Problem Set 1 Solution

The case $\alpha = \beta$ is not interesting, so assume $\beta > \alpha$. Let r and w be the (shadow) prices of capital and labor. The cost functions corresponding to the two technologies are $\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}$ and $\left(\frac{r}{\beta}\right)^{\beta}\left(\frac{w}{1-\beta}\right)^{1-\beta}$, respectively. So using technology one is cheaper if $\omega < \bar{\omega}$ where $\omega \equiv \frac{w}{r}$ and $\bar{\omega} \equiv \left[\frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}{\beta^{\beta}(1-\beta)^{1-\beta}}\right]^{\frac{1}{\beta-\alpha}}$. If labor is relatively cheap, it is cheaper to use the labor intensive technology. The capital-labor ratios used as a function of the relative price of labor are given by $\kappa_1(\omega) = \frac{\alpha}{1-\alpha}\omega$ and $\kappa_2(\omega) = \frac{\beta}{1-\beta}\omega$, respectively. We have $k_1 = \kappa_1(\bar{\omega})$ and $k_2 = \kappa_2(\bar{\omega})$. These are the capital-labor ratios employed if both technologies are used.

Now there are three regions for the capital-labor ratio. If $k \in [0, k_1)$, then $\kappa_1^{-1}(k) \leq \bar{\omega}$ and it is an equilibrium if only technology one is used: if technology one operates with capital-labor ratio k, this induces a relative price of labor given which it is cheaper to use technology one. Clearly it is not an equilibrium for both technologies to be used since the capital-labor ratios k_1 and k_2 are too large to be consistent with an aggregate ratio of k.

Similarly, if $k \in [k_2, +\infty)$, then $\kappa_2^{-1}(k) \geq \bar{\omega}$ and the only equilibrium is that only technology two is used.

Finally, if $k \in (k_1, k_2)$, then both technologies must be used in equilibrium. If only technology one is used, then $\kappa_1^{-1}(k) > \bar{\omega}$ and technology two is cheaper, a contradiction. If only technology two is used, then $\kappa_2^{-1}(k) < \bar{\omega}$ and technology one is cheaper. There is a unique equilibrium in which both technologies are used: the two equations $k_1 L_1 + k_2 L_2 = kL$ and $L_1 + L_2 = L$ have a unique solution and this solution has $k_1, L_1, k_2, L_2 > 0$.

We have determined the allocation of capital and labor for all three regions, which in turn pins down marginal products and factor prices.