# 6.003: Signals and Systems

## Discrete Approximation of Continuous-Time Systems

*September 29, 2011*

# Mid-term Examination #1

Wednesday, October 5, 7:30-9:30pm,

No recitations on the day of the exam.

Coverage: CT and DT Systems, Z and Laplace Transforms
Lectures 1–7
Recitations 1–7
Homeworks 1–4

Homework 4 will not collected or graded. Solutions will be posted.

Closed book: 1 page of notes ($8\frac{1}{2} \times 11$ inches; front and back).

No calculators, computers, cell phones, music players, or other aids.

Designed as 1-hour exam; two hours to complete.

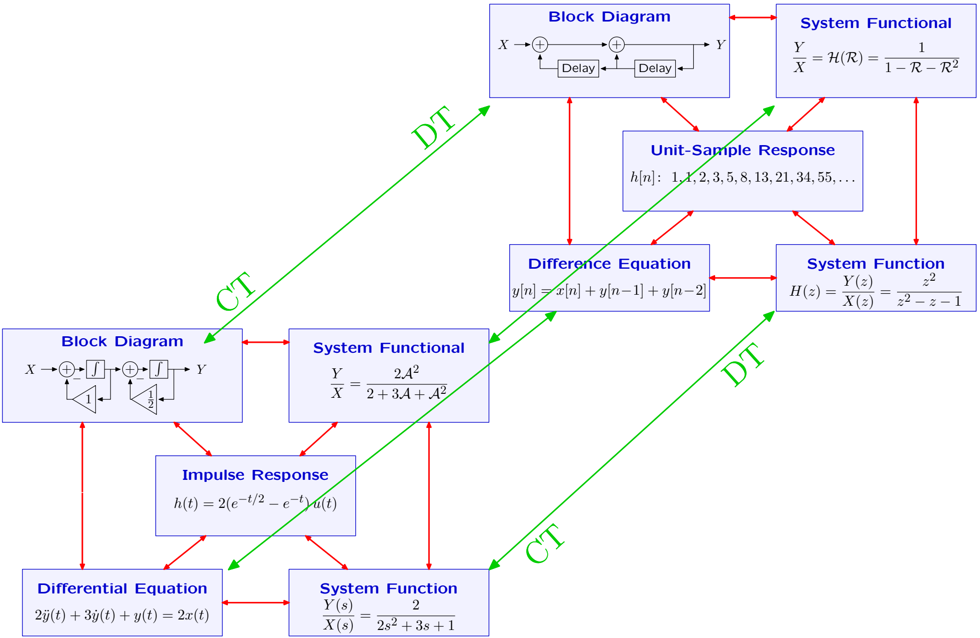
Review sessions during open office hours.

Conflict? Contact before Friday, Sept. 30, 5pm.

Prior term midterm exams have been posted on the 6.003 website.

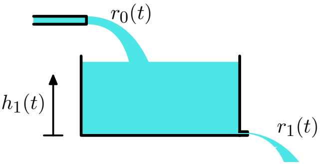
# Concept Map

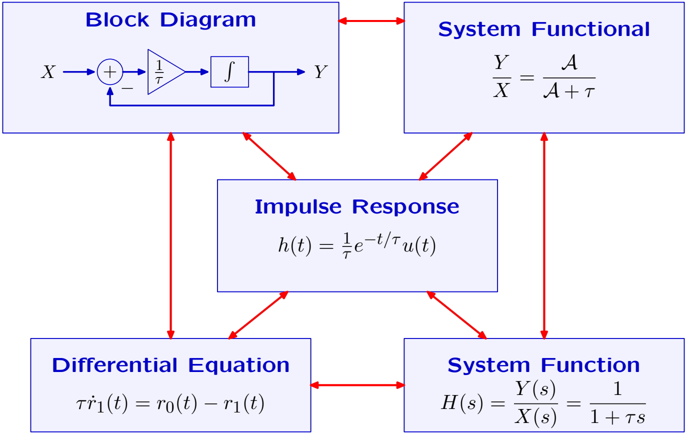
Today we will look at relations between CT and DT representations.



# Discrete Approximation of CT Systems

Example: leaky tank



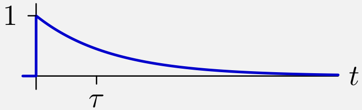




Today: compare step responses of leaky tank and DT approximation.

# Check Yourself (Practice for Exam)

What is the "step response" of the leaky tank system?


1.

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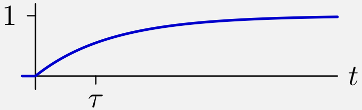


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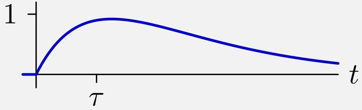


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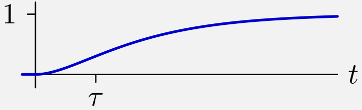



5. none of the above

## Check Yourself

What is the "step response" of the leaky tank system?

de: $\tau\,\dot{r}_1(t) = u(t) - r_1(t)$

$t < 0$: $r_1(t) = 0$

$t > 0$: $r_1(t) = c_1 + c_2 e^{-t/\tau}$

$\dot{r}_1(t) = -\frac{c_2}{\tau} e^{-t/\tau}$

Substitute into de: $\tau\left(-\frac{c_2}{\tau}\right) e^{-t/\tau} = 1 - c_1 - c_2 e^{-t/\tau} \quad \rightarrow \quad c_1 = 1$

Combine $t < 0$ and $t > 0$:

$$r_1(t) = u(t) + c_2 e^{-t/\tau} u(t)$$

$$\dot{r}_1(t) = \delta(t) + c_2\delta(t) - \frac{c_2}{\tau} e^{-t/\tau} u(t)$$

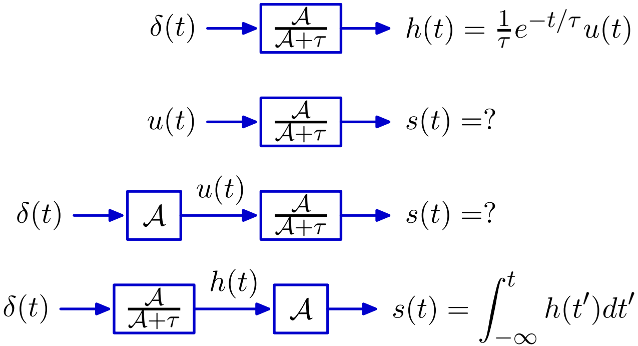
Substitute into de:

$$\tau(1 + c_2)\delta(t) - \tau\frac{c_2}{\tau} e^{-t/\tau} u(t) = u(t) - u(t) - c_2 e^{-t/\tau} u(t) \quad \rightarrow \quad c_2 = -1$$

$$r_1(t) = (1 - e^{-t/\tau})u(t)$$

# Check Yourself

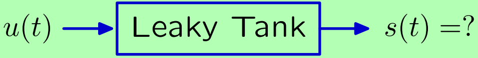
Alternatively, reason with systems!

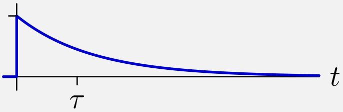


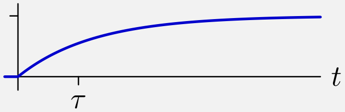

$$s(t) = \int_{-\infty}^{t} \frac{1}{\tau} e^{-t'/\tau} u(t')dt' = \int_{0}^{t} \frac{1}{\tau} e^{-t'/\tau} dt' = \left(1 - e^{-t/\tau}\right) u(t)$$

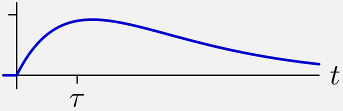
# Check Yourself

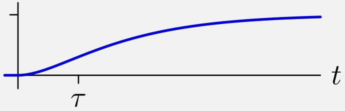
What is the "step response" of the leaky tank system? 2




1. 


2. 


3. 


4. 


5. none of the above

# Forward Euler Approximation

Approximate leaky-tank system using **forward** Euler approach.

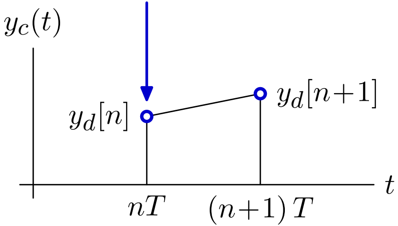
Approximate continuous signals by discrete signals:

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

Approximate derivative at $t = nT$ by looking **forward** in time:

$$\dot{y}_c(nT) = \frac{y_d[n+1] - y_d[n]}{T}$$



# Forward Euler Approximation

Approximate leaky-tank system using forward Euler approach.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big((n+1)T\big) - y_c\big(nT\big)}{T} = \frac{y_d[n+1] - y_d[n]}{T}$$

into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

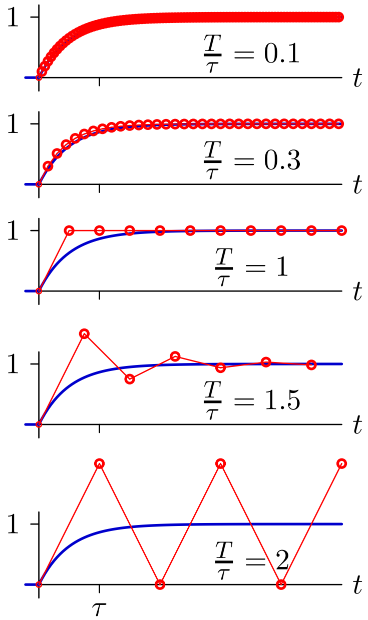
$$\frac{\tau}{T}\Big(y_d[n+1] - y_d[n]\Big) = x_d[n] - y_d[n]\,.$$

Solve:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

# Forward Euler Approximation

Plot.




Why is this approximation badly behaved for large $\frac{T}{\tau}$?

# Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole.

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

# Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Take the Z transform:

$$zY_d(z) - \left(1 - \frac{T}{\tau}\right) Y_d(z) = \frac{T}{\tau} X_d(z)$$

Solve for the system function:

$$H(z) = \frac{Y_d(z)}{X_d(z)} = \frac{\frac{T}{\tau}}{z - \left(1 - \frac{T}{\tau}\right)}$$

Pole at $z = 1 - \dfrac{T}{\tau}$.

# Check Yourself

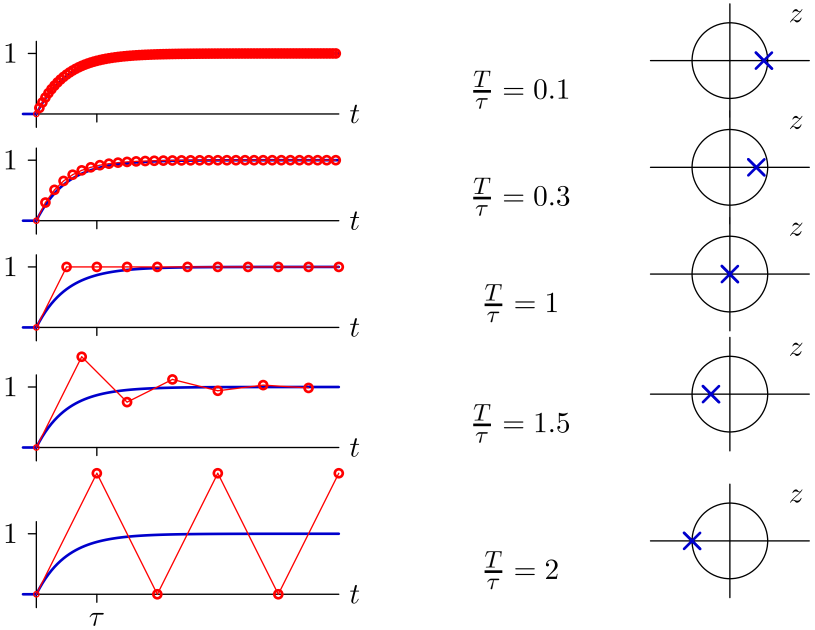
DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 2

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

# Dependence of DT pole on Stepsize




The CT pole was fixed ($s = -\frac{1}{\tau}$). Why is the DT pole changing?

# Dependence of DT pole on Stepsize

Dependence of DT pole on $T$ is generic property of forward Euler.

Approach: make a systems model of forward Euler method.

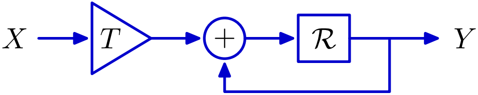
CT block diagrams: adders, gains, and integrators:


$$\dot{y}(t) = x(t)$$

Forward Euler approximation:

$$\frac{y[n+1]-y[n]}{T} = x[n]$$

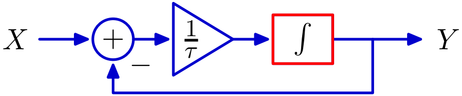
Equivalent system:



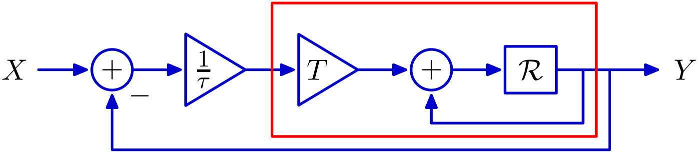

Forward Euler: substitute equivalent system for all integrators.

# Example: leaky tank system

Started with leaky tank system:




Replace integrator with forward Euler rule:




Write system functional:

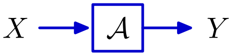
$$\frac{Y}{X} = \frac{\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}}{1+\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}} = \frac{\frac{T}{\tau}\mathcal{R}}{1-\mathcal{R}+\frac{T}{\tau}\mathcal{R}} = \frac{\frac{T}{\tau}\mathcal{R}}{1-\left(1-\frac{T}{\tau}\right)\mathcal{R}}$$

Equivalent to system we previously developed:

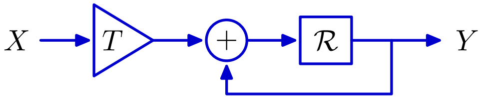
$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

# Model of Forward Euler Method

Replace every integrator in the CT system




with the forward Euler model:




Substitute the DT operator for $\mathcal{A}$:

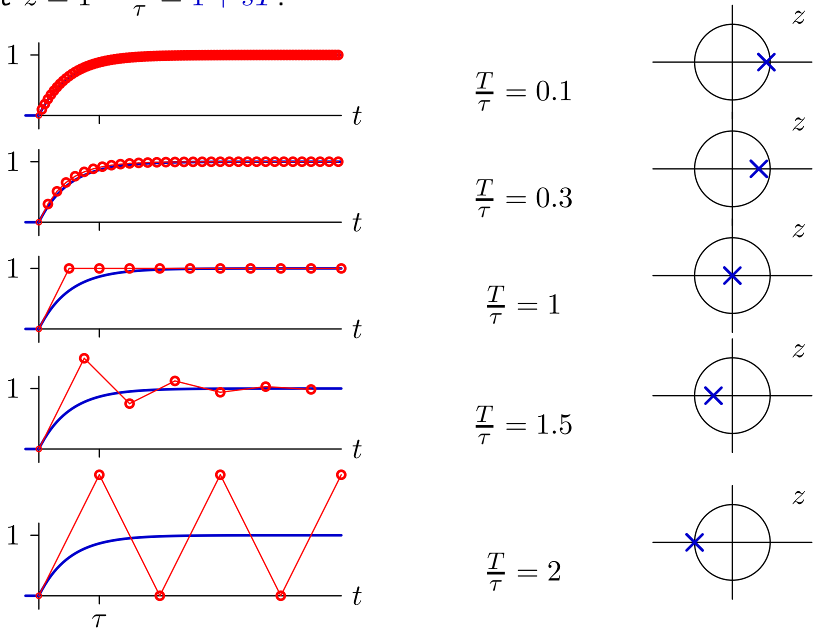
$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T\mathcal{R}}{1-\mathcal{R}} = \frac{\frac{T}{z}}{1-\frac{1}{z}} = \frac{T}{z-1}$$

Forward Euler maps $s \rightarrow \dfrac{z-1}{T}$.

Or equivalently: $z = 1 + sT$.

# Dependence of DT pole on Stepsize

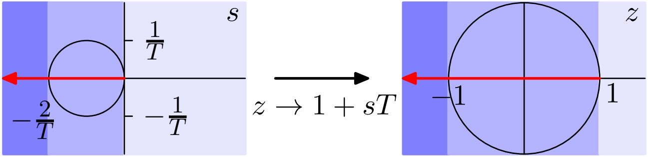
Pole at $z = 1 - \frac{T}{\tau} = 1 + sT$.



# Forward Euler: Mapping CT poles to DT poles

Forward Euler Map:

$$\begin{array}{ccc} s & \rightarrow & z = 1 + sT \\ 0 & & 1 \\ -\frac{1}{T} & & 0 \\ -\frac{2}{T} & & -1 \end{array}$$




DT stability: CT pole must be inside circle of radius $\frac{1}{T}$ at $s = -\frac{1}{T}$.

$$-\frac{2}{T} < -\frac{1}{\tau} < 0 \quad \rightarrow \quad \frac{T}{\tau} < 2$$

# Backward Euler Approximation

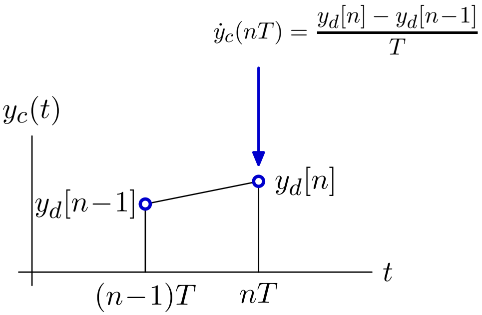
We can do a similar analysis of the **backward** Euler method.

Approximate continuous signals by discrete signals:

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

Approximate derivative at $t = nT$ by looking **backward** in time:



# Backward Euler Approximation

We can do a similar analysis of the backward Euler method.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big(nT\big) - y_c\big((n-1)T\big)}{T} = \frac{y_d[n] - y_d[n-1]}{T}$$

into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

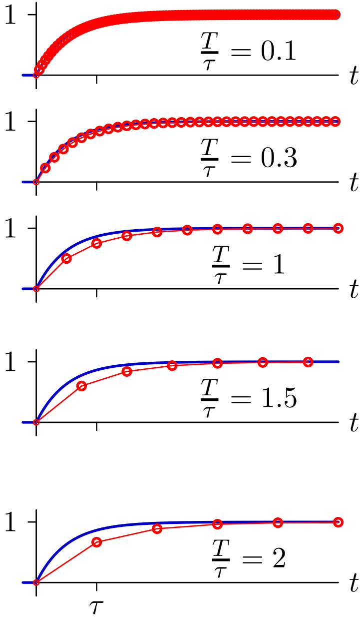
$$\frac{\tau}{T}\Big(y_d[n] - y_d[n-1]\Big) = x_d[n] - y_d[n]\,.$$

Solve:

$$\left(1 + \frac{T}{\tau}\right) y_d[n] - y_d[n-1] = \frac{T}{\tau} x_d[n]$$

# Backward Euler Approximation

Plot.




This approximation is better behaved. Why?

# Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Find the DT pole.

1. $z=\dfrac{T}{\tau}$
2. $z=1-\dfrac{T}{\tau}$
3. $z=\dfrac{\tau}{T}$
4. $z=-\dfrac{\tau}{T}$
5. $z=\dfrac{1}{1+\frac{T}{\tau}}$

# Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Take the Z transform:

$$\left(1+\frac{T}{\tau}\right)Y_d(z)-z^{-1}Y_d(z)=\frac{T}{\tau}X_d(z)$$

Find the system function:

$$H(z)=\frac{Y_d(z)}{X_d(z)}=\frac{\frac{T}{\tau}z}{\left(1+\frac{T}{\tau}\right)z-1}$$

Pole at $z=\dfrac{1}{1+\frac{T}{\tau}}$.

# Check Yourself

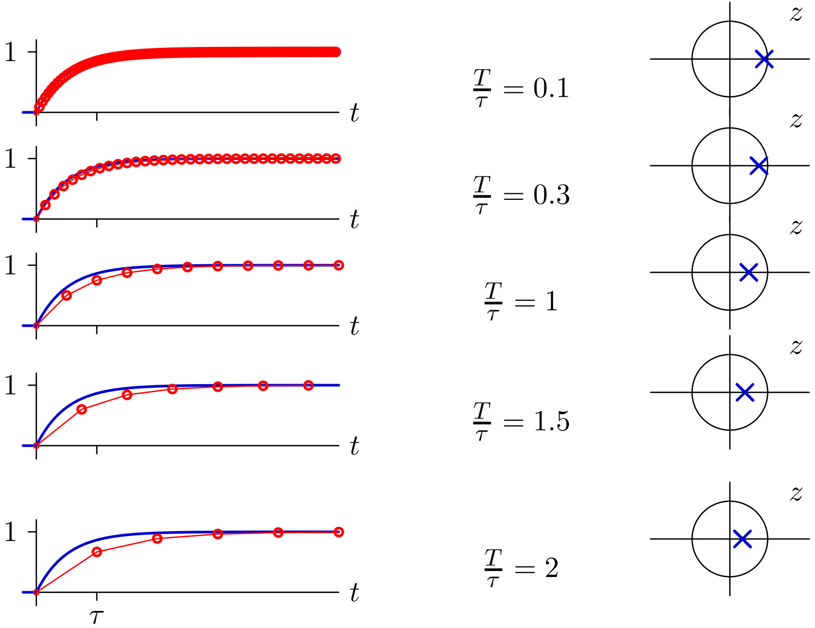
DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 5

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

# Dependence of DT pole on Stepsize




Why is this approximation better behaved?

# Dependence of DT pole on Stepsize

Make a systems model of backward Euler method.

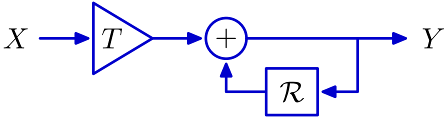
CT block diagrams: adders, gains, and integrators:


$$\dot{y}(t) = x(t)$$

Backward Euler approximation:

$$\frac{y[n] - y[n-1]}{T} = x[n]$$

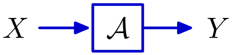
Equivalent system:



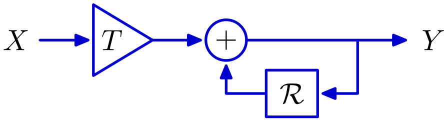

Backward Euler: substitute equivalent system for all integrators.

# Model of Backward Euler Method

Replace every integrator in the CT system




with the backward Euler model:



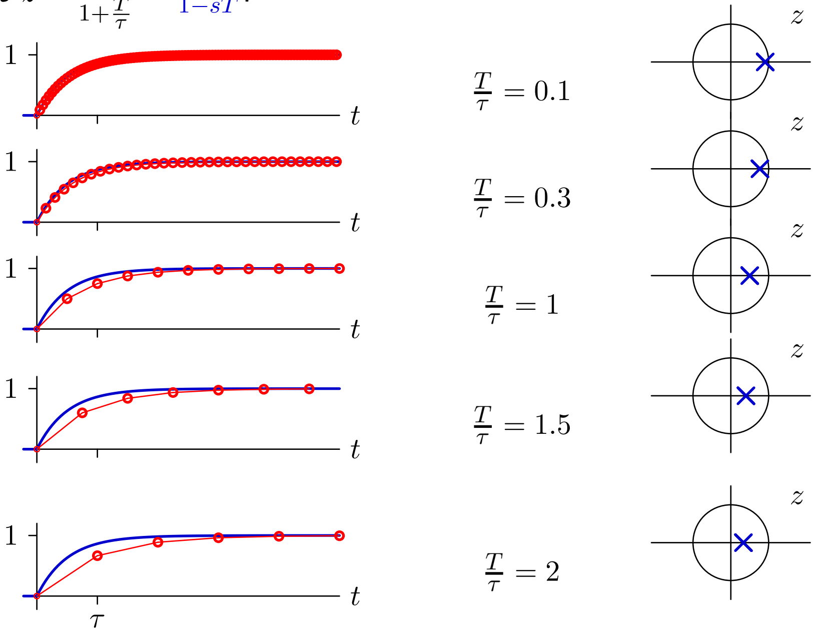

Substitute the DT operator for $\mathcal{A}$:

$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T}{1-\mathcal{R}} = \frac{T}{1-\frac{1}{z}}$$

Backward Euler maps $z \rightarrow \dfrac{1}{1-sT}$.

# Dependence of DT pole on Stepsize

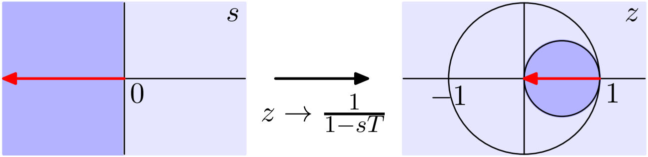
Pole at $z = \frac{1}{1+\frac{T}{\tau}} = \frac{1}{1-sT}$.



# Backward Euler: Mapping CT poles to DT poles

Backward Euler Map:

| $s$ | $\rightarrow$ | $z = \frac{1}{1-sT}$ |
| --- | --- | --- |
| $0$ | | $1$ |
| $-\frac{1}{T}$ | | $\frac{1}{2}$ |
| $-\frac{2}{T}$ | | $\frac{1}{3}$ |



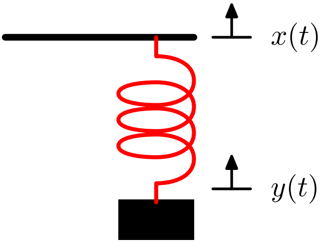

The entire left half-plane maps inside a circle with radius $\frac{1}{2}$ at $z = \frac{1}{2}$.

If CT system is stable, then DT system is also stable.

# Masses and Springs, Forwards and Backwards

In Homework 2, you investigated three numerical approximations to a mass and spring system:

- forward Euler
- backward Euler
- centered method



# Trapezoidal Rule

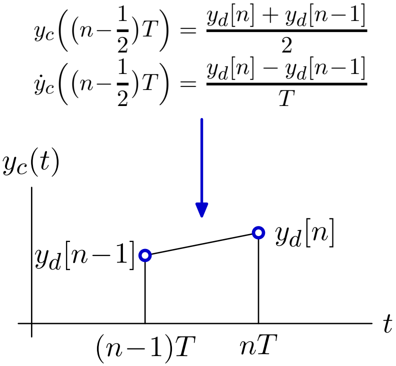
The trapezoidal rule uses centered differences.

Approximate CT signals at points between samples:

$$y_c\left((n-\frac{1}{2})T\right) = \frac{y_d[n] + y_d[n-1]}{2}$$

Approximate derivatives at points between samples:

$$\dot{y}_c\left((n-\frac{1}{2})T\right) = \frac{y_d[n] - y_d[n-1]}{T}$$



# Trapezoidal Rule

The trapezoidal rule uses centered differences.

$$\dot{y}(t) = x(t)$$

Trapezoidal rule:

$$\frac{y[n] - y[n-1]}{T} = \frac{x[n] + x[n-1]}{2}$$

Z transform:

$$H(z) = \frac{Y(s)}{X(s)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Map:

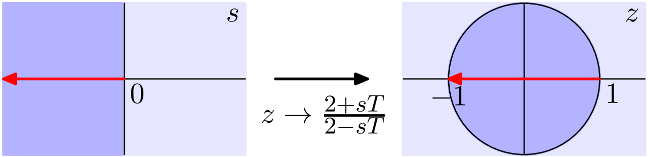
$$\mathcal{A} = \frac{1}{s} \to \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Trapezoidal rule maps $z \to \dfrac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$.

# Trapezoidal Rule: Mapping CT poles to DT poles

Trapezoidal Map:

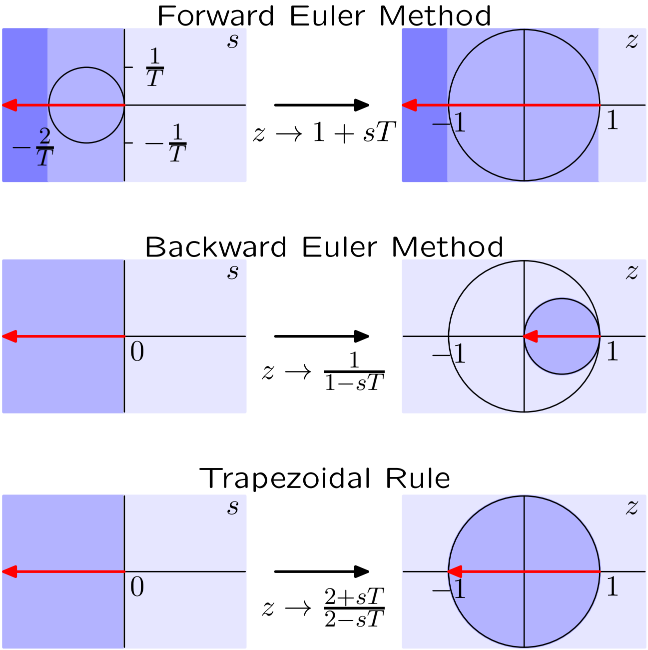
$$
\begin{array}{ccc}
s & \rightarrow & z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}} \\
0 & & 1 \\
-\frac{1}{T} & & \frac{1}{3} \\
-\frac{2}{T} & & 0 \\
-\infty & & -1 \\
j\omega & & \frac{2+j\omega T}{2-j\omega T}
\end{array}
$$



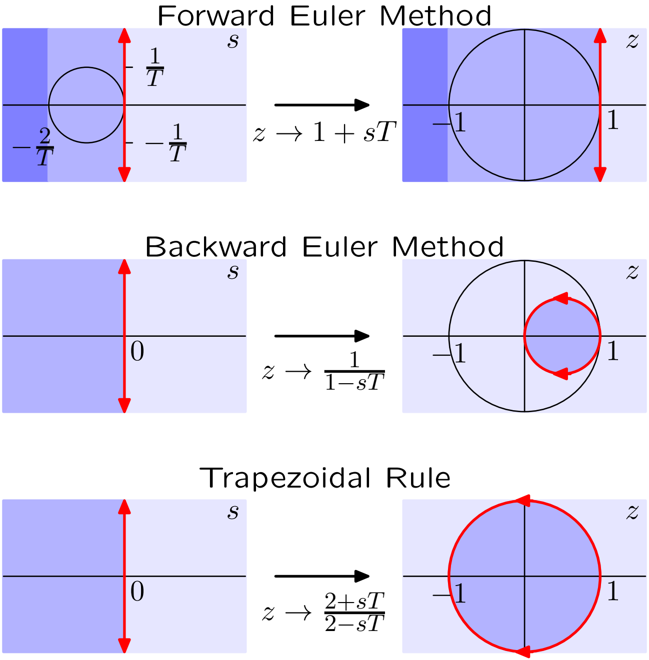

The entire left-half plane maps inside the unit circle.

The $j\omega$ axis maps onto the unit circle

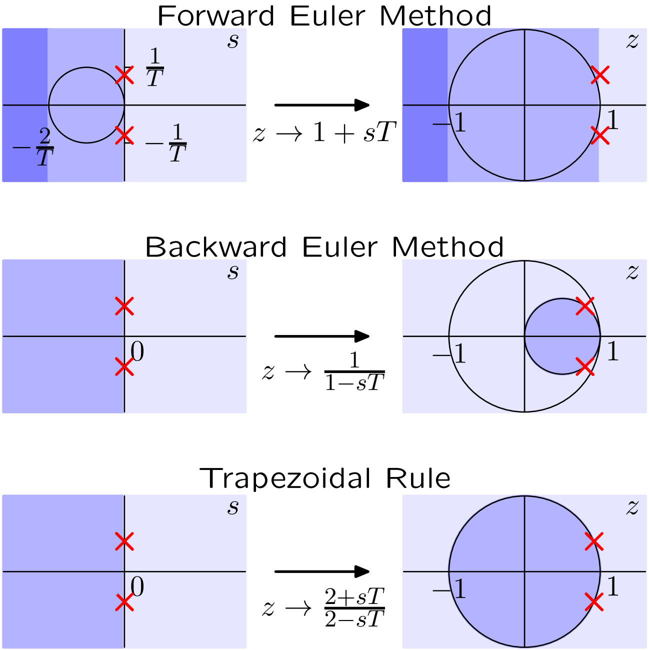
# Mapping s to z: Leaky-Tank System



# Mapping s to z: Mass and Spring System

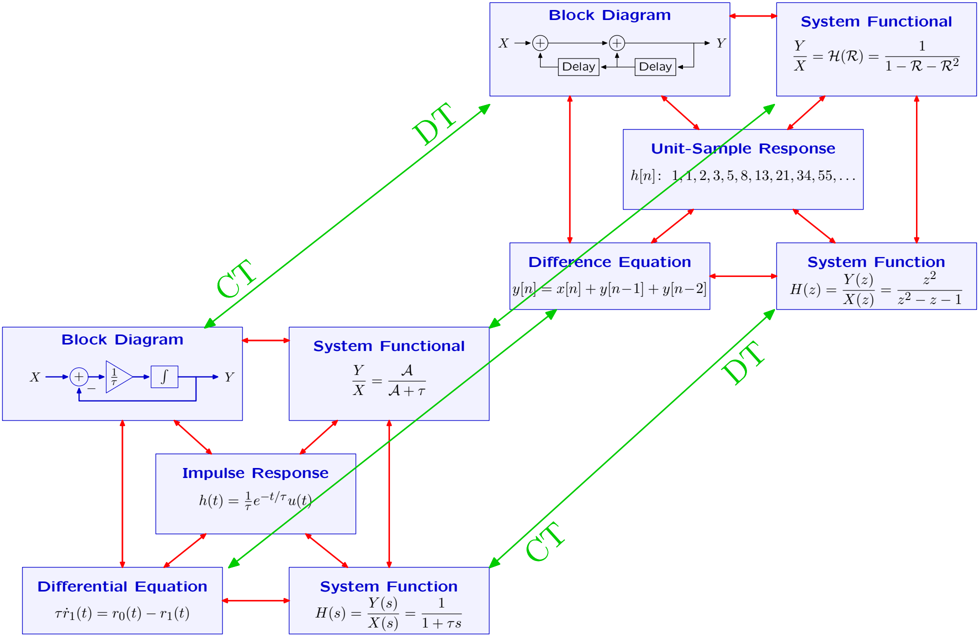


# Mapping s to z: Mass and Spring System



# Concept Map

Relations between CT and DT representations.



6.003 Signals and Systems
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