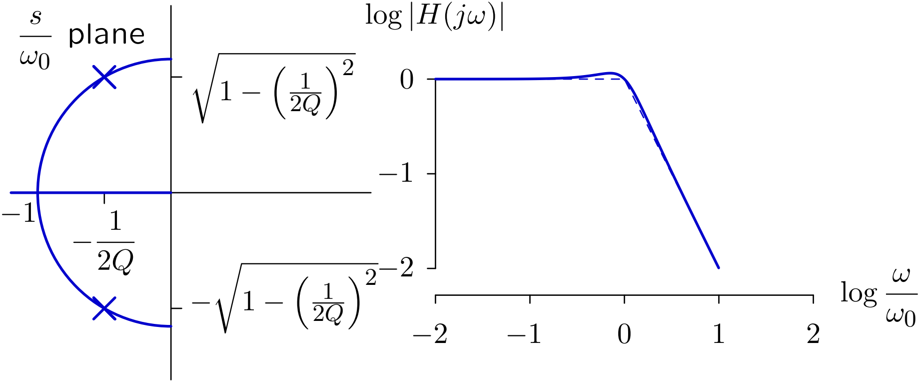


phase of 2 could be same as phase of 3: depends on sign of K

Frequency Response of a High-Q System

The frequency-response magnitude of a high-Q system is peaked.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



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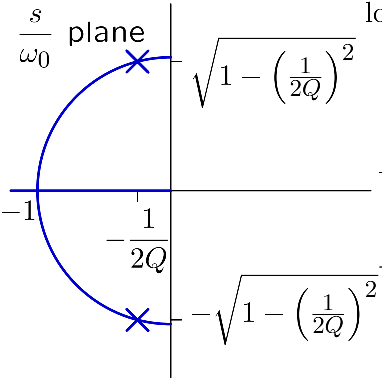

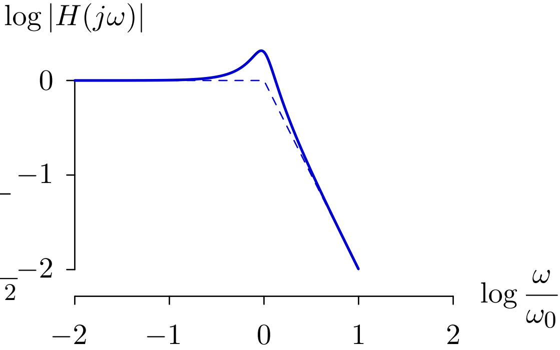

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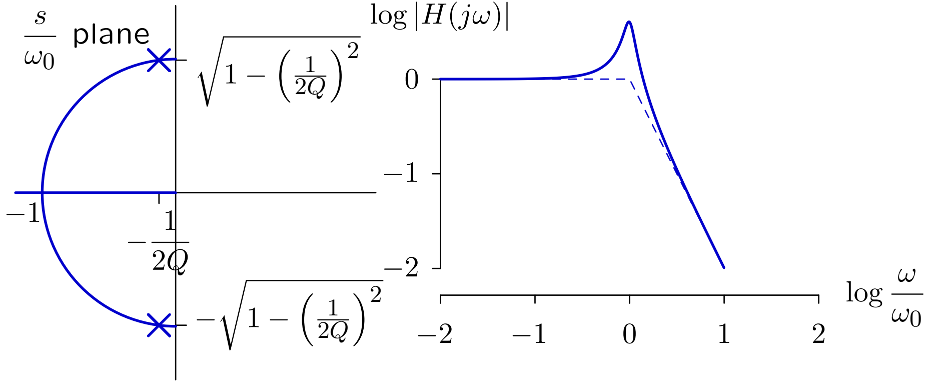

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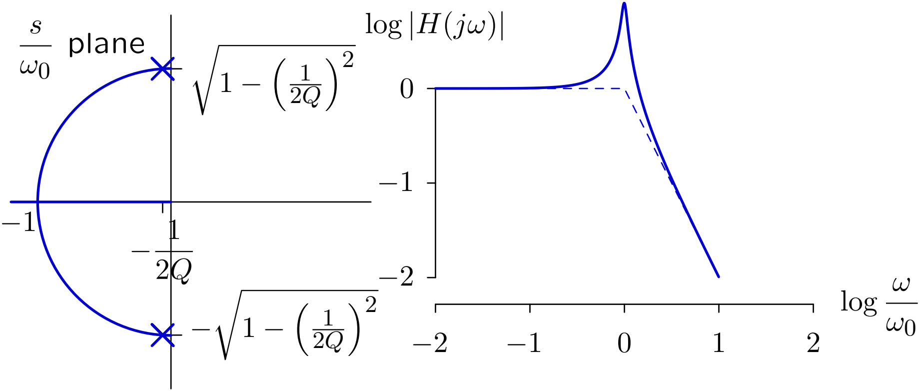

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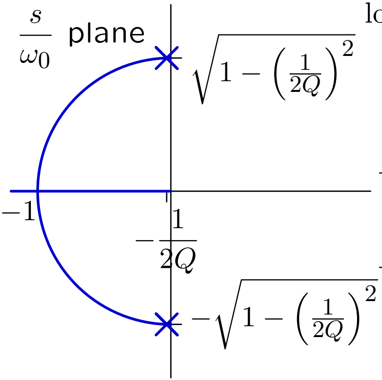
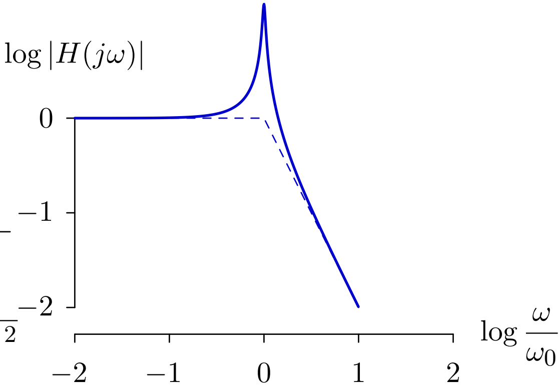
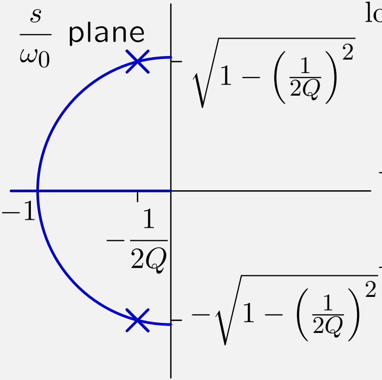


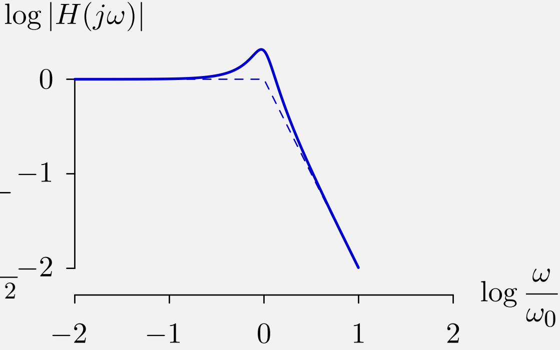


Check Yourself

Find dependence of peak magnitude on Q (assume $Q > 3$).

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



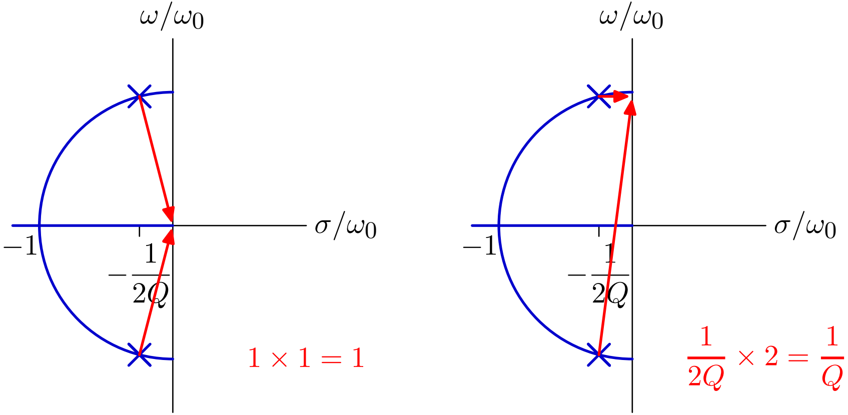


Check Yourself

Find dependence of peak magnitude on Q (assume $Q > 3$).

Analyze with vectors.

low frequencies high frequencies



Peak magnitude increases with Q!

Frequency Response of a High-Q System

As Q increases, the width of the peak narrows.

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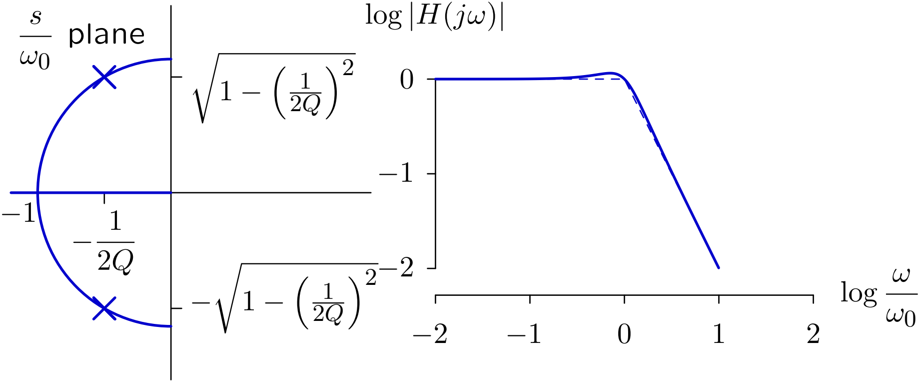

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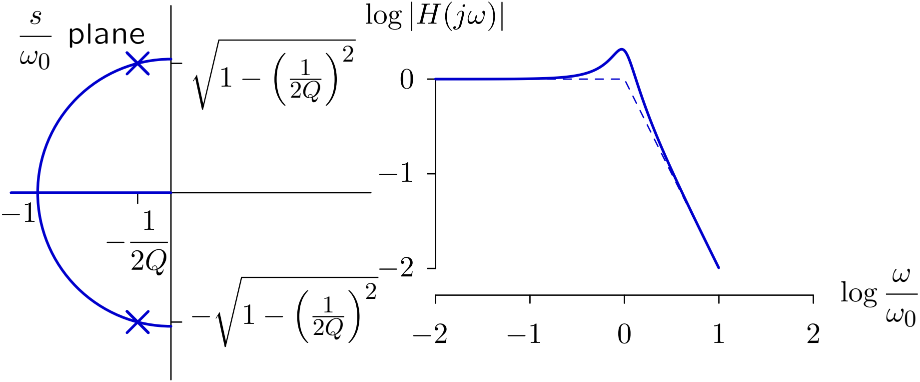

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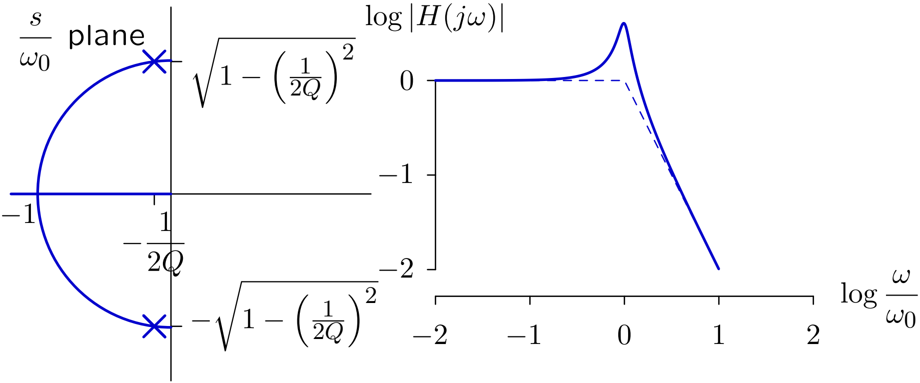

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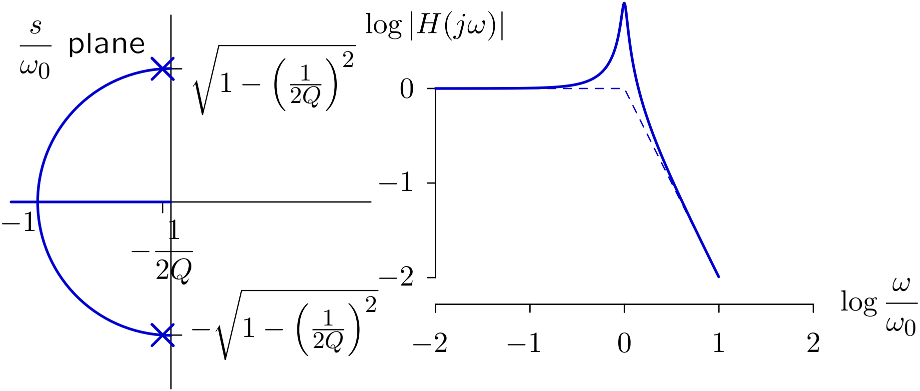

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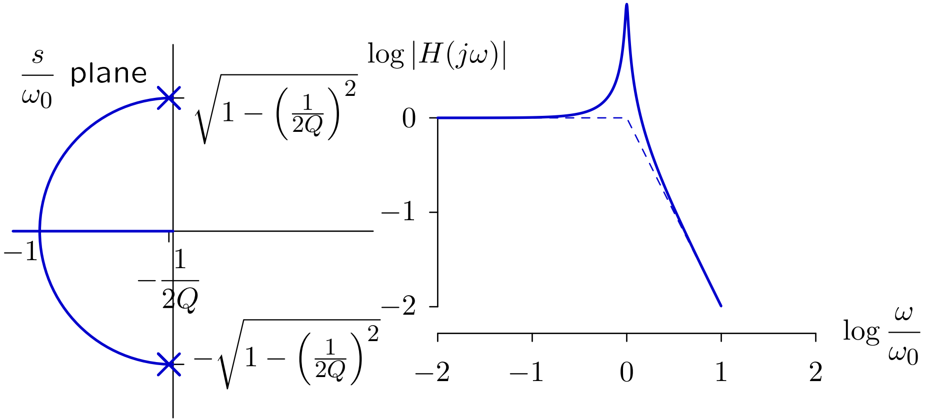
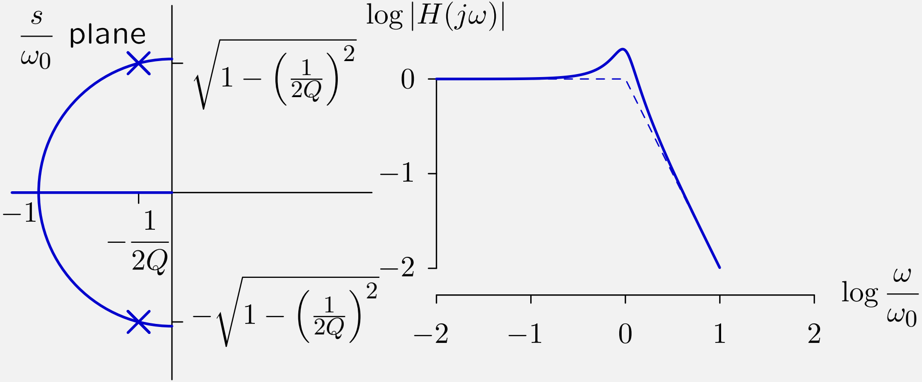


Check Yourself

Estimate the "3dB bandwidth" of the peak (assume $Q > 3$).

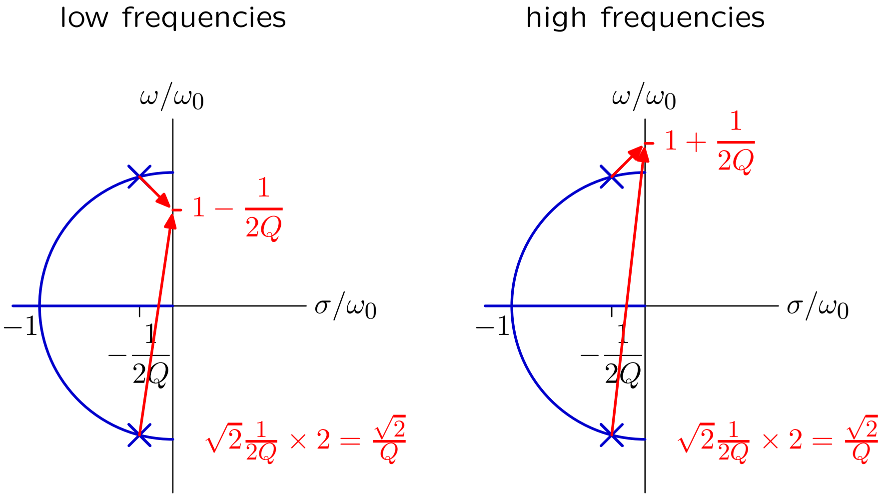
Let ω_l (or ω_h) represent the lowest (or highest) frequency for which the magnitude is greater than the peak value divided by $\sqrt{2}$. The 3dB bandwidth is then $\omega_h - \omega_l$.



Check Yourself

Estimate the "3dB bandwidth" of the peak (assume $Q > 3$).

Analyze with vectors.

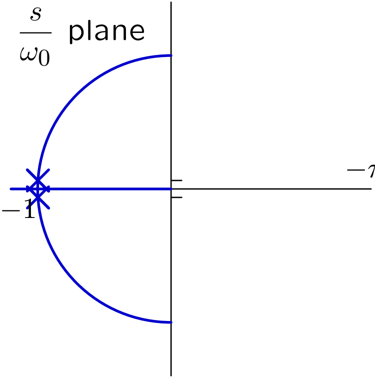


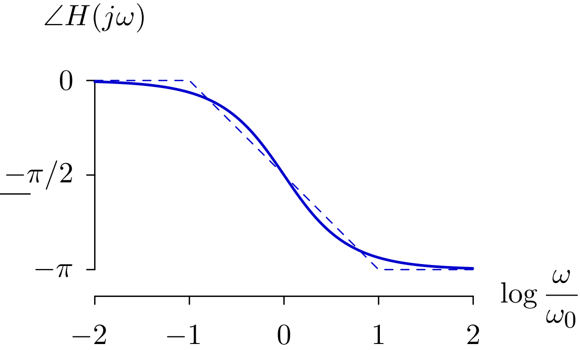
Bandwidth approximately $\frac{1}{Q}$

Frequency Response of a High-Q System

As Q increases, the phase changes more abruptly with ω.

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$





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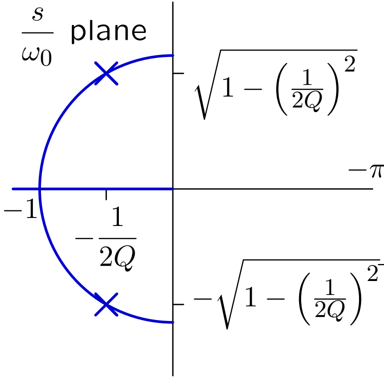

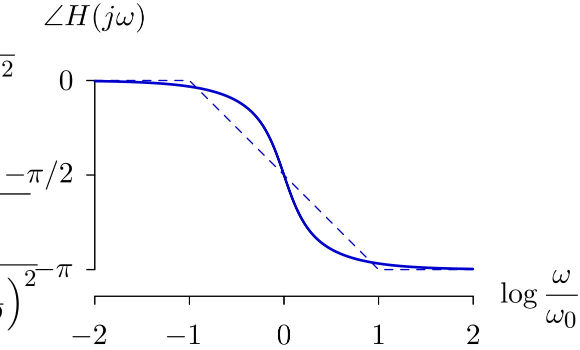

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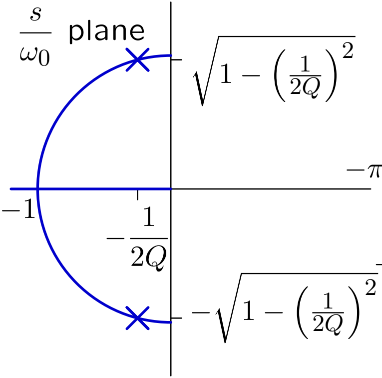

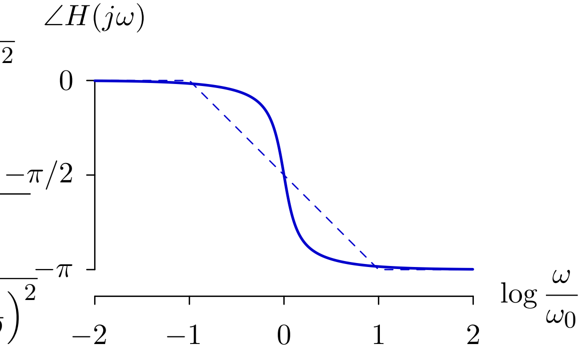

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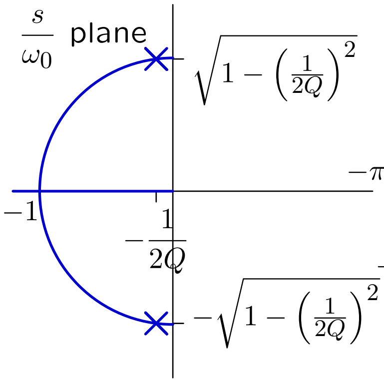

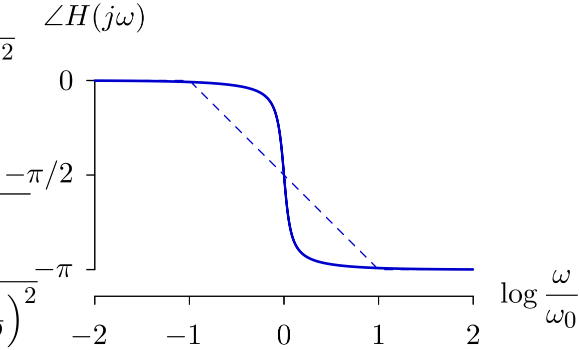

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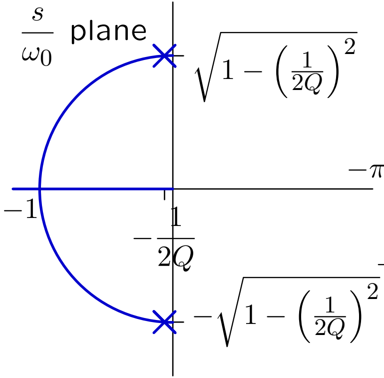
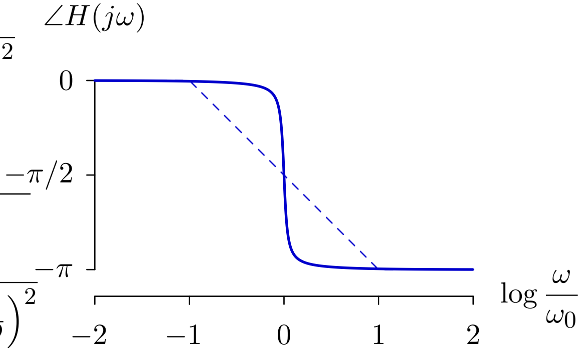
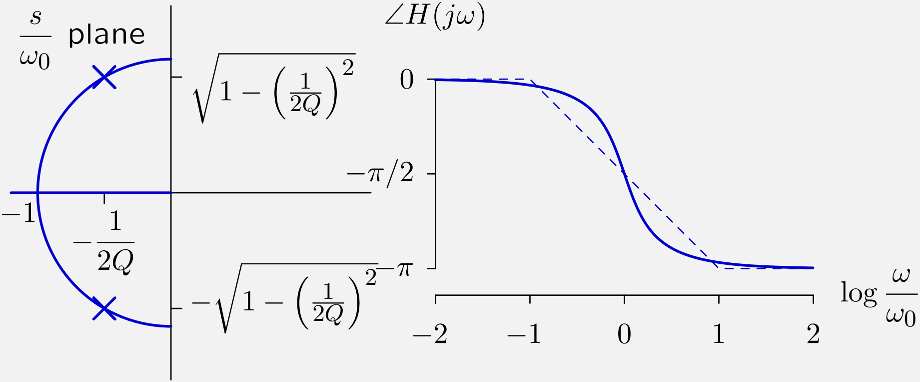




Check Yourself

Estimate change in phase that occurs over the 3dB bandwidth.

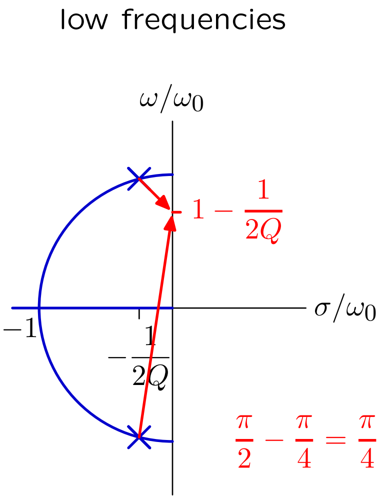
$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}$$



Check Yourself

Estimate change in phase that occurs over the 3dB bandwidth.

Analyze with vectors.



Change in phase approximately $\frac{\pi}{2}$.

Summary

The frequency response of a system can be quickly determined using Bode plots.

Bode plots are constructed from sections that correspond to single poles and single zeros.

Responses for each section simply sum when plotted on logarithmic coordinates.