14.462 Lecture Notes
Self Insurance and Risk Taking

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1 Self Insurance

- Exogenous stochastic income stream y_t. y_t is i.i.d., with support $[y_{\min}, y_{\max}]$, $y_{\max} > y_{\min} \geq 0$, and c.d.f. Ψ.
- Preferences:
$$\mathbb{E}_0 \mathcal{U} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$
where $U' > 0 > U''$; and, unless otherwise stated, $U'(0) = \infty$, $U'(\infty) = 0$.
- Budget and borrowing constraint:
$$c_t + a_t = (1+r)a_{t-1} + y_t$$
$$\begin{aligned} c_t &\geq 0 \\ a_t &\geq 0 \end{aligned}$$
which implies
$$c_t \leq (1+r)a_t + y_{t+1}$$

- Remark: We could relax borrowing constraint to

$$a_t \geq -\bar{b}_t$$

where $\bar{b}_t$ is the borrowing limit. Either exogenous to the economy; or endogenous.

E.g.:

$$\bar{b}_t = \inf_{\{y_{t+j}\}_{j=0}^{\infty}} \sum_{j=0}^{\infty} (1+r)^{-(j+1)} y_{t+j} = \frac{y_{\min}}{r}$$

- Define cash in hand as

$$z_t \equiv (1+r)a_t + y_t$$

It follows that

$$z_{t+1} = (1+r)(z_t - c_t) + y_{t+1}$$

and

$$0 \leq c_t \leq z_t$$

- We write the Belman equation as:

$$\begin{aligned} V(z) &= \max_{0 \leq c \leq z} U(c) + \beta \overbrace{\int V(\tilde{z})\, d\Psi(y)}^{\mathbb{E}V(\tilde{z})} \\ & \quad s.t. \quad \tilde{z} = (1+r)(z-c) + y \end{aligned}$$

We denote by $C(z)$ the arg max of the above and $A(z) = z - C(z)$.

- The value function V is the fixed point of the corresponding operator. Obviously, V inherits the properties of U. That is, $V' > 0 > V''$, $V'(0) = -\infty$, $V'(\infty) = 0$. Also, $C(z)$ and $A(z)$ are non-decreasing.

- The FOC:

$$U'(c_t) \geq \beta(1+r)\mathbb{E}_t V'(z_{t+1}), \quad = \ if \ c_t < z_t$$

The Envelope Condition:

$$V'(z_t) = U'(c_t)$$

Euler equation:

$$U'(c_t) \geq \beta(1+r)\mathbb{E}_t U'(c_{t+1}), \quad = \ if \ c_t < z_t$$

Alternatively

$$V'(z_t) \geq \beta(1+r)\mathbb{E}_t V'(z_{t+1}), \quad = \ if \ \mathbb{E}_t z_{t+1} > (1+r)z_t + \mathbb{E}_t y_{t+1}$$

1.1 Random Walk and Precautionary Motive

- Consider $\beta(1+r) = 1$, that is, that is, $r = \rho \equiv \beta^{-1} - 1$. If there were no uncertainty (and eventually no binding borrowing constraint), then

$$U'(c_t) = U'(c_{t+1}) \quad \text{or} \quad V'(z_t) = V'(z_{t+1})$$

implying

$$c_{t+1} = c_t = c^* \quad \text{and} \quad z_{t+1} = z_t = z^*$$

- Suppose now that there is risk in consumption, but there is no borrowing constraint and $r = \rho$. Then, the Euler condition impies

$$\mathbb{E}_t U'(c_{t+1}) = U'(c_t) \quad \text{and} \quad \mathbb{E}_t V'(z_{t+1}) = V'(z_t)$$

If in addition utility is quadratic, implying that U' and V' are linear, then

$$\mathbb{E}_t c_{t+1} = c_t \quad \text{and} \quad \mathbb{E}_t z_{t+1} = z_t$$

That is, consumption and wealth follow a random walk.

- But if $U''' > 0$ and $\text{Var}_t c_{t+1} > 0$, then $\mathbb{E}_t U'(c_{t+1}) = U'(c_t)$ implies

$$\mathbb{E}_t c_{t+1} > c_t$$

The precautionary motive for saving.

1.2 The U_c Supermartingale

- Consider again the general case. Define

$$M_t \equiv \beta^t (1+r)^t U'(c_t) = \beta^t (1+r)^t V'(z_t)$$

Then, by the Euler condition,

$$\mathbb{E}_t (M_{t+1} - M_t) \leq 0$$

That is, M_t is a supermartingale. Because M_t is non-negative (actually strictly positive), the supermartingale convergence theorem applies. The latter states that M_t converges almost surely to a non-negative random variable M_∞, $M_t \rightarrow_{a.s.} M_\infty$.

- Suppose $\beta(1+r) > 1$, that is, $r > \rho \equiv \beta^{-1} - 1$. Then, the fact that M_t converges a.s. while $\beta^t(1+r)^t$ diverges to $+\infty$ implies that $U'(c_t) = V'(z_t)$ must a.s. converge to 0. That is, c_t and z_t diverge a.s. to $+\infty$.

- Suppose next $\beta(1+r) = 1$, that is, $r = \rho \equiv \beta^{-1} - 1$. We want to argue again that c_t and z_t diverge a.s. to ∞. Suppose to the contrary that there is some upper limmit $z_{\max} < \infty$ such that $z_{t+1} \leq z_{\max} = (1+r)A(z_{\max}) + y_{\max}$. At $z_t = z_{\max}$, then

$$\begin{aligned} V'(z_t) &\geq \beta(1+r)\mathbb{E}_t V'(z_{t+1}) \Rightarrow \\ V'(z_{\max}) &\geq \mathbb{E}_t V'((1+r)A(z_{\max}) + y_{t+1}) \\ &> \inf_{y_{t+1}} \{V'((1+r)A(z_{\max}) + y_{t+1})\} = \\ &= V'((1+r)A(z_{\max}) + y_{\max}) = V'(z_{\max}). \end{aligned}$$

That is, $V'(z_{\max}) > V'(z_{\max})$, which is a contradiction. The resolution is $\mathrm{Var}_t V'(z_{t+1}) = 0$, which requires either the variance of y_{t+1} to vanish, or otherwise z_{t+1} to diverge a.s. to $+\infty$.

- Suppose finally $\beta(1+r) < 1$, that is, $r = \rho \equiv \beta^{-1} - 1$. Then, as long as $\text{Var}_t V'(z_{t+1}) = \text{Var}_t U'(c_{t+1})$ remains finite, then M_t will automatically converge a.s. to zero, and we are fine.

- We conclude that $A(z_0) = \infty$ if $r \geq \rho$, but $A(z_0)$ can be finite if $r < \rho$. With CARA, there is a unique $r < \rho$ for which $A(z_0)$ is finite. With deminishin ARA (such as CRRA), $A(z_0)$ is finite for every $r < \rho$.

- For stochastic r, Chamberlain and Wilson (1984/2000) prove that z diverges to infinite as long as $\mathbb{E}r$ exceeds ρ.

2 CARA-Normal Example

2.1 Individual behavior

- Suppose $\beta(1+r) < 1$.

- Suppose $y_t \sim N(\overline{y}, \sigma^2)$.

- Suppose CARA preferences,

$$\begin{aligned} U(c) &= -\frac{1}{\Gamma}\exp(-\Gamma c) \\ U'(c) &= \exp(-\Gamma c) \end{aligned}$$

- Show that there are $a, b, \hat{a}, \hat{b}$ such that

$$\begin{aligned} V(z) &= -\exp(-\hat{a}z - \hat{b}) \\ C(z) &= az + b \end{aligned}$$

- Because c is normal and U' is exponential,

$$\mathbb{E}_t U'(c_{t+1}) = U'(\mathbb{E}_t c_{t+1} - \Gamma \text{Var}_t(c_{t+1})/2)$$

- The Euler condition,
$$U'(c_t) = \beta(1+r)\mathbb{E}_t U'(c_{t+1}),$$
thus reduces to
$$\mathbb{E}_t c_{t+1} - c_t = \frac{1}{\Gamma}\ln[\beta(1+r)] + \frac{\Gamma}{2}\text{Var}_t(c_{t+1})$$

- Combining with $C(z) = az + b$ and $\text{Var}_t(z_{t+1}) = \sigma^2$, we infer $\text{Var}_t(c_{t+1}) = a^2\sigma^2$ and thus
$$\mathbb{E}_t c_{t+1} - c_t = \frac{1}{\Gamma}\ln[\beta(1+r)] + \frac{\Gamma}{2}a^2\sigma^2$$

- For a steady state, $\mathbb{E}_t c_{t+1} - c_t = 0$, we thus need
$$\ln[\beta(1+r)] = -\frac{(\Gamma a\sigma)^2}{2}$$
that is
$$r = \rho e^{-(\Gamma a\sigma)^2/2} < \rho$$

- Hence, the resolution to the risk-free rate puzzle.

2.2 Moving from CARA to CRRA

- A disturbing property of our CARA specification is that risk aversion is independent of wealth. Indeed, absolute risk aversion is Γ, but relative risk aversion is Γc_t. It is more reasonable to assume that relative rather than absolute risk aversion is constant. Therefore, lets us fix $\Gamma c_t = \gamma$, that is, calibrate Γ as $\Gamma = \gamma/c_t$, where γ measures relative risk aversion.

- Then, the Euler condition becomes
$$\frac{\mathbb{E}_t c_{t+1}}{c_t} - 1 = \frac{1}{\gamma}\ln[\beta(1+r)] + \frac{\gamma}{2}\frac{\text{Var}_t(c_{t+1})}{c_t^2}.$$

- Note that $\mathrm{Var}_t(c_{t+1}) = a^2\sigma^2$, $c_t^2 = (az_t+b)^2 \approx a^2 z_t^2$, and $\ln \beta(1+r) \approx r-\rho$ where $\rho \equiv 1/\beta - 1$. Letting $1/\gamma = \theta$ for the elasticity of intertemporal substitution, we conclude

$$\frac{\mathbb{E}_t c_{t+1}}{c_t} = 1 + \theta(r_t - \rho) + \frac{\gamma}{2}\left(\frac{\sigma}{z_t}\right)^2.$$

  That is, consumption growth (savings) are increasing in the difference between the interest rate and the dicount rate and increasing in the income risk relative to the level of wealth.

2.3 Towards General Equilibrium

- For consumption and wealth to be stationary, namely $\mathbb{E}_t c_{t+1}/c_t = 1$, we need

$$\theta(r_t - \rho) = -\frac{\gamma}{2}\left(\frac{\sigma}{z_t}\right)^2,$$

  which requires $r_t < \rho$. Equivalently,

$$z_t = \sqrt{\frac{\sigma^2/\gamma}{2\theta(\rho - r_t)}} \equiv Z(r_t).$$

- $Z(r)$ corresponds to the aggregate supply of savings: It say what is the stationary level of wealth for any given interest rate. Note that $Z(r) \in (0, \infty)$ and $Z'(r) > 0$ for all $r \in [0, \rho)$, with $Z(0) < \infty$ and $Z(r) \to \infty$ as $r \to \rho$.

- On the other hand, the optimal level of investment is pinned down by the equality of the MPK with the interest rate:

$$r_t = f'(K_t).$$

  Equivalently,

$$k_t = (f')^{-1}(r_t) \equiv K(r_t).$$

- $K(r)$ corresponds to the aggregate demand for capital. Note that $K(r) \in (0, \infty)$ and $K'(r) < 0$ for all $r \in (0, \rho]$, with $K(r) \to \infty$ as $r \to 0$ and $K(\rho) < \infty$.

- A steady state corresponds to an intersection of the curves $Z(r)$ and $K(r)$. That is, the steady-state interest rate and capital stock are given by r^* and k^* such that $Z(r^*) = K(r^*) = k^*$.

- By the properties of Z and K, the steady state exists and is unique.

- Moreover, for any $\sigma > 0$, the steady state is $r^* < \rho$ and $k^* > K(\rho)$. That is, the interest rate is lower and the capital stock is higher under incomplete markets than under complete markets.

- Finally, an increase in σ (labor income risk) increases the supply of savings $Z(r)$ without affecting the demand for investment $K(r)$. Therefore, r^* is decreasing in σ, and k^* is increasing in σ.

- The above analysis is a heuristic representation of the more formal and exact analysis in Aiyagari (1994).