6.207/14.15: Networks
Lecture 11: Giant Component, Generalized Random Graphs

Outline

- Emergence and size of a giant component in Erdös-Renyi graphs
- An application: contagion and diffusion
- Generalized random graph models
- Graphs with prescribed degrees – configuration model
- Emergence of a giant component in the configuration model

Reading:

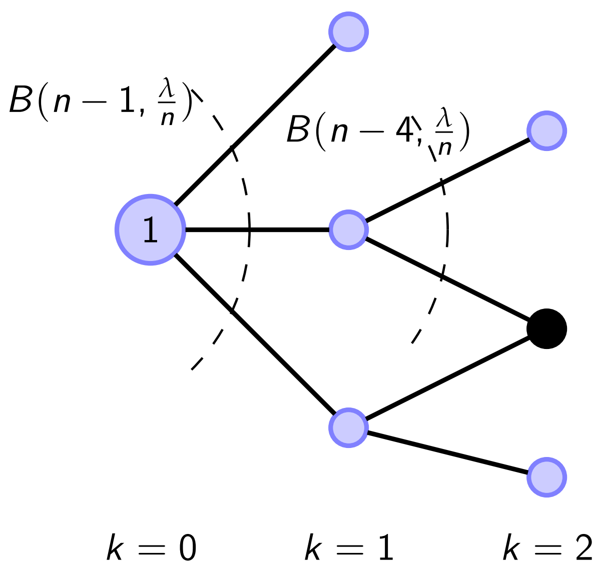
- Newman, Sections 12.1-12.5, 12.7-12.8.
- Newman, Sections 13.2 (skip 13.2.2), 13.3,13.4.

Giant Component

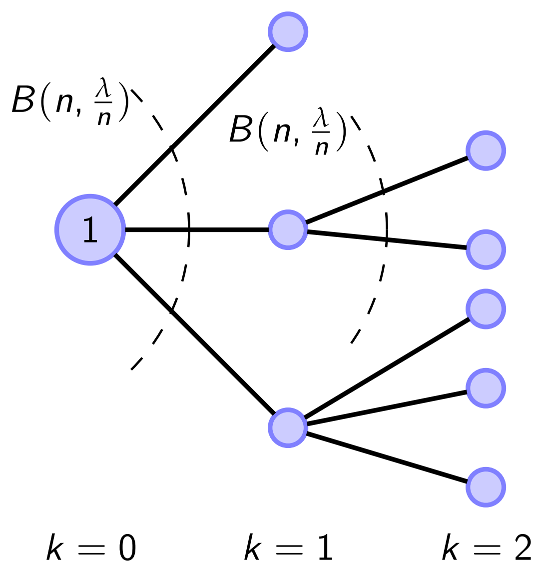
- We have shown that when $p(n) \ll \frac{\log(n)}{n}$, the Erdös-Renyi graph is disconnected with high probability.
- In cases for which the network is not connected, the component structure is of interest.
- We have argued that in this regime the expected number of isolated nodes goes to infinity. This suggests that the Erdös-Renyi graph should have an arbitrarily large number of components.
- We will next argue that the threshold $p(n) = \frac{\lambda}{n}$ plays an important role in the component structure of the graph.
  - For $\lambda < 1$, all components of the graph are "small".
  - For $\lambda > 1$, the graph has a (unique) giant component, i.e., a component that contains a constant fraction of the nodes.

Emergence of the Giant Component—1

- We will analyze the component structure in the vicinity of $p(n) = \frac{\lambda}{n}$ using a branching process approximation.
- We assume $p(n) = \frac{\lambda}{n}$.
- $B(n, \frac{\lambda}{n})$: binomial random variable with parameters $n, \frac{\lambda}{n}$.
- Consider starting from node 1 and exploring the graph.



(a) Erdos-Renyi graph process.



(b) Branching Process Approx.

Emergence of the Giant Component—2

- We first consider the case when $\lambda < 1$.
- Let Z_k^G and Z_k^B denote the number of individuals at stage k for the graph process and the branching process approximation, respectively.
- In view of the "overcounting" feature of the branching process, we have
$$Z_k^G \leq Z_k^B \qquad \text{for all } k.$$
- From branching process analysis (see Lecture 3 notes), we have
$$\mathbb{E}[Z_k^B] = \lambda^k,$$
(since the expected number of children is given by $n \times \frac{\lambda}{n} = \lambda$).
- Let S_1 denote the number of nodes in the Erdös-Renyi graph connected to node 1, i.e., the size of the component which contains node 1.
- Then, we have
$$\mathbb{E}[S_1] = \sum_k \mathbb{E}[Z_k^G] \leq \sum_k \mathbb{E}[Z_k^B] = \sum_k \lambda^k = \frac{1}{1-\lambda}.$$

Emergence of the Giant Component—3

- The preceding result suggests that for $\lambda < 1$, the sizes of the components are "small".

Theorem

Let $p(n) = \frac{\lambda}{n}$ and assume that $\lambda < 1$. For all (sufficiently large) $a > 0$, we have

$$\mathbb{P}\left(\max_{1 \leq i \leq n} |S_i| \geq a \log(n) \right) \to 0 \quad \text{as } n \to \infty.$$

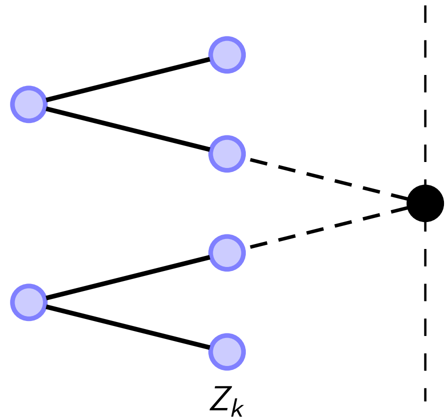
Here $|S_i|$ is the size of the component that contains node i.

- This result states that for $\lambda < 1$, all components are small [in particular they are of size $O(\log(n))$].
- Proof is beyond the scope of this course.

Emergence of the Giant Component—4

- We next consider the case when $\lambda > 1$.
- We claim that $Z_k^G \approx Z_k^B$ when $\lambda^k \leq O(\sqrt{n})$.
- The expected number of conflicts at stage $k+1$ satisfies

$$\mathbb{E}[\text{number of conflicts at stage } k+1] \approx np^2\mathbb{E}[Z_k^2] = n\frac{\lambda^2}{n^2}\mathbb{E}[Z_k^2].$$



- We assume for large n that Z_k is a Poisson random variable and therefore $\text{var}(Z_k) = \lambda^k$. This implies that

$$\mathbb{E}[Z_k^2] = \text{var}(Z_k) + \mathbb{E}[Z_k]^2 = \lambda^k + \lambda^{2k} \approx \lambda^{2k}.$$

- Combining the preceding two relations, we see that the conflicts become non-negligible only after $\lambda^k \approx \sqrt{n}$.

Emergence of the Giant Component—5

- Hence, there exists some $c > 0$ such that $\mathbb{P}(\text{there exists a component with size } \geq c\sqrt{n} \text{ nodes}) \to 1$ as $n \to \infty$.
- Moreover, between any two components of size $\sqrt{n}$, the probability of having a link is given by

$$\mathbb{P}(\text{there exists at least one link}) = 1 - (1 - \frac{\lambda}{n})^n \approx 1 - e^{-\lambda},$$

  i.e., it is a positive constant independent of n.
- This argument can be used to see that components of size $\leq \sqrt{n}$ connect to each other, forming a connected component of size qn for some $q > 0$, a giant component.

Size of the Giant Component

- Form an Erdös-Renyi graph with $n-1$ nodes with link formation probability $p(n) = \frac{\lambda}{n}$, $\lambda > 1$.
- Now add a last node, and connect this node to the rest of the graph with probability $p(n)$.
- Let q be the fraction of nodes in the giant component of the $n-1$ node network. We can assume that for large n, q is also the fraction of nodes in the giant component of the n-node network.
- The probability that node n is not in the giant component is given by

$$\mathbb{P}(\text{node } n \text{ not in the giant component}) = 1 - q \equiv \rho.$$

- The probability that node n is not in the giant component is equal to the probability that none of its neighbors is in the giant component, yielding

$$\rho = \sum_{k=0}^{n-1} p_k \rho^k \equiv \Phi(\rho).$$

- Like before, this equation has a fixed point $\rho^* \in (0,1)$.

An Application: Contagion and Diffusion

- Consider a society of n individuals.
- A randomly chosen individual is infected with a contagious virus.
- Assume that the network of interactions in the society is described by an Erdös-Renyi graph with link probability p.
- Assume that any individual is immune with a probability π.
- We would like to find the expected size of the epidemic as a fraction of the whole society.
- The spread of disease can be modeled as:
    - Generate an Erdös-Renyi graph with n nodes and link probability p.
    - Delete πn of the nodes uniformly at random.
    - Identify the component that the initially infected individual lies in.
- We can equivalently examine a graph with $(1-\pi)n$ nodes with link probability p.

An Application: Contagion and Diffusion

- We consider 3 cases:
- $p(1-\pi)n < 1$:

$$\mathbb{E}[\text{size of epidemic as a fraction of the society}] \leq \frac{\log((1-\pi)n)}{n} \approx 0.$$

- $1 < p(1-\pi)n < \log((1-\pi)n)$:

$$\mathbb{E}[\text{size of epidemic as a fraction of the society}] = \frac{qq(1-\pi)n + (1-q)\log((1-\pi)n))}{n} \approx q^2(1-\pi),$$

where q denotes the fraction of nodes in the giant component of the graph with $(1-\pi)n$ nodes, i.e., $q = 1 - e^{-q(1-\pi)np}$.

- $p > \frac{\log((1-\pi)n)}{(1-\pi)n}$:

$$\mathbb{E}[\text{size of epidemic as a fraction of the society}] = (1-\pi).$$

Configuration Model—1

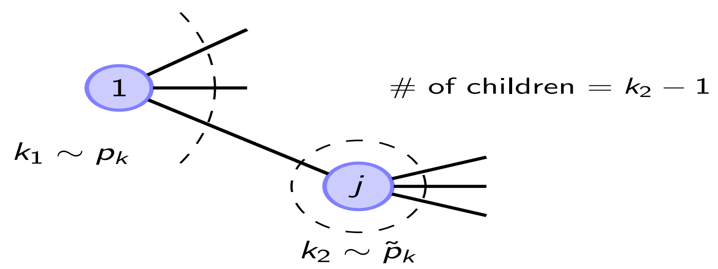
- We have seen that the Erdös-Renyi model has a Poisson degree distribution, which falls off very fast.
- Our next goal is to generate random networks with a "given degree distribution".
- One of the most widely method used for this purpose is the configuration model developed by Bender and Canfield in 1978.
- The configuration model is specified in terms of a degree sequence, i.e., for a network of n nodes, we have a desired degree sequence $(k_1, \ldots, k_n)$, which specifies the degree k_i of node i, for $i = 1, \ldots, n$.
  - Given a degree distribution p_k, we can generate the degree sequence for n nodes by sampling the degrees independently from the distribution p_k, i.e., $k_i \sim p_k$.
  - A law of large numbers argument establishes that the frequency of degrees $p_k^{(n)}$ converges to the degree distribution p_k as n goes to infinity.

Configuration Model—2

- Given the degree k_i for node i for all $i = 1, \ldots, n$, we create a random network with these degrees as follows:
- We give each node i, k_i "stubs" sticking out of it, which are ends of edges-to-be (there are a total of $\sum_i k_i = 2m$ stubs, where m is the number of edges).
- We choose two stubs uniformly at random and create an edge between the corresponding nodes.
- We choose another pair from the remaining $2m - 2$ stubs, connect those and continue until all the stubs are used up.
- Remarks:
  - This process generates each possible matching of stubs with equal probability.
  - The sum of degrees needs to be even (or else an entry will be left out at the end).
  - It is possible to have self-edges and multiedges.

Distribution of the Degree of a Neighboring Node—1

- We will use a branching process approximation to study the giant component in the configuration model.
- For this we need to understand the distribution of the degree of a neighboring node, i.e., given some node i with degree d_i, consider a neighbor j. What is the degree distribution of node j?



- Naive intuition: Same distribution as node i.
- Example: Consider a graph with 4 nodes and links {1,2}, {2,3}, {3,4}.
  - We have $p_1 = p_2 = 1/2$. Pick a link at random, then randomly pick an end of it, there is a 2/3 chance of finding a node with degree 2 and 1/3 chance of finding a node with degree 1.
  - Higher degree nodes are involved in a higher percentage of the links.

Distribution of the Degree of a Neighboring Node—2

- The degree of a node we reach by following a randomly chosen edge is not given by p_k.
- In the configuration model, an edge emerging from a node has equal chance of terminating at any of the stubs.
- Since there are $2m$ stubs in total, the probability of this edge ending at any particular node of degree k is $k/2m$.
- Since the total number of nodes with degree k is given by np_k, the probability of the edge attaching to a node with degree k is given by

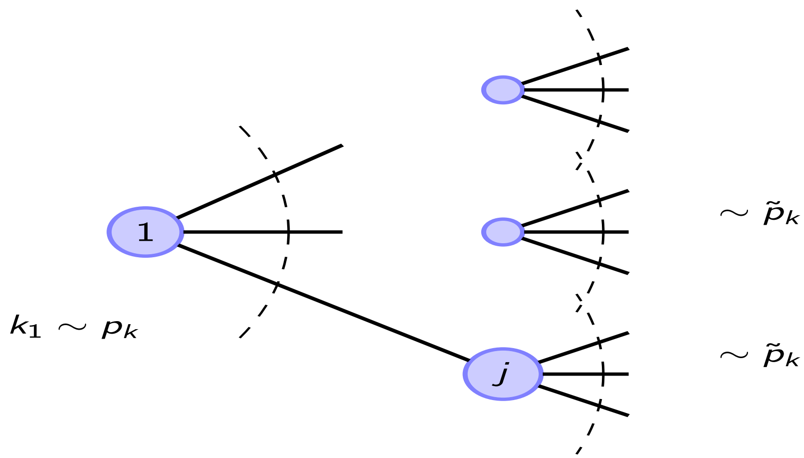
$$\frac{k}{2m} n p_k = \frac{k p_k}{\langle k \rangle},$$

where $\langle k \rangle$ is the expected degree in the network and the equality follows from the relation $2m = n\langle k \rangle$.

Distribution of the Degree of a Neighboring Node—3

- Intuitively, there are k edges that arrive at a node of degree k, we are k times as likely to arrive at that node than another node that has degree 1.
- Thus, the degree distribution of the neighboring node $\tilde{p}_k$ is proportional to kp_k,

$$\tilde{p}_k = \frac{kp_k}{\sum_j jp_j} = \frac{kp_k}{\langle k \rangle}.$$



Emergence of a Giant Component in the Configuration Model—1

- We will use a branching process approximation to analyze the emergence of the giant component.
  - We ignore self loops (can be shown to have small probability) and conflicts (do not matter until the graph grows to a substantial size).
- Note that we have

$$
\begin{aligned}
\mu &= \tilde{\mathbb{E}}[\text{number of children}] = \tilde{\mathbb{E}}[k-1] \\
&= \sum_k k\tilde{p}_k - 1 \\
&= \sum_k \frac{k^2 p_k}{\langle k \rangle} - 1 \\
&= \frac{\langle k^2 \rangle}{\langle k \rangle} - 1.
\end{aligned}
$$

Emergence of a Giant Component in the Configuration Model—2

- Using the branching process analysis, this yields the following threshold for the emergence of the giant component:

  Subcritical: $\mu < 1$, or equivalently

  $$\frac{\langle k^2 \rangle}{\langle k \rangle} < 2 \quad \Leftrightarrow \quad \langle k(k-2) \rangle < 0.$$

  Supercritical: $\mu > 1$, or equivalently

  $$\langle k(k-2) \rangle > 0.$$

- In the case of an Erdös-Renyi graph, we have $\langle k^2 \rangle = \langle k \rangle + \langle k \rangle^2$, and so the giant component emerges when

  $$\langle k \rangle^2 > \langle k \rangle \quad \Leftrightarrow \quad \langle k \rangle > 1.$$

- Since $\langle k \rangle = (n-1)p$ in the Erdös-Renyi graph, this indeed yields the threshold function $t(n) = \frac{1}{n}$ for the emergence of the giant component.