6.003 Homework #2

*Due at the beginning of recitation on **September 21, 2011**.*

Problems

1. Finding outputs

Let $h_i[n]$ represent the n^{th} sample of the unit-sample response of a system with system functional $H_i(\mathcal{R})$. Determine $h_i[2]$ and $h_i[119]$ for each of the following systems:

a. $H_1(\mathcal{R}) = \dfrac{\mathcal{R}}{1 - \frac{3}{4}\mathcal{R}}$

$h_1[2] = \boxed{}$ $\quad h_1[119] = \boxed{}$

b. $H_2(\mathcal{R}) = \dfrac{1 - \frac{1}{16}\mathcal{R}^4}{1 - \frac{1}{2}\mathcal{R}}$

$h_2[2] = \boxed{}$ $\quad h_2[119] = \boxed{}$

c. $H_3(\mathcal{R}) = \dfrac{1}{(1-\frac{1}{2}\mathcal{R})(1-\frac{1}{4}\mathcal{R})}$

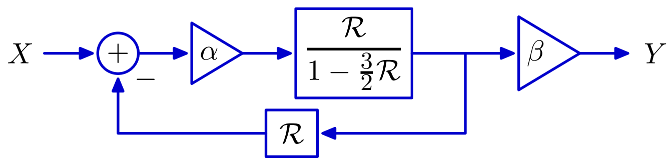
$h_3[2] =$ $\boxed{}$ $h_3[119] =$ $\boxed{}$

d. $H_4(\mathcal{R}) = \dfrac{1}{(1-\mathcal{R})^2}$

$h_4[2] =$ $\boxed{}$ $h_4[119] =$ $\boxed{}$

2. Feedback

Consider the following system.



Assume that X is the unit-sample signal, $x[n] = \delta[n]$. Determine the values of α and β for which $y[n]$ is the following sequence (i.e., $y[0]$, $y[1]$, $y[2]$, ...):

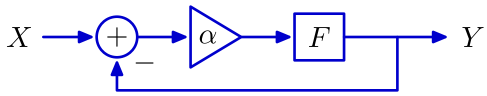
$$0\,,\ 1\,,\ \frac{3}{2}\,,\ \frac{7}{4}\,,\ \frac{15}{8}\,,\ \frac{31}{16}\,,\ \ldots$$

Enter the values of α and β in the boxes below. Enter **none** if the value cannot be determined from the information provided.

$\alpha =$ ☐ $\beta =$ ☐

3. Mystery Feedback

Consider the following feedback system where F is the system functional for a system composed of just adders, gains, and delay elements.



If $\alpha = 10$ then the closed-loop system functional is known to be

$$\left.\frac{Y}{X}\right|_{\alpha=10} = \frac{1+\mathcal{R}}{2+\mathcal{R}}$$

Determine the closed-loop system functional when $\alpha = 20$.

$$\left.\frac{Y}{X}\right|_{\alpha=20} = \boxed{}$$

4. Ups and Downs

The unit-sample response of a linear, time-invariant system is given by

$$h[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0, 3, 6, 9, ... \\ 2 & n = 1, 4, 7, 10, ... \\ 3 & n = 2, 5, 8, 11, ... \end{cases}$$

a. Determine a closed-form expression for the system functional for this system.

$H(\mathcal{R}) =$

b. Enter the poles of the system in the box below.

5. Characterizing a system from its unit-sample response

The first 30 samples of the unit-sample response of a linear, time-invariant system are given in the following table.

n	$h[n]$	n	$h[n]$
0	1	15	10761680
1	2	16	32285041
2	7	17	96855122
3	20	18	290565367
4	61	19	871696100
5	182	20	2615088301
6	547	21	7845264902
7	1640	22	23535794707
8	4921	23	70607384120
9	14762	24	211822152361
10	44287	25	635466457082
11	132860	26	1906399371247
12	398581	27	5719198113740
13	1195742	28	17157594341221
14	3587227	29	51472783023662

Determine the poles of this system. Enter the number of poles and list the pole locations below. If a pole is repeated k times, then enter that pole location k times. If there are more than 5 poles, enter just 5 of the pole locations. If there are fewer than 5 poles, leave the unused entries blank.

# of poles:

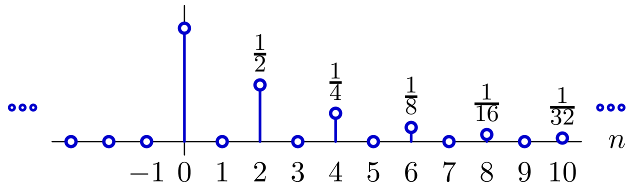
locations:

Engineering Design Problems

6. Unit-sample response

Consider a linear, time-invariant system whose unit-sample response $h[n]$ is shown below.

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^{n/2} & n = 0, 2, 4, 6, 8, \ldots, \infty \\ 0 & \text{otherwise} \end{cases}$$



Part a. Is it possible to represent this system with a finite number of poles?

Yes or **No**:

If yes, enter the number of poles and list the pole locations below. If a pole is repeated k times, then enter that pole location k times. If there are more than 5 poles, enter just 5 of the pole locations. If there are fewer than 5 poles, leave the unused entries blank.

\# of poles:

locations:

If no, briefly explain why not.

Part b. Is it possible to implement this system with a finite number of adders, gains, and delays (and no other components)?

Yes or **No**:

If yes, sketch a block diagram for the system in the following box.

If no, briefly explain why not.

7. Repeated Poles

Consider a system H whose unit-sample response is

$$h[n] = \begin{cases} n+1 & \text{for } n \geq 0 \\ 0 & \text{otherwise}. \end{cases}$$

a. Determine the poles of H.

b. H can be written as the cascade of two identical subsystems, each called G. Determine the difference equation for G.

c. Draw a block diagram for H using just adders, gains, and delays. Use the block diagram to explain why the unit-sample response of H is the sequence $h[n] = n + 1$, $n \geq 0$.

d. Because the system functional has two poles at the same location, the unit-sample response of H cannot be expressed as a weighted sum of geometric sequences,

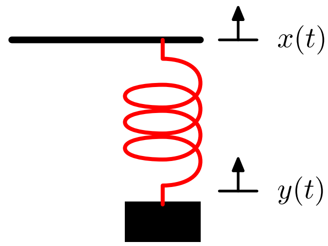
$$h[n] = a_0 z_0^n + a_1 z_1^n\,; \quad n \geq 0\,.$$

However, h can be written in the previous form if the poles of H are displaced from their true positions by a small amounts (e.g., one pole by $+\epsilon$ and the other by $-\epsilon$). Determine a_0, a_1, z_0, and z_1 as functions of ϵ.

e. Compare the results of the approximation in part d for different values of ϵ.

8. Masses and Springs, Forwards and Backwards

The following figure illustrates a mass and spring system. The input $x(t)$ represents the position of the top of the spring. The output $y(t)$ represents the position of the mass.



The mass is $M = 1\,\text{kg}$ and the spring constant is $K = 1\,\text{N/m}$. Assume that the spring obeys Hooke's law and that the reference positions are defined so that if the input $x(t)$ is equal to zero, then the resting position of $y(t)$ is also zero.

a. Determine a differential equation that relates the input $x(t)$ and output $y(t)$.

b. Calculate the step response of the system.

c. The differential equation in part a contains a second derivative (if you did part a correctly). We wish to develop a forward-Euler approximation for this derivatve. One method is to write the second-order differential equation in part a as a part of first order differential equations. Then apply the forward-Eular approximation to the first order derivatives:

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} \approx \frac{y[n+1]-y[n]}{T}\,.$$

Use this approach to find a difference equation to approximate the behavior of the mass and spring system. Determine the step response of the system and compare your results to those in part b.

d. An alternative to the forward-Euler approximation is the backward-Euler approximation:

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} \approx \frac{y[n]-y[n-1]}{T}\,.$$

Repeat the exercise in the previous part, but using the backward-Euler approximation instead of the forward-Euler approximation.

e. The forward-Euler method approximates the second derivative at $t = nT$ as

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{y[n+2]-2y[n+1]+y[n]}{T^2}\,.$$

The backward-Euler method approximates the second derivative at $t = nT$ as

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{y[n]-2y[n-1]+y[n-2]}{T^2}\,.$$

Consider a compromise based on a centered approximation:

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{y[n+1]-2y[n]+y[n-1]}{T^2}\,.$$

Use this approximation to determine the step response of the system. Compare your results to those in the two previous parts of this problem.