Problem Set #1 (14.453)

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1 Business Cycles Costs

Let utility be given by:

$$E_{-1}\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)$$

where the instantaneous utility takes the standard CRRA specification $u\left(c_t\right)=c_t^{1-\gamma}/(1-\gamma)$.

1.1 AR(1)

The consumption process is

$$c_t=c_{t-1}^{\alpha}\varepsilon_t\exp\left(\mu\right)$$

where

$$\mu=-\frac{\sigma_\varepsilon^2\left(1-\alpha\right)}{2\left(1-\alpha^2\right)}\qquad\qquad \log\varepsilon_t\sim N\left(0,\sigma_\varepsilon^2\right)\qquad \text{and i.i.d.}$$

so that the logarithm of consumption follows an AR(1) with parameter α:

$$\log c_t=\mu+\alpha\log c_{t-1}+\log\varepsilon_t.$$

(a) In the long run, $\log c_t$ converges (in distribution) to $N\left(\frac{\mu}{1-\alpha},\frac{\sigma_\varepsilon^2}{1-\alpha^2}\right)$, which we call the invariant distribution. Show that the normalization proposed is such that the mean consumption is equal to one.

(b) Assuming that the consumption at time 0 (c_0) is random and distributed according to the invariant distribution defined in point 1, find the value function

$$V\left(\lambda,\sigma_\varepsilon^2\right)\equiv E_{-1}\sum_{t=0}^{\infty}\beta^t u\left[c_t\left(1+\lambda\right)\right].$$

where λ is a positive constant.

(c) Compute the value of λ as a function of α, σ_ε^2 and γ such that

$$V\left(\lambda, \sigma_\varepsilon^2\right) = V(0,0).$$

What does λ represent? Determine how λ depends on α, σ_ε^2 and γ and interpret your results. Find a nice approximation (hint: try to use $\log(1+x) \approx x$, etc...). Compare to the *i.i.d.* case, *i.e.* $\alpha = 0$.

(d) What happens when $\alpha \rightarrow 1$? Is it sensible to think at λ as the cost of "business cycles"?

(e) Suppose now that c_0 is known and not random. Why is the previous analysis that assumed c_0 was random an upper bound to this case? Moreover, assume now that $u(c_t) = \log(c_t)$. Calculate λ (remember: now c_0 is given and not random) and again use approximations to express λ. What happen when $\alpha \rightarrow 1$? And when $\alpha \rightarrow 0$?

1.2 Random walk

Now assume the following consumption process

$$c_t = c_{t-1}\varepsilon_t \exp\left(-\frac{\sigma_\varepsilon^2}{2}\right)$$

where $\log \varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ and i.i.d. The logarithm of consumption follows a random walk and $E(\varepsilon_t) = 1$.

(a) Write welfare recursively as

$$V\left(c; \lambda, \sigma_\varepsilon^2\right) = u\left[c(1+\lambda)\right] + \beta E\left[V\left(c'; \lambda, \sigma_\varepsilon^2\right)\right]$$

where c denotes the consumption of today, c' the consumption of tomorrow and λ is a positive constant. Guess and verify the value function. (Hint: try the guess $V(c; \lambda, \sigma_\varepsilon^2) = A\frac{c^{1-\gamma}}{1-\gamma}$ where A is a constant).

(b) Compute the value of λ such that

$$V\left(c; \lambda, \sigma_\varepsilon^2\right) = V(c; 0, 0).$$

Show that λ is independent of c and is a function of β, σ_ε^2 and γ. Determine how λ depends on each of its arguments and explain the results.

(c) Explore λ for some reasonable values of the parameters. Comment. Compare with the *i.i.d.* case and with the autoregressive case of the previous exercise.

2 Intertemporal Elasticity of Substitution, Risk Aversion and the Cost of Fluctuations

This problem will familiarize you with the non-expected utility framework of Kreps and Porteus (1978), Epstein and Zin (1991) and Weil (1990).

Let remaining lifetime utility *after the resolution of current consumption* at time t be given by v_t. Assume v_t satisfies:

$$v_t = \left[(1-\beta)\, c_t^{\rho} + \beta \left(E_t v_{t+1}^{\alpha} \right)^{\frac{\rho}{\alpha}} \right]^{\frac{1}{\rho}} \tag{1}$$

for $1 \geq \rho \neq 0$ and $\alpha \leq 1$. Think of this equation as defining a stochastic process for $\{v_t\}$ given a stochastic process for $\{c_t\}$. In any period t *before the resolution of current consumption* we can define lifetime utility by u_t as:

$$u_t = \left(E_t v_t^{\alpha} \right)^{\frac{1}{\alpha}} \tag{2}$$

(a) Show that if $\{v_t\}$ is implied by $\{c_t\}$ using (1) then $\{\lambda v_t\}$ is implied by $\{\lambda c_t\}$ using (1) for any $\lambda > 0$. Show that if $c_t = \psi$ for some constant ψ then v_t and u_t are also constant and equal to ψ.[1]

(b) For any stationary process $c = \{c_t\}$, with associated utility u_0, let $\bar{c}$ denote the deterministic and constant sequence of consumption $(Ec_t, Ec_t, \cdots)$. Define our measure of welfare loss from the variability of $\{c_t\}$ to be that constant percentage increase, η, in the process c that is required to make the individual indifferent between $(1+\eta)\, c$ and $\bar{c}$.

Use your results in (a) to show that:

$$1 + \eta = \frac{Ec_t}{u_0}$$

(c) Show that if $\alpha = \rho$ then:

$$v_0 = \left[(1-\beta) \left(c_0^{\rho} + E_0 \sum_{t=1}^{\infty} \beta^t c_t^{\rho} \right) \right]^{\frac{1}{\rho}} \tag{3}$$

which is just a monotonic transformation of the more familiar

$$\frac{c_0^{\rho}}{\rho} + E_0 \sum_{t=1}^{\infty} \beta^t \frac{c_t^{\rho}}{\rho}$$

utility specification.

[1] Given these results we could say that v_t (u_t) represents the "certainty equivalent constant consumption" from t on, after (before) the resolution of time t consumption.

(d) (i) Show that if $\{c_t\}$ is deterministic then (3) holds (without the need for the expectation operator of course). Use this to compute the value of utility obtained if $c_{lh} = (c_l, c_h, c_l, c_h, ...)$, denote this lifetime utility by $v_{0,l}$. and $v_{0,h}$, $c_{hl} = (c_h, c_l, c_h, c_l, ...)$. How does your answer depend on the parameters α and ρ?

(ii) It is useful to think of the limit as $\beta \to 1$. Show that $v_{0,h}$ and $v_{0,l}$ converge to the same value. What is the intuition for this? Compute the welfare loss as a fraction of the constant consumption stream that equals the average $\bar{c} \equiv \frac{1}{2}c_h + \frac{1}{2}c_l$. How does your answer depend on the parameters α and ρ?

(iii) Compute the value of u_0 if the sequences c_{lh} and c_{hl} occur with probability 1/2 (which is revealed at the beginning of period 0) as $\beta \to 1$. This should be very easy given your answer to (*ii*). How does your answer depend on the parameters α and ρ?

(e) (note: we no longer work with the limit $\beta \to 1$ here, we are back to β bounded away from 1) Find an expression for utility, u_0, when $\{c_t\}$ is such that $c = (c_h, c_h, ...)$ with probability 1/2 and $c = (c_l, c_l, ...)$ with probability 1/2. How does your answer depend on the parameters α, ρ?

(f) Let c_t be *i.i.d.* distributed over time with outcomes c_l and c_h each with probability 1/2. Combine (1) and (2) and the fact that with i.i.d. consumption $u_t = u_0$ to show that u_0 must solve of the form:

$$u_0 = \left[\frac{1}{2} \left[(1-\beta) c_l^\rho + \beta u_0^\rho \right]^{\frac{\alpha}{\rho}} + \frac{1}{2} \left[(1-\beta) c_h^\rho + \beta u_0^\rho \right]^{\frac{\alpha}{\rho}} \right]^{\frac{1}{\alpha}} \tag{4}$$

Think of this equation as implicitly defining u_0 (whether it does so uniquely is an interesting question I am not asking). Why does this equation involve both of the parameters α and ρ?

(g) Use the equation (4) found in part (f) to compute u_0 and then the welfare loss, η, as defined in (b) of consuming the i.i.d. sequence described in (f).

Set the rest of the parameters in all cases as follows: $\beta = .95$ and $\log c_h = 1.02$ and $\log c_l = 0.98$. Using a computer find the welfare loss for all nine combinations of the parameters: $\alpha = 1,\ 1/2,\ -1$ and $\rho = 1,\ 1/2,\ -1$. Display your results for η on a table.

Hint. To solve you can iterate on this equation: plug in a good initial guess (such as $u_0 = 1$) on the right hand side and compute the LHS, then use this new number on the right hand side. Repeat this process until the numbers converge (when there absolute value difference is less than 10^{-10} say). You can work with any program for this: Matlab or Excel will certainly do.