14.452. Topic 2. Consumption/Saving and Productivity shocks

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April 2007

1. What starting point?

Want to start with a model with at least two ingredients:

- Shocks, so uncertainty. (Much of what happens is unexpected).

  Natural shocks if we want to get good times, bad times:

  Productivity shocks.

  Why not taste (discount rate) shocks?

- Basic intertemporal choice: Consumption/saving

Natural choice. **Ramsey** model, add technological shocks, and by implication uncertainty. (Think of it as basic version of Arrow-Debreu).

- Clear limits. Infinite horizons. No heterogeneity. No movements in employment. No money, etc

- Still good starting point:

  Shocks/propagation mechanisms

  Consumption smoothing.

- Good playground for conceptual and technical issues.

  Equivalence between centralized and decentralized eco

  Solving such models.

2. The optimization problem

$$\max E[\sum_0^\infty \beta^i U(C_{t+i})|\Omega_t]$$

subject to:

$$C_{t+i} + S_{t+i} = Z_{t+i}F(K_{t+i}, 1)$$

$$K_{t+i+1} = (1-\delta)K_{t+i} + S_{t+i}$$

(K_t capital at the beginning of period t).

- Central planning problem. Later decentralized interpretation.

- Infinite horizon. Separability. Exponential discounting. Assumptions of convenience.

- CRS. $N_t \equiv 1$. Z_t random variable, with mean Z.

  No growth. Could easily introduce Harrod neutral progress: $Z_t F(K_t, A_t N_t)$. Then, do all in efficiency units (divided by A_t).

- No separate saving/investment decisions. Same letter. What would be needed?

- Goal? Dynamic effects of Z on Y, C, S.

3. Deriving the first order conditions

Put the two constraints together:

$$K_{t+i+1} = (1-\delta)K_{t+i} + Z_{t+i}F(K_{t+i}, 1) - C_{t+i}$$

Easiest way: Lagrange multiplier(s). Associate $\beta^i \lambda_{t+i}$ with the constraint at time $t+i$ (Why do that rather than use just μ_{t+i} associated with the constraint at time $t+i$?)

The Lagrangian is given by:

$E[U(C_t)+\beta U(C_{t+1})-\lambda_t(K_{t+1}-(1-\delta)K_t-Z_tF(K_t,1)+C_t)-\beta\lambda_{t+1}(K_{t+2}-$
$(1-\delta)K_{t+1} - Z_{t+1}F(K_{t+1}, 1) + C_{t+1}) + \ldots \mid \Omega_t]$

The first order conditions for $t+i$ are given by:

$$C_{t+i} \ : \ E[\, U'(C_{t+i}) = \lambda_{t+i} \mid \Omega_t]$$

$$K_{t+i+1} \ : \ E[\, \lambda_{t+i} = \beta \lambda_{t+i+1}(1 - \delta + Z_{t+i+1} F_K(K_{t+i+1}, 1) \mid \Omega_t]$$

Define $R_{t+i+1} \equiv 1 - \delta + Z_{t+i+1} F_K(K_{t+i+1}, 1)$. Apply to time t, and use the fact that C_t, λ_t are known at time t, to get:

$$U'(C_t) = \lambda_t$$

$$\lambda_t = E[\beta R_{t+1} \lambda_{t+1} \mid \Omega_t]$$

4. Interpreting the two first order conditions

$$U'(C_t) = \lambda_t$$

$$\lambda_t = E[\beta R_{t+1}\lambda_{t+1} \mid \Omega_t]$$

Interpretation:

- The marginal utility of consumption must equal to the marginal value of capital. (wealth)
- The marginal value of capital must be equal to the expected value of the marginal value of capital tomorrow times the gross return on capital, times the subjective discount factor.

Merging the two:

$$U'(C_t) = E\left[\ \beta R_{t+1}U'(C_{t+1}) \mid \Omega_t\right]$$

This is the **Keynes-Ramsey condition**: **Smoothing** and **tilting**.

The Keynes-Ramsey condition. A variational argument

$$U'(C_t) = E\left[\ \beta R_{t+1} U'(C_{t+1}) \mid \Omega_t \right]$$

- Decrease consumption by Δ today, at a loss of $U'(C_t)\Delta$ in utility.
- Invest, to get $R_{t+1}\Delta$ next period
- Worth $E[\beta U'(C_{t+1}) R_{t+1}\Delta \, |\Omega_t]$ in terms of utility
- Along an optimal path, must be indifferent. Gives the condition.

Smoothing and tilting: A useful special case.

Use the constant elasticity function (which, with separability, in the context of uncertainty, also corresponds to the CRRA function):

$$U(C) = \frac{\sigma}{\sigma - 1} C^{(\sigma-1)/\sigma}$$

Then:

$$C_t^{-1/\sigma} = E[\ \beta R_{t+1} C_{t+1}^{-1/\sigma}\ |\ \Omega_t]$$

Or, as C_t is known at time t:

$$E[(\frac{C_{t+1}}{C_t})^{-1/\sigma} \beta R_{t+1} | \Omega_t] = 1$$

$$E[(\frac{C_{t+1}}{C_t})^{-1/\sigma} \beta R_{t+1} | \Omega_t] = 1$$

- Finance: Implications for equilibrium asset returns given consumption growth. (C-CAPM)
- Macro: Implications for consumption growth given asset returns (Often take R as non stochastic)
- Both endogenous. Two sides of the same coin.

Ignore uncertainty, so:

$$\frac{C_{t+1}}{C_t} = (\beta R_{t+1})^{\sigma}$$

As $\sigma \to 0$, then $C_{t+1}/C_t \to 1$.

As $\sigma \to \infty$, then $C_{t+1}/C_t \to +\infty/0$.

- Smoothing. If $\beta R = 1$, then $C_{t+1} = C_t$
- Tilting. Depends on σ

5. The effects of shocks. Using the FOCs and intuition

Look at the non stochastic steady state, Z constant:

$$C_t = C_{t+1} \;\Rightarrow\; R = 1/\beta \;\Rightarrow (1 - \delta + ZF_K(K^*, 1)) = 1/\beta \;\Rightarrow K^*$$

This is the modified golden rule. (Define θ as the discount rate, so $\beta = 1/(1+\theta)$. Then, the formula above becomes:

$$ZF_K(K^*, 1) - \delta = \theta$$

The other condition is simply:

$$ZF(K^*, 1) - \delta K^* = C$$

Effects of an unexpected permanent increase in Z?

Using intuition: Two effects on C:

- Level effect. C increases (by less than the increase in production. Why? Capital higher in steady state)
- Slope effect. ZF_K higher. Tilt consumption towards future. So C decreases.

- Net effect? On C: ambiguous (depends on σ). On S, I: unambiguous. On Y: increase, and further increase over time.

- If increase in Z is transitory. C up less, S, I up more, for less time.

- Positive co-movements. Good news.

- Taste shocks. (Decrease in β). Consumption up, Investment down. (Same production).

6. The effects of shocks. Actually solving the model

Solving the model is tough. Various approaches.

- Find special cases which solve explicitly.
- Ignore uncertainty, go to continuous time, and use a phase diagram.
- Linearize or log linearize, and get an explicit solution (numerically, or analytically).
- Set it up as a stochastic dynamic programming problem, and solve numerically.

Each is useful in its own right. Each one has shortcomings.

- The first (special cases) may be misleading.
- The second (ignoring uncertainty) evacuates the interesting effects of uncertainty.
- The third loses the non-linearities.
- The fourth may not work: There may be no SDP problem to which this is a solution.

7. A useful special case

$$U(C_t) = \log C_t$$

$$Z_t F(K_t, 1) = Z_t K_t^{\alpha} \text{ (Cobb Douglas)}$$

$$\delta = 1 \text{ (Full depreciation)}$$

The last assumption clearly the least palatable. Under these assumptions: (this is true whatever the process for Z_t)

$$C_t = (1 - \alpha\beta) Z_t K_t^{\alpha}$$

$$I_t = \alpha\beta \;\; Z_t K_t^{\alpha}$$

- A positive shock affects investment and consumption in the same way. Both increase in proportion to the shock.
- Response is independent of expectations of Z_t, whether the shock is transitory or permanent. Why?

7. Continuous time, ignoring uncertainty

Set up the model in continuous time. Pretend that people act as if they were certain.
Can then use a phase diagram to characterize the dynamic effects of shocks. Often very useful. (BF, Chapter 2)
The optimization problem:

$$\max \int_0^\infty e^{-\theta t} U(C_t)$$

subject to:

$$\dot{K_t} = ZF(K_t, 1) - \delta K_t - C_t$$

Then Keynes-Ramsey FOC (Use the maximum principle):

$$\dot{C_t} \; / Ct = \sigma(C)(ZF_K(K_t, 1) - \delta - \theta)$$

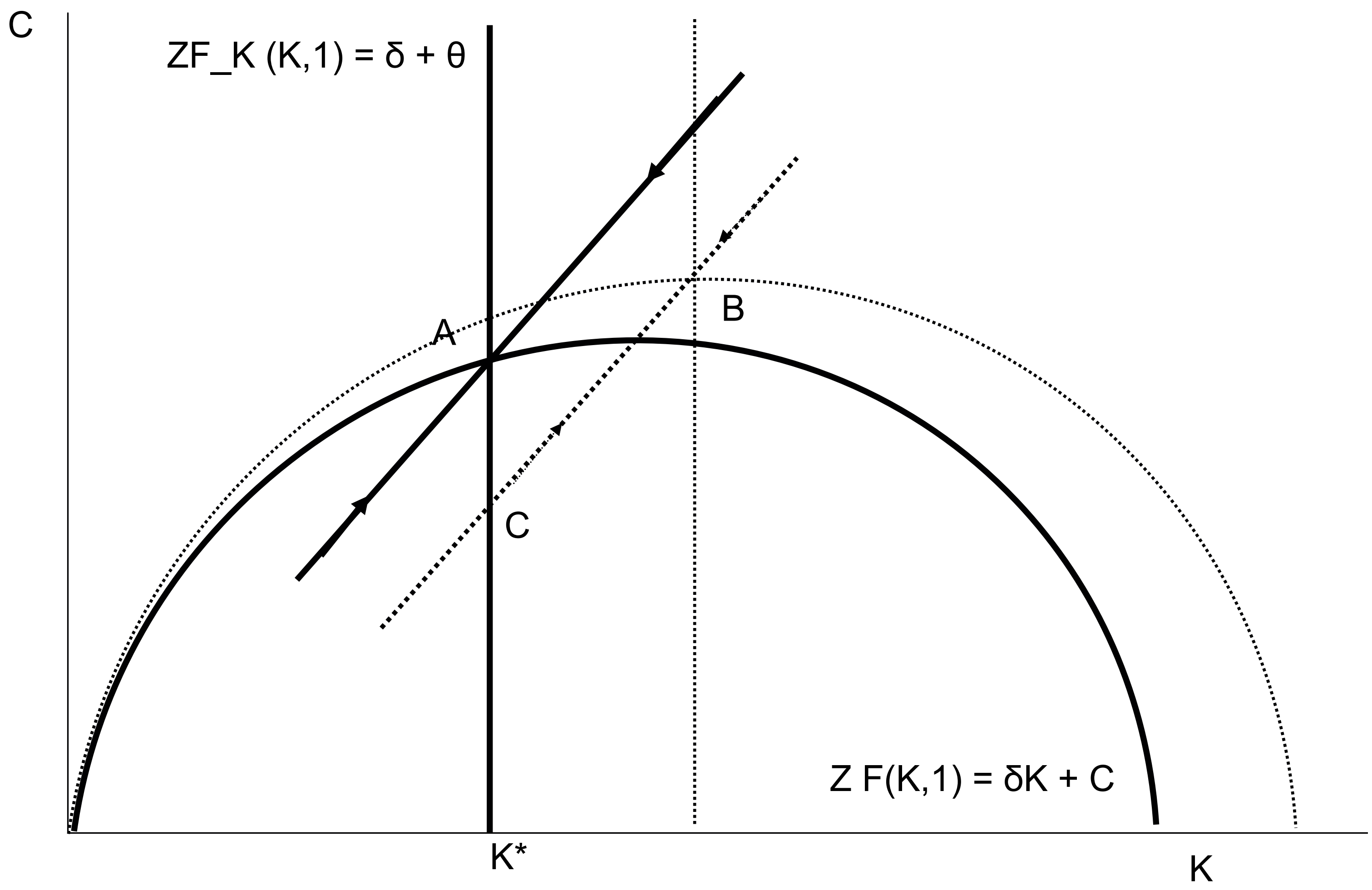
where $\sigma(C)$ is the elasticity of substitution evaluated at C. If CRRA, then σ is constant.

- Phase diagram. Keynes-Ramsey rule, and budget constraint. Saddle point, saddle path.

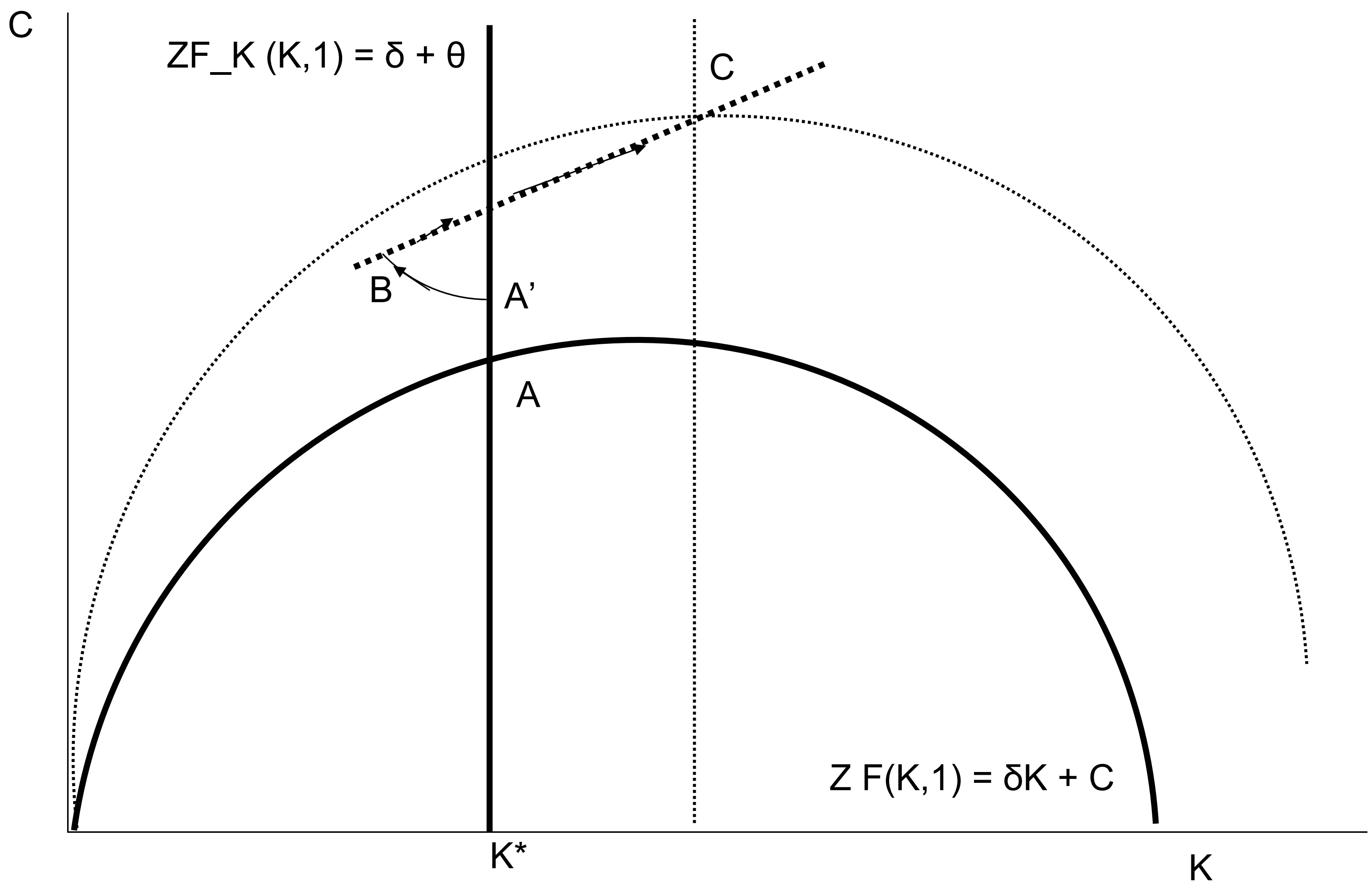
- Show the effect of a permanent (unexpected) increase in Z. Show whether C goes up or down is ambiguous and depends on σ.

- Can also look at the effects of an anticipated increase in Z, or a temporary increase. Make sure you know how to do it.

Equilibrium, and dynamics of the Ramsey model



Equilibrium, and dynamics of the Ramsey model.
The effects of an anticipated technological shock



8. Linearization or log linearization

The (original) FOC are a non linear difference system in K_t and C_t with forcing variable Z_t:
If linearize (or loglinearize, often a more attractive approximation. elasticities instead of derivatives), easy to solve.
Log linearizing around the steady state gives:

$$c_t = E[c_{t+1}|\Omega_t] - \sigma E[r_{t+1}|\Omega_t]$$

$$(R/F_K)E[r_{t+1}|\Omega_t] = (F_{KK}K/F_K)k_{t+1} + E[z_{t+1}|\Omega_t]$$

$$k_{t+1} = Rk_t - (C/K)c_t + (F/K)z_t$$

where small letters denote proportional deviations from steady state.

Then, replacing Er_{t+1} by the second expression, and replacing k_{t+1} by its value from the third expression, we get a linear system in c_t, $E[c_{t+1} \mid \Omega_t]$, k_{t+1}, k_t, $E[z_{t+1} \mid \Omega_t]$, and z_t.

$$\begin{bmatrix} Ec_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_t \\ k_t \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} z_t \\ Ez_{t+1} \end{bmatrix}$$

c_t: co-state, or jump variable. k_t: state variable.
One root of A inside the circle, one root outside.

This difference system can be solved in a number of ways.

Methods of solution

- In simple cases: undetermined coefficients. Guess:

$$c_t \text{ linear in } k_t, z_t, E[z_{t+1} \mid \Omega_t], E[z_{t+2} \mid \Omega_t]...$$

  Can solve for c_t as a function of any sequence of current and expected shocks.

- In general, solve explicitly, using matrix algebra: BK. Dynare as a nice MATLAB package.

- Advantage over SDP: Speed (no iteration). Can solve for arbitrary sequences of Ez_{t+i}, for example, the effect of an anticipated increase in Z in 50 quarters.

- If

$$z_t = \rho z_{t-1} + \epsilon_t$$

Then, all expectations of the future depend only on z_t. So, consumption is given by:

$$c_t \text{ linear in } k_t, z_t$$

Consumption rule: Consumption log linear in k_t and z_t. But the true consumption function is unlikely to be loglinear.

9. Stochastic dynamic programming.

Basic idea: Reduce to a two-period optimization problem.
If Z_t follows (for example) a first order AR(1), then all we need to predict future values of Z is Z_t. Then the value of the program depends only on K_t and Z_t. (Why?) So write it as $V(K_t, Z_t)$. Then rewrite the optimization problem as:

$$V(K_t, Z_t) = \max_{C_t, K_{t+1}} [U(C_t) + \beta E[V(K_{t+1}, Z_{t+1}) | \Omega_t]$$

subject to:

$$K_{t+1} = (1 - \delta)K_t + Z_t F(K_t, 1) - C_t$$

If we knew the form of the value function, then would be straightforward. We would get the rule:

$$C_t = C(K_t, Z_t)$$

We obviously do not know the value function. Easy to derive it numerically:

- Start with any function $V(.,.)$, call it $V_0(.,.)$.
- Use it as the function on the right hand side. Solve for optimal $C_0(.,)$.
- Solve for the implied $V_1(.,.)$ on the left hand side
- Use $V_1(.,.)$ on the right hand side, derive $C_1(.,.)$, and iterate.

Under fairly general conditions, this is a contraction mapping and the algorithm will converge to the value function and the optimal consumption rule. Various numerical issues/tricks. Need a grid for K, Z. But conceptually straightforward.

10. The decentralized economy

Many ways to describe the decentralized economy:

- Capital rented or bought by firms?
- Financing of firms through equity, bonds, retained earnings?

Assume capital owned by consumers, who rent it to firms. (Explore alternative: Purchase of capital by firms, financed through a mix of bonds, equities, and retained earnings.)

The goods, labor, capital services markets are competitive.

Consumers

- Each one has the same preferences as above.
- Each supplies one unit of labor inelastically in a competitive labor market, at wage W_t
- Each one can save by accumulating capital. Capital is rented out to firms every period in a competitive market for rental services, at net rental rate (rental rate net of depreciation) r_t (not the same r_t as in the log linearization).
- Each one owns an equal share of all firms in the economy, As firms operate under constant returns, profits are zero.
- Net supply of bonds is zero. Can ignore it. We could allow them to buy/sell bonds. In equilibrium, this would allow us to price bonds (equivalently determine the riskless rate).

Consumers, continued

- The budget constraint of consumers is therefore given by:

$$K_{t+1} = (1 + r_t)K_t + W_t - C_t$$

- So the first order condition is:

$$U'(C_t) = E[\ (1 + r_{t+1})\beta U'(C_{t+1}) \mid \Omega_t]$$

- And trivially (by assumption)

$$N_t = 1$$

Firms

- Firms have the same technology as above, namely $Y_t = Z_t F(K_t, N_t)$.
- Firms rent labor and capital. Their profit is therefore given by

$$\pi_t = Y_t - W_t N_t - (r_t + \delta) K_t$$

  The last term in parentheses is the gross rental rate.

- The value of a firm is given by:

$$\max E[\sum_{i=0}^{\infty} \beta^i \frac{U'(C_{t+i})}{U'(C_t)} \pi_{t+i} \mid \Omega_t]$$

- Given assumptions above, firms face a static choice, that of maximizing profit each period.

Firms, continued

- (Value) Profit maximization implies:

$$W_t = F_N(K_t, N_t)$$

$$r_t + \delta = Z_t F_K(K_t, N_t)$$

Equivalence

Using the relation between rental rate, and marginal product of capital, and replacing in the first order condition of consumers:

$$U'(C_t) = E[(1 - \delta + F_K(K_{t+1}, 1))\beta U'(C_{t+1} \mid \Omega_t]$$

Using the expressions for the wage and the rental rate in the budget constraint of consumers gives:

$$K_{t+1} = (1 - \delta + F_K(K_t, 1))K_t + F_N(K_t, 1) - C_t$$

Or

$$K_{t+1} = (1 - \delta)K_t + F(K_t, 1) - C_t$$

What is learned? Gives a different interpretation. Think about the consumers after a positive shock to Z_t. They anticipate higher wages, but also higher interest rates. What do they do?

Could not solve explicitly for consumption before (in the central planning problem), cannot now. But again, can cheat or consider special cases.

For example, ignore uncertainty, assume log utility and show (along the lines of BF, p50) that:

$$C_t = (1-\beta)[(1+r_t)K_t + H_t] \text{ where}$$

$$H_t \equiv [W_t + \sum_{i=1}^{\infty} \Pi_{j=1}^{j=i} (1+r_{t+j})^{-1} W_{t+j}]$$

A consumption function: Consumers look at **human wealth**, the present value of wages, plus **non human wealth**, capital.
They then consume a constant fraction of that total wealth. Whatever they do not consume, they save.

(What is the difference between this "consumption function" and the Keynes-Ramsey equation?)

On the derivation of the intertemporal budget constraint.
Make sure you understand how to go from a **dynamic budget constraint** to an **intertemporal budget constraint**]

- What additional condition is needed?
- Why does the derivation not go through under uncertainty?

Now think again about the effects of a technological shock. What are the effects at work in the two equations above?

11. Summary

- Consumption-saving: The Keynes-Ramsey condition
- Smoothing-tilting
- Productivity shocks, consumption, and investment.