# 6.003 Homework #3

*Due at the beginning of recitation on **September 28, 2011**.*

## Problems

### 1. Complex numbers

**a.** Evaluate the real and imaginary parts of $j^j$.

Real part = | | Imaginary part = | |

**b.** Evaluate the real and imaginary parts of $(1 - j\sqrt{3})^{12}$.

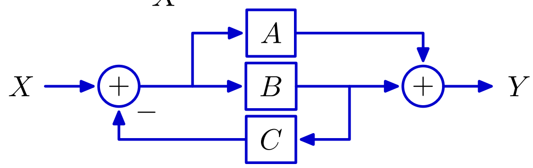
Real part = | | Imaginary part = | |

**c.** Express the real part of $e^{5j\theta}$ in terms of $\sin\theta$ and $\cos\theta$.

Real part = | |

## 2. Yin-Yang

Determine the system functional $\frac{Y}{X}$ for the following system




where $A$, $B$, and $C$ represent the system functionals for the boxed subsystems.

$\frac{Y}{X} =$ [ ]

## 3. Z transforms

Determine the Z transform (including the region of convergence) for each of the following signals:

**a.** $x_1[n] = \left(\frac{1}{2}\right)^n u[n-3]$

$X_1 =$ | |
---|---

ROC: | |
---|---

**b.** $x_2[n] = (1+n)\left(\frac{1}{3}\right)^n u[n]$

$X_2 =$ | |
---|---

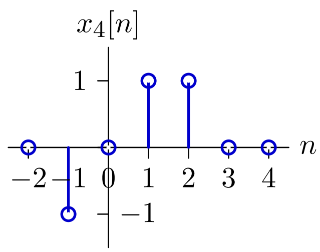
ROC: | |
---|---

**c.** $x_3[n] = n\left(\frac{1}{2}\right)^{|n|}$

$X_3 =$

ROC:

**d.**




$X_4 =$

ROC:

## 4. Inverse Z transforms

Determine all possible signals with Z transforms of the following forms.

**a.** $X_1(z) = \dfrac{1}{z-1}$

Enter expressions (or numbers) in the following table to discribe the possible signals. Each row should correspond to a different signal. If there are fewer signals than rows, enter **none** in the remaining rows.

| | $n < -1$ | $n = -1$ | $n = 0$ | $n = 1$ | $n > 1$ |
|---|---|---|---|---|---|
| $x_1[n] =$ | | | | | |
| or | | | | | |
| or | | | | | |

**b.** $X_2(z) = \dfrac{1}{z(z-1)^2}$

| | $n < -1$ | $n = -1$ | $n = 0$ | $n = 1$ | $n > 1$ |
|---|---|---|---|---|---|
| $x_2[n] =$ | | | | | |
| or | | | | | |
| or | | | | | |

**c.** $X_3(z) = \dfrac{1}{z^2 + z + 1}$

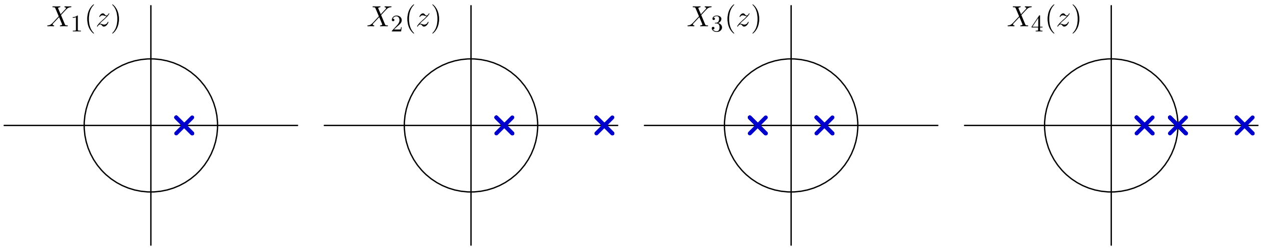
| | $n=-2$ | $n=-1$ | $n=0$ | $n=1$ | $n=2$ |
|---|---|---|---|---|---|
| $x_3[n] =$ | | | | | |
| or | | | | | |
| or | | | | | |

**d.** $X_4(z) = \left(\frac{1-z^2}{z}\right)^2$

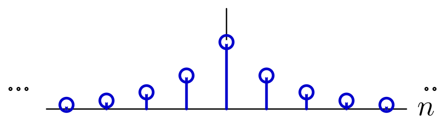
| | $n=-2$ | $n=-1$ | $n=0$ | $n=1$ | $n=2$ |
|---|---|---|---|---|---|
| $x_4[n] =$ | | | | | |
| or | | | | | |
| or | | | | | |

## 5. Poles

The following diagrams represent systems with poles (indicated by x's) but no zeros. The scale for each diagram is indicated by the circle, which has unit radius.




**a.** Which (if any) of $X_1(z)$ through $X_4(z)$ could represent a system with the following unit-sample response?




Enter a subset of the numbers 1 through 4 (separated by spaces) to represent $X_1(z)$ through $X_4(z)$ in the answer box below. If none of $X_1(z)$ through $X_4(z)$ apply, enter **none**.

1 and/or 2 and/or 3 and/or 4 or **none**:

**b.** Which (if any) of the systems could be stable?

Hint: A system is stable iff the region of convergence of its Z transform includes the unit circle.[1]

1 and/or 2 and/or 3 and/or 4 or **none**:

**c.** Which (if any) of systems could be causal?

Hint: A linear, time-invariant system is causal if its unit-sample response is zero for $t < 0$.

1 and/or 2 and/or 3 and/or 4 or **none**:

**d.** Which (if any) of the systems could be both causal and stable?

1 and/or 2 and/or 3 and/or 4 or **none**:

[1] If we decompose a system function using partial fractions, then we can consider the unit-sample response of the system as a sum of components that each correspond to one of the poles of the system. If the contribution of a pole is right-sided, then its Z transform converges for all $z$ with magnitudes bigger than that of the pole. To be stable, the magnitude of that pole must be less than 1. It follows that the region of convergence includes the unit circle. A similar argument holds for left-sided contributions.

**6. Periodic system**

Consider this variant of the Fibonacci system:

$$y[n] = y[n-1] - y[n-2] + x[n]$$

where $x[n]$ represents the input and $y[n]$ represents the output.

**a.** Compute the unit-sample response and show that it is periodic. Enter the period in the box below.

period = [ ]

**b.** Enter the poles of the system in the box below (separated by spaces).

poles = 

**c.** Decompose the system functional into partial fractions, and use the result to determine a closed-form expression for $h[n]$, the unit-sample response. Enter your expression in the box below.

$h[n] =$ 

# Engineering Design Problems

**7. Scaling time**

A system containing only adders, gains, and delays was designed with system functional

$$H = \frac{Y}{X}$$

which is a ratio of two polynomials in $\mathcal{R}$. When this system was constructed, users were dissatisfied with its responses. Engineers then designed three new systems, each based on a different idea for how to modify $H$ to improve the responses.

**System $H_1$:** every delay element in $H$ is replaced by a cascade of two delay elements.

**System $H_2$:** every delay element in $H$ is replaced by a gain of $\frac{1}{2}$ followed by a delay.

**System $H_3$:** every delay element in $H$ is replaced by a cascade of three delay elements.

For each of the following parts, evaluate the truth of the associated statement and enter **yes** if the statement is always true or **no** otherwise.

**a.** If $H$ has a pole at $z = j = \sqrt{-1}$, then $H_1$ has a pole at $z = e^{j5\pi/4}$.

**Yes** or **No**: [ ]

Explain.

**b.** If $H$ has a pole at $z = p$ then $H_2$ has a pole at $z = 2p$.

**Yes** or **No**:

Explain.

**c.** If $H$ is stable then $H_3$ is also stable (where a system is said to be stable if all of its poles are inside the unit circle).

**Yes** or **No**:

Explain.

**8. Complex Sum**

Each diagram below shows the unit circle in the complex plane, with the origin labeled with a dot.

Each diagram illustrates the sum

$$S = \sum_{n=0}^{100} \alpha^n .$$

Determine the diagram for which $\alpha = 0.8 + 0.2j$.

diagram =