6.207/14.15: Networks
Problem Set 5
Answer Keys

Problem 1

(a) Assume $h(x) > 0$. A consumer with v_i is better off purchasing the good if $v_i h(x) - c \geq 0$. Thus, the best response is $\hat{x} = 1 - F(c/h(x))$. Note that the map $x \mapsto \hat{x}$ is continuous and maps a compact interval $[0, 1]$ to itself. By Brouwer's fixed-point theorem, an equilibrium exists.

(b) Omitted.

Problem 2

(a) It describes a good that a consumer want some people to possess but not many. For example, a party venue that gets better with more attendance, but gets worse when it is too crowded. The value v_i measures how much player i likes to party. The value p is a cover charge for the club; if it is too high there is no equilibrium with postivie attendance.

(b) First, we need to check end points: $x = 0$ is an equilibrium since $u_i < 0$, while $x = 1$ is not. Second, we check interior solutions. Note that $x \geq 1/2$ cannot be an equilibrium since then $g(x) \leq 0$. Let $\bar{v}$ be such that consumers $[\bar{v}, 1]$ purchase and others do not. Since $v_i \sim U[0, 1]$, we have $x^* = 1 - \bar{v}$ in interior equilibria. Thus, x^* solves $(1 - x^*)g(x^*) - p = 0$. The solutions that satisfy $x^* < 1/2$ are $\frac{1-\sqrt{1-4p}}{2}(< \frac{1}{4})$ and $\frac{3-\sqrt{1+16p}}{4}(> \frac{1}{4})$.

(c) 0 and $\frac{3-\sqrt{1+16p}}{4}$ are stable as small deviation will induce incentives to correct it; $\frac{1-\sqrt{1-4p}}{2}$ is not since any small deviation will shift the equilibrium to either of the previous two.

(d) Suppose consumers with values higher than $1-x$ purchase the good. Then the social welfare is

$$\int_{1-x}^{1} [vg(x) - p]dv = \frac{x(2-x)}{2}g(x) - px.$$

This is maximized at $x^* = 1/4$. Therefore, no equilibrium attains the social optimum.

Problem 3

The circle graph with four players have the adjacency matrix of

$$G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Recall that agent i's best response function is

$$\mathrm{BR}_i(x_{-i}) = \max\left\{0,\, 1 - \delta \sum_{j \neq i} g_{ij} x_j \right\}.$$

First, consider an equilibrium where everyone is active. The condition is

$$\begin{aligned} x_1 &= 1 - \delta(x_2 + x_3) > 0, \\ x_2 &= 1 - \delta(x_1 + x_4) > 0, \\ x_3 &= 1 - \delta(x_1 + x_4) > 0, \\ x_4 &= 1 - \delta(x_2 + x_3) > 0. \end{aligned}$$

This yields $(x_1^*, x_2^*, x_3^*, x_4^*) = (\frac{1}{1+2\delta}, \frac{1}{1+2\delta}, \frac{1}{1+2\delta}, \frac{1}{1+2\delta})$ for any $\delta \geq 0$. Second, consider an equilibrium with three active agents.

$$\begin{aligned} x_1 &= 1 - \delta(x_2 + x_3) > 0, \\ x_2 &= 1 - \delta x_1 \qquad\quad > 0, \\ x_3 &= 1 - \delta x_1 \qquad\quad > 0, \\ x_4 &= 1 - \delta(x_2 + x_3) = 0, \end{aligned}$$

which is impossible. Third, consider an equilibrium with two active agents. By the same exercise, we know it is impossible to have agents 1 and 2 active. For the case with agents 1 and 4 active, we have

$$\begin{aligned} x_1 &= 1, \\ x_2 &= 1 - \delta(x_1 + x_4) \leq 0, \\ x_3 &= 1 - \delta(x_1 + x_4) \leq 0, \\ x_4 &= 1. \end{aligned}$$

This yields $(x_1^*, x_2^*, x_3^*, x_4^*) = (1, 0, 0, 1)$ as long as $\delta \geq 1/2$. Note also that its rotation $(0, 1, 1, 0)$ is also an equilibrium.

Finally, we can verify that there is no equilibrium with one or zero active agent. Thus, there are two equilibria as derived above.

Problem 4

Consider the equilibrium strategy in which player 1 plays C, D, C, D, ... as long as player 2 plays D, C, D, C, ..., and vice versa. If the opponent deviates, then each player commits to play D forever. In period 1, player 1's anticipated payoff along the given equilibrium path is

$$-1 + 6\delta - 1\delta^2 + 6\delta^3 - \cdots = \sum_{k=0}^{\infty}(-1 + 6\delta)\delta^{2k} = \frac{-1 + 6\delta}{1 - \delta^2}.$$

By deviating to D, he obtains

$$0 + 0\delta + 0\delta^2 + \cdots = 0.$$

Thus, we need $\delta \geq 1/6$. It is easy to check that player 2 in period 1 (or player 1 in period 2) has no incentive to deviate if $\delta \geq 1/6$. Also, it is easy to see that if either has deviated (so they are in an off-path state), then there is no incentive for either to deviate from playing D forever. Hence, the given strategies constitute an equilibrium if $\delta \geq 1/6$.

Recall that the equilibrium payoff by cooperation in every period is $2+2\delta+2\delta^2+\cdots = \frac{2}{1-\delta}$. Since $\frac{-1+6\delta}{1-\delta^2} + \frac{6-\delta}{1-\delta^2} - \frac{2}{1-\delta} - \frac{2}{1-\delta} = \frac{1}{1-\delta} > 0$, we see that this alternating equilibrium earns higher welfare.

Problem 5

(a) The pure strategy equilibria are (B, B) and (C, C).

(b) In the second period, the highest payoff attainable is 1 at (B, B) since it is the last period. In the first period, the highest possible payoff is 3 at (A, A). I argue that payoff 3 in the first stage is attainable. Consider the strategy in which a player takes A in the first period, and dependeing on the opponent's action in the first period, the player determines the second-stage action; in particular, he takes B in the second period if the opponent took A in the first period, and takes C otherwise. The pair of this strategy earns a payoff of 4, while if one deviates, one can at most obtain $4 - 1 = 3$. Therefore, there is no incentive to deviate and hence it is a subgame perfect equilibrium. Thus, the highest welfare attainable in a SPE is $2(3 + 1) = 8$.