1.5 Long run contracts with limited enforcement

- See Hopenhayn and Clementi (2006) for something close to the moral-hazard model of Holmstrom and Tirole

- Here follow Lorenzoni and Walentin (2007)

- Process $A_t = \Gamma\left(A_{t-1}, \epsilon_t\right)$

- Consumers and entrepreneurs, unit mass of each

- Consumers have endowment of labor l_C

- Risk neutral, discount factor $\beta_E < \beta_C$

- Entrepreneurs die and replaced with prob γ

- Newborn entrepreneur supply l_E, start with $n = wl_E$

- Offer long-term financial contract $\{d_t\}_{t=t_0}^{\infty}$

- Market for used capital q_t^o

- Total adjustment cost $G\left(k_{t+1}, k_t^o\right)$

Budget constraint:

- first period of life

$$c_t^E + G\left(k_{t+1}, k_t^o\right) + q_t^o k_t^o \leq w_t l_E - d_t$$

- continuation period

$$c_t^E + G\left(k_{t+1}, k_t^o\right) + q_t^o \left(k_t^o - k_t\right) \leq A_t F\left(k_t, l_t\right) - w_t l_t - d_t$$

- last period

$$c_t^E = A_t F\left(k_t, l_t\right) - w_t l_t + q_t^o k_t - d_t.$$

Same trick as above (CRS)

- first period of life

$$c_t^E + q_t^m k_{t+1} + q_t^o k_t^o \leq w_t l_E - d_t$$

- continuation period

$$\begin{aligned} c_t^E + q_t^m k_{t+1} &\leq [A_t F(k_t, l_t) - w_t l_t + q_t^o k_t] - d_t \\ &= R_t k_t - d_t \end{aligned}$$

- last period

$$c_t^E = R_t k_t - d_t.$$

1.5.1 Limited enforcement

- Entrepreneur controls firm's assets

- In each period, can run away, diverting a fraction $(1-\theta)$

- If he does so, he re-enters the financial market as a young entrepreneur, with initial wealth

$$(1-\theta)\, R_t k_t$$

  and no debt

1.5.2 Recursive competitive equilibrium

- Aggregate state variables

$$X_t \equiv (A_t, K_t, B_t)$$

- Conjecture: positive consumers' consumption

- Present value of the liabilities of individual entrepreneur

$$b_t = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta_C^s d_{t+s} \right]$$

- B_t economy-wide aggregate of these liabilities

- Recursive CE: law of motions for the endogenous state variables

$$
\begin{aligned}
K_t &= \mathcal{K}\left(X_{t-1}\right), \\
B_t &= \mathcal{B}\left(X_{t-1}, \epsilon_t\right),
\end{aligned}
$$

and maps

$$
w\left(X_t\right), q^o\left(X_t\right)
$$

- Use compact notation

$$
X_t = H\left(X_{t-1}, \epsilon_t\right).
$$

1.5.3 Optimal financial contracts

- Continuing entrepreneur, in state X, who controls a firm with capital k and outstanding liabilities b

- $V(k, b, X)$ expected utility computed
  - after production takes place
  - assuming the entrepreneur has chosen no default
  - before new investment and consumption

- Budget constraint

$$c^E + q^m(X)\,k' \leq R(X)\,k - d$$

- Promise-keeping constraint

$$b = d + \beta_C\left((1-\gamma)\sum \pi\left(\epsilon'\right) b'\left(\epsilon'\right) + \gamma \sum \pi\left(\epsilon'\right) b'_L\left(\epsilon'\right)\right)$$

- $b'(\epsilon')\,, b'_L(\epsilon')$ PV of liabilities next period

- No-default condition

$$V\left(k', b'\left(\epsilon'\right), X'\right) \geq V\left((1-\theta)\, k', 0, X'\right)$$

for all ϵ' and $X' = H\left(X, \epsilon'\right)$.

- If tomorrow final period, no-default

$$R\left(X'\right) k' - b'_L\left(\epsilon'\right) \geq (1-\theta)\, R\left(X'\right) k'$$

Conjecture: linear value function

$$V(k,b,X) = \phi(X)(R(X)k - b)$$

linear in *net worth*:

$$R(X)k - b$$

- Then no default becomes

$$\begin{aligned} b'\left(\epsilon'\right) &\leq \theta R\left(H\left(X,\epsilon'\right)\right)k' \\ b'_L\left(\epsilon'\right) &\leq \theta R\left(H\left(X,\epsilon'\right)\right)k' \end{aligned}$$

  for all ϵ'

Bellman equation

$$
\begin{aligned}
V\left(k,b,X\right) \;=\; & \max_{c^E,k',b'(.),b'_L(.)} c^E + \beta_E \left(1-\gamma\right) \sum \pi\left(\epsilon'\right) V\left(k', b'\left(\epsilon'\right), H\left(X,\epsilon'\right)\right) \\
& + \beta_E \gamma \sum \pi\left(\epsilon'\right) \left[R\left(H\left(X,\epsilon'\right)\right) k' - b'_L\left(\epsilon'\right)\right]
\end{aligned}
$$

$$
c^E + q^m\left(X\right) k' \leq R\left(X\right) k - d
$$

$$
b = d + \beta_C \left(\left(1-\gamma\right) \sum \pi\left(\epsilon'\right) b'\left(\epsilon'\right) + \gamma \sum \pi\left(\epsilon'\right) b'_L\left(\epsilon'\right)\right)
$$

$$
\begin{aligned}
b'\left(\epsilon'\right) &\leq \theta R\left(H\left(X,\epsilon'\right)\right) k' \\
b'_L\left(\epsilon'\right) &\leq \theta R\left(H\left(X,\epsilon'\right)\right) k'
\end{aligned}
$$

Assumptions

- profitability

$$\beta_E \mathbb{E}\left[R\left(H\left(X,\epsilon'\right)\right)\right] > q^m(X) \qquad \text{(a)}$$

- limited pledgeability

$$\theta\beta_C \mathbb{E}\left[R\left(H\left(X,\epsilon'\right)\right)\right] < q^m(X) \qquad \text{(b)}$$

- finite utility

$$\frac{(1-\gamma)(1-\theta)\mathbb{E}\left[R\left(H\left(X,\epsilon'\right)\right)\right]}{q^m(X)-\theta\beta_C\mathbb{E}\left[R\left(H\left(X,\epsilon'\right)\right)\right]} < 1 \qquad \text{(c)}$$

Then find

$$\phi(X) = \frac{\beta_E (1-\theta) \mathbb{E}\left[\left(\gamma + (1-\gamma)\phi\left(H\left(X,\epsilon'\right)\right)\right) R\left(H\left(X,\epsilon'\right)\right)\right]}{q^m(X) - \theta\beta_C \mathbb{E}\left[R\left(H\left(X,\epsilon'\right)\right)\right]}$$

and guess and verify that:

$$V(k,b,X) = \phi(X)\left(R(X)k - b\right)$$

Optimal solution

- no consumption until final date

$$c^E = 0,$$

- maximum borrowing

$$b'\left(\epsilon'\right) = b'_L\left(\epsilon'\right) = \theta R\left(H\left(X, \epsilon'\right)\right) k'.$$

- dynamics for capital accumulation

$$k' = \frac{R(X)\,k - b}{q^m(X) - \theta\beta_C \mathbb{E}\left[R\left(H\left(X, \epsilon'\right)\right)\right]}$$

Need one extra assumption (no delay):

$$\phi(X) > \frac{\beta_E}{\beta_C}\phi\left(H\left(X, \epsilon'\right)\right) \qquad \text{(d)}$$

1.5.4 Aggregation

$$N_t = (1-\gamma)(R_t K_t - B_t) + \gamma w_t l_E$$

$$K_{t+1} = \frac{(1-\gamma)(R_t K_t - B_t) + \gamma w_t l_E}{q_t^m - \theta \beta_C \mathbb{E}_t [R_{t+1}]}$$

$$B_{t+1} = \beta_C \theta R_{t+1} K_{t+1}$$

$$w_t = A_t \frac{\partial F(K_t, 1)}{\partial L_t}$$

$$q_t^o = -\frac{\partial G(K_{t+1}, K_t)}{\partial K_t}$$

- This confirms that the state variables in X are sufficient to characterize the dynamics of prices

- Aggregation relies on linearity

- Trades of used capital:

  - entering entrepreneurs buy

$$\frac{\gamma w_t l_E}{q_t^m - \theta \beta_C \mathbb{E}_t [R_{t+1}]} \frac{k_t^o}{k_{t+1}}$$

  - exiting entrepreneurs sell

$$(1-\gamma) K_t$$

Computation

2nd order stoch. difference equation in K_t

$$K_{t+1} = \frac{(1-\gamma)(1-\theta) R_t K_t + \gamma w_t l_E}{q_t^m - \theta \beta_C \mathbb{E}_t [R_{t+1}]}$$

with

$$w_t = A_t \frac{\partial F(K_t, 1)}{\partial L_t}$$

$$\begin{aligned} R_t &= A_t \frac{\partial F(K_t, 1)}{\partial K_t} - \frac{\partial G(K_{t+1}, K_t)}{\partial K_t} \\ q_t^m &= \frac{\partial G(K_{t+1}, K_t)}{\partial K_{t+1}} \end{aligned}$$

- Remember to check that (a)-(d) are satisfied!

$$
\begin{aligned}
A_t F(k_t, l_t) &= A_t k_t^{\alpha} l_t^{1-\alpha}, \\
G(k_{t+1}, k_t) &= k_{t+1} - (1-\delta) k_t + \frac{\xi}{2} \frac{(k_{t+1} - k_t)^2}{k_t}.
\end{aligned}
$$

1.5.5 Steady state

$$A_t = 1$$

$$\left(1 - \theta\beta_C R^S\right) K^S = (1-\gamma)(1-\theta) R^S K^S + \gamma w^S l_E$$

$$R^S = \alpha \left(K^S\right)^{\alpha - 1} + 1 - \delta$$

$$K^S = \left(\frac{\alpha(\theta\beta_C + (1-\gamma)(1-\theta)) + \gamma(1-\alpha) l_E}{1 - (\theta\beta_C + (1-\gamma)(1-\theta))(1-\delta)}\right)^{\frac{1}{1-\alpha}}$$

Parameters such that (a)

$$\beta_E R^S > 1$$

then (b) and (c)

$$\theta\beta_C R^S < 1 \text{ and } \frac{(1-\gamma)(1-\theta)R^S}{1-\theta\beta_C R^S} < 1$$

follow from $\gamma w^S l_E > 0$

$$1 - \beta_C\theta R^S - (1-\gamma)(1-\theta)R^S > 0.$$

In steady state $\phi(X)$ is

$$\phi^S = \frac{(1-\theta)\beta_E R^S}{1-\theta\beta_C R^S}\left(\gamma + (1-\gamma)\phi^S\right)$$

Condition (d) is why $\beta_E < \beta_C$ is needed.

1.5.6 Frictionless benchmark

Very close to Hayashi (1982)

- entrepreneurs consume $w_t l_E$ in first period of life
- all investment financed with outside funds
- capital stock dynamics

$$\beta_C \mathbb{E}_t [R_t] = q_t^m$$

-

$$q_t^m = q_t$$

1.5.7 Q theory

- value of the firm (end of period)

$$\begin{aligned} p_t &= V(k_t, b_t, X_t) + b_t - c_t^E - d_t = \\ &= \phi_t (R_t k_t - b_t) + b_t - d_t \\ &= (\phi_t - 1)(R_t k_t - b_t) + q_t^m k_{t+1} \end{aligned}$$

- Tobin's q

$$q_t = (\phi_t - 1) \frac{R_t k_t - b_t}{k_{t+1}} + q_t^m > q_t^m.$$

Recall that ϕ_t is forward looking variable capturing future excess returns

$$\phi_t = \frac{\beta_E (1 - \theta) \mathbb{E}_t \left[\left(\gamma + (1 - \gamma) \phi_{t+1} \right) R_{t+1} \right]}{q_t^m - \theta \beta_C \mathbb{E}_t [R_{t+1}]}$$

	a_1	a_2
Model with financial friction	0.018	0.444
Frictionless model	0.118	0.000
Gilchrist and Himmelberg (1995)	0.033 (0.016)	0.242 (0.038)

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1.5.8 Wrapping up on q theory

- financial frictions can help explain failure of q-theory equations

- disconnect between when funds available and when profitable investment opportunities arise

- related ideas: growth options (Abel and Eberly)

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