14.30 Exam #1 Solutions
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Question 1:

A. True/False/Uncertain:

i. False. Two events can be neither disjoint nor exhaustive. For example, consider the outcome of a single fair die: $\{1,2,3,4,5,6\}$. The events $A = \{1,2\}$ and $B = \{1,3\}$ are not disjoint, since: $A \cap B = \{1\} \neq \phi$. They are also not exhaustive because: $A \cup B = \{1,2,3\} \neq \{1,2,3,4,5,6\}$

ii. False. For example consider two (independent) tosses of a fair coin, where A is an indicator for heads in the first toss and B is an indicator for heads in the second toss. A and B are clearly independent, but: $\Pr(A \cap B) = \frac{1}{4} \neq 0$. Also: $\Pr(A \cup B) = \frac{3}{4}, \Pr(A) = \Pr(B) = \frac{1}{2}$ so: $\frac{3}{4} = \Pr(A \cup B) \neq \Pr(A) + \Pr(B) = 1$.

iii. False. Using events A and B defined in the previous answer $\Pr(A) = \frac{1}{2}, \Pr(A \cap B) - \Pr(A \cap B^c) = \frac{1}{4} - \frac{1}{4} = 0$ so: $\Pr(A) \neq \Pr(A \cap B) - \Pr(A \cap B^c)$.

iv. True. $1 = \int_0^1 \int_0^1 f_{X,Y}(x,y)\, dxdy = \int_0^1 \int_0^1 ke^{x+y} dxdy = k(e-1)^2 \Rightarrow f_{X,Y}(x,y) = \begin{cases} (e-1)^{-2} e^{x+y} \text{ if } x, y \in [0,1] \\ 0 \text{ otherwise} \end{cases}$. For $x \in [0,1] : f_X(x) = \int_0^1 (e-1)^{-2} e^{x+y} dy = e^x (e-1)^{-1}$ and otherwise $f_X(x) = 0$. Similarly for $y \in [0,1]$: $f_Y(y) = e^y (e-1)^{-1}$ and otherwise: $f_Y(y) = 0$. Thus: $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ so X and Y are independent. A shortcut also exists: notice that you can write: $f_{X,Y}(x,y) = ke^{x+y} = g_X(x) h_Y(y) = (k_1 e^x)(k_2 e^y)$ so X and Y are independent.

v. False. The statement is true for the pmf but not for a pdf. For example if $X \sim U[0,1]$ then $f_X(0.5) = 1$ but $\Pr(X = 0.5) = 0$.

vi. False. Suppose: $\Pr(X = 0) = \Pr(X = 1) = 0.5$ and $Y = X$ then: $F_X(0.5) = 0.5 \neq F_{X|Y}(0.5|1) = 1$. The statement is true only for two independent random variables.

B. To calculate the first probability we need to get the correct limits of integration: $P(X + Y < 1 \text{ and } Y < 0.5) = \int_0^{0.5} \int_0^y (8xy)\, dxdy$. The second probability requires integrating over two sections: $P(X + Y < 1 \text{ and } Y < 0.6) = \int_0^{0.5} \int_0^y (8xy)\, dxdy + \int_{0.5}^{0.6} \int_0^{1-y} (8xy)\, dxdy$.

Question 2:

a. For the first two places: sample two letters with repetition: 26^2. For the last three places: sample three numbers with repetition: 10^3. Total number of different plates: $26^2 10^3 = 676\,000$.

b. Sample two letters without repetition: $\binom{26}{2}$. Sample three numbers without repetition: $\binom{10}{3}$. Arrange the five symbols: 5!. Total number of different plates: $\binom{26}{2}\binom{10}{3}5! = 4680\,000$.

c. We have three possible locations for "777": beginning, middle or end. For each of these locations we need to sample two letters with repetition 26^2. so the numerator is:: $3 \cdot 26^2$. In the denominator we have to choose two slots for letters: $\binom{5}{2}$. For each slot with a letter there are 26 possible letters and for each number slot there are 10 possible numbers. The denominator is therefore: $\binom{5}{2}26^2 10^3 = 6.76 \times 10^6$. The answer is that the probability of "777" is: $\frac{3 \cdot 26^2}{\binom{5}{2}26^2 10^3} = \frac{3}{10\,000}$.

Question 3:

a. $\Pr(B_1|G) = \frac{\Pr(G|B_1)\Pr(B_1)}{\Pr(G|B_1)\Pr(B_1)+\Pr(G|B_2)\Pr(B_2)+\Pr(G|B_3)\Pr(B_3)} = \frac{1\cdot(1/3)}{1\cdot(1/3)+(1/2)\cdot(1/3)+x\cdot(1/3)} = \frac{2}{3+2x}$

b. $(16/30) = \Pr(G) = \Pr(G|B_1)\Pr(B_1) + \Pr(G|B_2)\Pr(B_2) + \Pr(G|B_3)\Pr(B_3) = 1 \cdot (1/3)+(1/2)\cdot(1/3)+x\cdot(1/3) = (1/2)+(x/3) \Rightarrow x/3 = (16-15)/30 = 1/30 \Rightarrow x = 0.1$

c. Barber 1 never gives a bad haircut, so if we received one bad haircut (or more) we could not have gotten him, so: $\Pr(B_1|G_1 \cap G_2 \cap G_3 \cap NG_4 \cap G_5 \cap G_6) = 0$.

d. $\Pr(B_1|G_1) = \frac{10}{16}$ (using the result from part a. and $x = 0.1$). Similarly: $\Pr(B_2|G_1) = \frac{\Pr(G_1|B_2)\Pr(B_2)}{\Pr(G_1|B_1)\Pr(B_1)+\Pr(G_1|B_2)\Pr(B_2)+\Pr(G_1|B_3)\Pr(B_3)} = \frac{(1/2)\cdot(1/3)}{1\cdot(1/3)+(1/2)\cdot(1/3)+x\cdot(1/3)} = \frac{1}{2x+3} = \frac{5}{16}$. Since B_1, B_2 and B_3 are exhaustive and mutually exclusive: $\Pr(B_1|G_1) + \Pr(B_2|G_1) + \Pr(B_3|G_1) = 1$ and it follows that: $\Pr(B_3|G_1) = 1 - \frac{10}{16} - \frac{5}{16} = \frac{1}{16}$. In other words, having seen G_1 you update the probability that you will see each of the barbers, so that the probability of meeting barber 1 is $\frac{10}{16}$, in which case you will get a good haircut with probability 1. Your updated probability of meeting barber 2 is $\frac{5}{16}$, in which case you will get a good haircut with probability $\frac{1}{2}$. Finally, your probability of meeting barber 3 is $\frac{5}{16}$ and if you do you will get a good haircut with probability 0.1. Therefore: $\Pr(G_2|G_1) = \frac{10}{16} \cdot 1 + \frac{5}{16} \cdot \frac{1}{2} + \frac{1}{16} \cdot \frac{1}{10} = \frac{63}{80} = 0.787\,5$

Question 4:

a. Denote the support of X by $[0, A]$ then the integral over the cdf is: $\frac{1}{2}\cdot\frac{1}{2}\cdot 1+\frac{1}{2}\cdot\left(A-\frac{1}{2}\right)\cdot 1=1 \Rightarrow A=2$. So: $f_X(x)=\begin{cases} 2x \text{ if } x\in[0,0.5] \\ \frac{4-2x}{3} \text{ if } x\in(0.5,2] \\ 0 \text{ otherwise}\end{cases}$, $F_X(x)=\begin{cases} 0 \text{ if } x<0 \\ x^2 \text{ if } x\in[0,0.5] \\ \frac{4x-x^2-1}{3} \text{ if } x\in(0.5,2] \\ 1 \text{ if } x>2\end{cases}$

b. Intuitively, all we have to do is multiply the pdf by 0.8 and add the discrete probability: $f_Y(y)=\begin{cases} 1.6y \text{ if } y\in[0,0.5] \\ \frac{8(2-y)}{15} \text{ if } y\in(0.5,2] \\ 0.2 \text{ if } y=10 \\ 0 \text{ otherwise}\end{cases}$. Now we can easily calculate the cdf: $F_Y(y)=\begin{cases} 0 \text{ if } y<0 \\ 0.8y^2 \text{ if } y\in[0,0.5] \\ \frac{4\left(4y-y^2-1\right)}{15} \text{ if } y\in(0.5,2] \\ 0.8 \text{ if } y\in(2,10) \\ 1 \text{ if } y>10\end{cases}$