14.452 Recitation #4: Solving Endogenous Growth Models

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Logistics

- Problem Set 4 due Monday at **noon**
- Final recitation next week

Outline for today

1. A roadmap for solving endogenous growth models
2. Knowledge spillovers
   - Case where $\phi = 1$ and $n = 0$
   - Case where $\phi < 1$ and $n > 0$
3. Schumpeterian growth model
4. Final remarks
5. More examples
   - Simple NGM with population growth
   - Lab Equipment model
   - Problem Set 4, Question 4

Section 1

A roadmap for solving endogenous growth models

Five steps

1. Identify the **state variable(s) X** of the model.
2. **Solve the production side** of the model conditional on the state variables, possibly up to an undetermined variable ℓ.
3. Write down the **"four equations"**.
4. **BGP:** Assume constant growth rates & solve the four equations for g and r.
5. **Trans. dynamics:** Exist if const. growth rate solution only holds if
    - a condition on **X**'s holds
    - $t \to \infty$

The four equations

- Law of motion of the state variable(s)

$$\dot{\mathbf{X}} = H(\mathbf{X}, \ell, C)$$

- Value(s) of innovation

$$rV = \pi(\mathbf{X}, \ell) + \dot{V}$$

- Free entry equation(s)

$$G(\mathbf{X}, V) = 0$$

- Euler equation

$$\dot{C} = C\frac{1}{\theta}(r - \rho)$$

Side remark on fourth step

- When solving for BGP, often useful to split equations into two blocks:
- First three equations: **"Demand for funds"** $r = r^d(g)$
  - given r, how much g will be generated?
- Euler equation: **"Supply of funds"** $r = r^s(g) = \rho + \theta g$
  - given g, how much r do households charge?

Section 2

Knowledge spillovers

Model

- Output

$$Y = \frac{1}{1-\beta}\left[\int_0^N x_\nu^{1-\beta} d\nu\right] L_E^\beta$$

- Intermediate goods produced at marg cost $\psi = 1 - \beta$
- Labor L is either employed (L_E) or does research (L_R), and grows at rate n
- Competitive production of ideas

$$\dot{N} = \eta N^\phi L_R$$

with L_R paid marginal product (if $L_R > 0$)

$$w = \eta N^\phi V$$

First step

- State variables are L and N.
- Given those, solve the model as much as possible $\rightarrow$ second step.

Second step: Intermediates

- Demand for intermediate goods

$$p_\nu = \frac{\partial Y}{\partial x_\nu} = x_\nu^{-\beta} L_E^{\beta} \;\Rightarrow\; x_\nu = L_E p_\nu^{-1/\beta}$$

- Thus

$$p_\nu = \underbrace{\frac{\beta^{-1}}{\beta^{-1} - 1}}_{\frac{1}{1-\beta}} \psi \equiv 1 \;\Rightarrow\; x_\nu = L_E$$

and profits are

$$\pi = \left(\frac{1}{1-\beta} - 1 \right) \psi x_\nu = \beta L_E$$

Second step: Aggregating

- This given output

$$Y = \frac{1}{1-\beta} N L_E$$

  and wages

$$w = \frac{\partial Y}{\partial L_E} = \frac{\beta}{1-\beta} N$$

- This determines the economy given state variables N, L **up to the fraction of employed workers** $\ell \equiv L_E / L$

Third step: Four equations

- Law of motion of the state variables

$$\dot{N} = \eta N^{\phi}(1-\ell)L$$

$$\dot{L} = nL$$

- Value of innovation

$$rV = \beta \ell L + \dot{V}$$

- Free entry equation

$$\frac{\beta}{1-\beta} N = w = \eta N^{\phi} V$$

- Euler equation

$$\dot{C} = C \frac{1}{\theta}(r-\rho)$$

Subsection 1

Case where $\phi = 1$ and $n = 0$

Fifth step: BGP for $\phi = 1$

- Have $L = const$. Set $\dot{N} = g_N N$. Then first three equations:

$$g_N = \eta(1-\ell)L$$

$$rV = \beta \ell L$$

$$V = \eta^{-1}\frac{\beta}{1-\beta}\ell L$$

which implies **"demand for funds"**

$$r = (1-\beta)\eta L \ell = (1-\beta)\left(\eta L - g\right)$$

- Euler: **"supply of funds"**

$$r = \rho + \theta g$$

- Can combine the two to get r and g

Sixth step: Transitional dynamics for $\phi = 1$?

- Transitional dynamics? $\rightarrow$ did not find any conditions on state variables N, L here.
- $\rightarrow$ **No transitional dynamics!**

Subsection 2

Case where $\phi < 1$ and $n > 0$

Fifth step: BGP for $\phi < 1$

- Already have $\dot{L} = nL$. Set $\dot{N} = g_N N$. Then first three equations imply

$$g_N = \eta N^{\phi-1}(1-\ell)L$$

$$V = \frac{\beta}{1-\beta}\eta^{-1}N^{1-\phi}$$

with derivative

$$\frac{\dot{V}}{V} = (1-\phi)g_N$$

and thus

$$r = \frac{\pi}{V} + \frac{\dot{V}}{V} = \frac{\beta\ell L}{\frac{\beta}{1-\beta}\eta^{-1}N^{1-\phi}} + (1-\phi)g_N$$

- Note: **Per capita** consumption grows at rate $g_C = g_Y - n = g_N \equiv g$. Vertical demand curve for funds. Euler $\Rightarrow$ r and g

Sixth step: Transitional dynamics for $\phi < 1$

- Other conditions for BGP:

$$g = \eta N^{\phi-1}(1-\ell)L$$

$$r = \frac{\beta \ell L}{\frac{\beta}{1-\beta}\eta^{-1}N^{1-\phi}} + (1-\phi)g$$

- These two equations together give you ℓ and a condition involving N and L

$$\ell = \frac{r-n}{r-n+(1-\beta)g}$$

$$\eta N^{\phi-1}L = \frac{r-n}{1-\beta} + g$$

- $\Rightarrow$ **Transitional dynamics!**

Section 3

Schumpeterian growth model

Model

- Output

$$Y = \frac{1}{1-\beta} \int_0^1 q_\nu x_\nu^{1-\beta} d\nu L^\beta$$

- Intermediate good produced at marg cost $\psi q_\nu = (1-\beta) q_\nu$
- **Quality ladder:** $q_\nu(t) = \lambda^{n_\nu(t)} q_\nu(0)$
- Competitive production of better quality: **flow rate z_ν of success**

$$z_\nu = \frac{\eta}{q_\nu} Z_\nu$$

with **free entry**

$$\frac{\eta}{q_\nu} \lambda V_\nu = 1$$

(holds if λ suff. large – "drastic regime")

First and second step

- State variables: q_ν. But: Will see that only $Q \equiv \int_0^1 q_\nu d\nu$ matters!
- Given q_ν, solve for the production side:
- Demand for intermediates

$$x_\nu = q_\nu^{1/\beta} p_\nu^{-1/\beta} L$$

hence the optimal price is

$$p_\nu = \frac{1}{1-\beta}\psi q_\nu \equiv q_\nu,$$

quantities are

$$x_\nu = L,$$

and profits are

$$\pi_\nu = \beta q_\nu L$$

Second step (cont'd)

- Aggregating things up ...

$$Y = \frac{1}{1-\beta} \int_0^1 q_\nu d\nu \quad L \equiv \frac{1}{1-\beta} QL$$

$$w = \frac{\partial Y}{\partial L} = \frac{\beta}{1-\beta} Q$$

- These only depend on Q (rather than q_ν)
- Law of motion of Q:

$$Q_{t+dt} = zdt \cdot \lambda Q + (1 - zdt) \cdot Q \;\; \Rightarrow \; \dot{Q} = z(\lambda - 1) Q$$

where $z \equiv \eta \int Z_\nu d\nu / Q$

Third step: Four equations (1)

- Valuation and free entry:
- Valuation

$$rV_\nu = \pi_\nu + z_\nu(0 - V_\nu) + \dot{V}_\nu$$

$$\frac{\eta}{q_\nu}\lambda V_\nu = 1$$

- Notice that V_ν scales in q_ν. Define $v \equiv \frac{V_\nu}{q_\nu}$. Gives

$$rv = \beta L + z(0 - v) + \dot{v}$$

$$\eta\lambda v = 1$$

- This is why we're okay using Q instead of the q_ν's in the four equations!

Third step: Four equations (2)

- Law of motion

$$\dot{Q} = (\lambda - 1)zQ$$

- Valuation

$$rv = \beta L + z(0 - v) + \dot{v}$$

- Free entry

$$\eta \lambda v = 1$$

- Euler:

$$\dot{C} = C\frac{1}{\theta}(r - \rho)$$

Fourth step: BGP

- Assume constant growth rates then:

$$g_Q = (\lambda - 1)z$$

- Valuation + free entry $\Rightarrow$

$$r = \eta\lambda\beta L - z$$

- **Demand for funds**

$$r = \eta\lambda\beta L - \frac{g}{\lambda - 1}$$

- **Supply of funds** is standard

$$r = \rho + \theta g$$

- Together $\Rightarrow r, g$

Fifth step: Transitional dynamics?

- We did not end up with any conditions on state variables $\Rightarrow$ **no transitional dynamics**

Section 4

Final remarks

Remarks

- Link between g_X and g_C is provided by **per capita output** $y = Y(X)/L$ that we solve for in second step
- If you want to find $C(0)$, add the resource constraint to the four equations
- In all these models, parameters need to ensure **TVC** and **positive growth**

$$r > g_Y$$

$$g > 0$$

- Today: focus on cases where there exists an "exact BGP" (rather than only asymptotic BGP)
  - could extend method to allow for asymptotic BGP

Section 5

More examples

Subsection 1

Simple NGM with population growth

First and second step

- State variables are K, L.
- Given $K, L \Rightarrow$ ouput is given by $Y = F(K, L)$

Third step: Four equations

- Laws of motion

$$\dot{K} = F(K, L) - \delta K - C$$
$$\dot{L} = nL$$

- Valuation

$$rV = F_K(K, L) - \delta + \dot{V}$$

- Free entry

$$V = 1$$

- Euler

$$\dot{C} = C\frac{1}{\theta}(r - \rho)$$

Fourth and fifth step: BGP

- Assuming constant growth rates:

$$g_K = F(1, L/K) - \delta - C/K$$

which means $n = g_L = g_K$ giving **demand for funds**

$$g = n$$

- **Supply of funds** is standard

$$r = \rho + \theta g$$

- In addition: For BGP need

$$r = F_K(K, L) - \delta$$

which is a restriction of the two state variables K and L! $\Rightarrow$ **transitional dynamics!**

Subsection 2

Lab Equipment model

First and second step

- State variable N
- Solving the production side gives (see lecture notes)

$$Y = \frac{1}{1-\beta} NL$$

$$X^{interm} = (1-\beta) NL$$

$$w = \frac{\beta}{1-\beta} N$$

$$\pi = \beta L$$

Third step: Four equations

- Law of motion

$$\dot{N} = \eta Z \;\; (= \eta(Y - X - C)$$

- Value of innovation

$$rV = \beta L + \dot{V}$$

- Free entry

$$\eta V = 1$$

- Euler

$$\dot{C} = C\frac{1}{\theta}(r - \rho)$$

Fourth step: BGP

- Assuming constant growth rates:

$$g_N N = \eta Z$$

- Value of innovation + free entry $\Rightarrow$ **demand for funds**

$$r = \eta \beta L$$

- **Supply of funds** is standard

$$r = \rho + \theta g$$

- No condition on values of state variable N $\Rightarrow$ **no transitional dynamics!**

Subsection 3

Problem Set 4, Question 4

First and second step

- State variables N and L
- Solving the production side gives

$$Y = N^{(1-\beta)/\beta} L_E$$

$$w = \beta N^{(1-\beta)/\beta}$$

$$\pi = (1-\beta)\frac{Y}{N}$$

- Denote $\ell = L_E / L$

Third step: Four equations

- Laws of motion of N and L

$$\dot{N} = \eta N^{\phi}(1-\ell)L$$

$$\dot{L} = nL$$

- Value of innovation

$$rV = (1-\beta)\frac{Y}{N} + \dot{V}$$

- Free entry of labor

$$\beta N^{(1-\beta)/\beta} = w = \eta N^{\phi} V$$

- Euler

$$\dot{C} = C\frac{1}{\theta}(r-\rho)$$

- Rest is similar to knowledge spillover section above!