Exam 3

14.30 Fall 2003

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Instructions: This exam is closed-book and closed-notes. You may use a simple calculator. Please read through the exam in order to ask clarifying questions and to allocate your time appropriately. You **must show** all your calculations. You have approximately 120 minutes to complete the exam (90 points). Good Luck!

1. (12 points, 4 each) True, False, or Uncertain. Please explain briefly (but, no explanation no points).

a. If an estimator is consistent, it is also unbiased. Thus, maximum likelihood estimators are unbiased.

b. A interval estimation is just the probability that your estimator (say $\bar{x}$) is contained in a random interval with probability $1 - \alpha$.

c. The definition of the power function is: $\pi(\theta|\delta) = P(\text{rejecting} H_0 | \theta \in \Omega_0)$. Thus, $\alpha_\theta + \beta_\theta = 1$ only when $\theta \in \Omega_0$.

d. The standard error is an estimator of the standard deviation of the population.

2. (17 points) Assume a random sample of size n from a population $f_X(x)$, where $E(X_i) = \theta_1$ and $Var(X_i) = \theta_2$. Define 4 statistics as follows:

$$T_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad T_2 = \frac{1}{n-3}\sum_{i=1}^{n} X_i \qquad T_3 = \frac{1}{n}\sum_{i=1}^{n/2} X_i \qquad T_4 = \frac{1}{n^2}\sum_{i=1}^{n} X_i$$

(Remember to show your work in detail.)

a.(3 points) What is the expected value of the statistics T_2 and T_3? Are they unbiased?

b.(4 points) Are the statistics T_2, T_3 and T_4 consistent? Prove your result. (You can

use any of the two limiting result we saw in class without proving them).

c.(4 points) Among statistics T_1 and T_3, which has a smaller variance?

d.(6 points) Assume $n = 3$ and the sample $x_1 = 1, x_2 = 5, x_3 = 3$. Compute the sample variance, the variance of the sample mean, and the method of moment estimator for θ_2.

3. (17 points) Assume a random sample of size n from a population $f_Y(y)$, where:

$$f_Y(y) = \begin{cases} \theta y^{\theta-1} & \text{if } 0 < y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and θ is an unknown parameter such that $\theta \in (0, \infty)$.

a.(7 points) Find the MLE estimator of the mean of the distribution.

b.(10 points) Now assume that the database you have is incomplete in the following sense. For every observation lower or equal than 0.4 the value reported is 0.4. Otherwise, for every value higher than 0.4, the true value is reported (assume that not all the sample is in the range of $(0, 0.4]$). Find the MLE estimator of θ based on the limited information you have at hand. (Hint: Think in the intuitive approach we saw in class to the likelihood function.)

4. (9 points) Assume the following 2 random samples, where only the parameters σ_X^2 and σ_Y^2 are known:

$$X_i \sim N(\mu_X, \sigma_X^2) \quad \text{of sample size } n_X$$
$$Y_i \sim N(\mu_Y, \sigma_Y^2) \quad \text{of sample size } n_Y,$$

Perform an interval estimation for $\mu_X + \mu_Y$ with a confidence level of $1 - \alpha = 95\%$. Assume $n_X \neq n_Y$. (Show your work in detail.)

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5. (23 points) Professor H. Testing is trying to study the level of personal net worth of a low income population. In particular he is interested in testing the hypothesis that the mean level of net worth is $0 (net worth can be positive or negative and is measured in thousands of dollars). He does not know the distribution of the level of net worth of the population. For his study, you (his RA) collected a sample from 100 persons asking them their level of net worth. The results were $\bar{x} = 3$ and $s^2 = 100$. (Assume the following unless you are told otherwise: you choose the persons randomly, they report their true net worth, and the population is sufficiently large.)

You want to impress Professor Testing. For that, you would like to give him not only the values you got from the random sample, but a complete report on whether his hypothesis is rejected or not, as well as some properties of the test you did...I will help you a bit with some instructions:

(i) Provide a simple hypothesis test to evaluate Professor Testing's hypothesis. Specify your decision rule such that the probability of Type I Error is 5%. Do not use the t-distribution. (You can use any of the two limiting result we saw in class without proving them.)

(ii) Generally, researchers choose between two options with respect to the distribution you used in part (i). Describe how these 2 distributions are graphically different. How would you justify the use of the t-distribution? How important is this argument for your sample size?

(iii) Test the null hypothesis using the test defined in part (i) and the results of the random sample. State whether or not there is enough statistical evidence that the hypothesis defined by Professor H. Testing can be disregarded. Explain what you mean by 'enough statistical evidence'.

(iv) Find the p-value of the test you performed.

(v) Write an expression for the probability of Type II Error for your test. On what will it depend?

(vi) Collecting the data you suspect that 80% of the people that answered your question about their net worth, was not very accurate in their answer. Based on the answers

you trust, you think that assuming a normally distributed population is not a bad assumption (although you do not know any of the two parameters). For robustness, you would like to present the result of another test, using only the 20 observations you believe are 100% reliable. Assume that with these 20 observations you got the same sample mean and sample variance as before: $\bar{x} = 3$ and $s^2 = 100$. Do the corresponding test and see if you reject the null. Explain the intuition of your result in light of your result in part (iii).

6. (12 points) Assume a random sample of size n from a $N(\mu, 1)$, where μ is unknown. Suppose the following hypotheses:

$$H_0 : \mu = 0$$
$$H_1 : \mu = 1$$

a.(8 points) Define the Decision Rule and the sample size needed (n) in order to perform a Simple 1-Sided Test with a size of 1% and a 5% probability of Type II Error. (Express your results in decimals.)

b.(4 points) What can you say about the optimality of your test? Explain.

The End...

...and Happy Holidays!!!